Fibred coarse embeddings of metric spaces and higher index problems

Qin Wang

Research Center for Operator Algebras
East China Normal University, Shanghai, China

2014 ICM Satellite Conference on Operator Algebras and Applications
Cheongpung, Korea August 8, 2014

Motivation: Index theory

Let M be a complete Riemannian manifold, and let D be an elliptic differential operator on M.

Case 1. M is compact.

- D is Fredholm, i.e., invertible modulo compact operators $\mathcal{K}(\mathcal{H})$.
- The Atiyah-Singer index theorem computes $\mathrm{Index}(D)$ by topological data:

$$\begin{aligned} \text{"topological data"} &= \mathrm{Index}(D) \\ &= \dim(\ker(D)) - \dim(\ker(D^*)) \\ &\in \mathbb{Z} = K_0(\mathcal{K}(H)) \end{aligned}$$

Index theory on non-compact manifolds

Case 2. M is non-compact.

- D is no longer Fredholm in the usual sense.
- D is "generalized Fredholm", i.e., invertible modulo a C^*-algebra $C^*(M)$, the Roe algebra of M, which is generated by locally compact operators with finite propagation on M.

Then, the standard procedure in C^*-algebra K-theory defines a generalized Fredholm index of D in the K-theory groups of $C^*(M)$.

Higher index of D

$\mathrm{Index}(D) \in K_*(C^*(M))$

Remark: If M is compact, then $C^*(M) \cong \mathcal{K}(H)$.

Operators on metric spaces: discrete case

Let X be a discrete metric space with bounded geometry, i.e.

$$\forall r > 0 \ \exists N > 0 \ s.t. \ \#B(x,r) < N, \ \forall x \in X.$$

Let H_0 be a separable Hilbert space. A bounded linear operator T on $\ell^2(X) \otimes H_0$ has a matrix form

$$T = [T_{x,y}]_{x,y \in X}$$

with entries $T_{x,y} \in \mathcal{B}(H_0)$.

- T is locally compact if $T_{x,y} \in \mathcal{K}(H_0)$ for all $x, y \in X$.
- T has finite propagation if $\exists \ R > 0$ such that $T_{x,y} = 0$ whenever $d(x,y) > R$.

Finite propagation

A locally compact, finite propagation operator T on $\ell^2(X) \otimes H_0$

$$T = \begin{bmatrix} * & * & * & * & & & & & & & & \\ * & * & * & * & * & & & & & & & \\ * & * & * & * & * & * & & & & & & \\ * & * & * & * & * & * & * & & & & & \\ & * & * & * & * & * & * & * & & & & \\ & & * & * & * & * & * & * & * & & & \\ & & & * & * & * & * & * & * & * & & \\ & & & & * & * & * & * & * & * & * & \\ & & & & & * & * & * & * & * & * & * \\ & & & & & & * & * & * & * & * & * \\ & & & & & & & * & * & * & * & * \\ & & & & & & & & * & * & * & * \end{bmatrix}_{X \times X}$$

with the entries $* \in \mathcal{K}(H_0)$.

The Roe algebra

Let

$$\mathbb{C}[X] = \left\{ T \in \mathcal{B}(\ell^2(X) \otimes H_0) \;\middle|\; \begin{array}{l} \text{locally compact,} \\ \text{finite propagation} \end{array} \right\}.$$

Definition. The Roe algebra of X is defined to be

$$C^*(X) = \overline{\mathbb{C}[X]}^{\|\cdot\|}.$$

The maximal Roe algebra

Suppose X has bounded geometry. The following is well-defined:

Definition. The maximal Roe algebra of X is defined to be

$$C^*_{\max}(X) = \overline{\mathbb{C}[X]}^{\|\cdot\|_{\max}},$$

where

$$\|T\|_{\max} = \sup_{\phi} \left\|\phi(T)\right\|_{\mathcal{B}(H_\phi)}$$

The Coarse Baum-Connes conjecture

There is an assembly map

$$\mu : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(C^*(X)).$$

The Coarse Baum-Connes conjecture

μ is an isomorphism.

Implications:

- The Novikov conjecture.
- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.

The Coarse Novikov conjecture

There is an assembly map

$$\mu : \lim_{d \to \infty} K_*(P_d(X)) \to K_*(C^*(X)).$$

The Coarse Novikov conjecture

μ is injective.

Implications:

- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.

The maximal Coarse Baum-Connes conjecture

Similarly, one can define a maximal higher index map

$$\mu_{\max} : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(C^*_{\max}(X)).$$

The maximal Coarse Baum-Connes conjecture

$\mu_{\max}$ *is an isomorphism.*

Same implications as the Coarse Baum-Connes Conjecture:

- The Novikov conjecture.
- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.

The maximal Coarse Novikov conjecture

$$\mu_{\max} : \lim_{d\to\infty} K_*(P_d(X)) \to K_*(C^*_{\max}(X)).$$

The maximal Coarse Novikov conjecture

$\mu_{\max}$ *is injective.*

Same implications as the Coarse Novikov Conjecture:

- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.

Relations

The canonical map $\lambda : C^*_{\max}(X) \to C^*(X)$ induces the following commutative diagram

$$\begin{tikzcd} & K_*(C^*_{\max}(X)) \arrow[d, "\lambda_*"] \\ \lim_{d\to\infty} K_*(P_d(X)) \arrow[ur, "\mu_{\max}"] \arrow[r, "\mu"] & K_*(C^*(X)) \end{tikzcd}$$

Fibred coarse embedding into Hilbert space

Coarse embedding into Hilbert space

Definition (M. Gromov): A map

$$f : X \to H$$

from X to a Hilbert space H is a *coarse embedding* if there exist non-decreasing functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \rho_\pm(r) = \infty$ such that

$$\rho_-(d(x,y)) \leq \left\| f(x) - f(y) \right\| \leq \rho_+(d(x,y))$$

for all $x, y \in X$.

Gromov's suggestion

Let $\Gamma = \pi_1(M)$ the fundamental group of a closed manifold M, equipped with the word-length metric.

M. Gromov (1993):

Coarse embeddability of Γ into Hilbert space would be helpful to attack the Novikov conjecture for M.

Guoliang Yu

Let X be a metric space with bounded geometry.

Theorem: (G. Yu, Invent. Math. 2000)

If X is coarsely embeddable into Hilbert space, then the coarse Baum-Connes conjecture holds for X.

Applications:

- The Novikov conjecture.
- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.
- $\cdots$

Yu's Property A

Definition. A discrete metric space X has Property A if for every $\varepsilon > 0$ and every $R > 0$ there is a family $\{A_x\}_{x \in X}$ of finite subsets of $X \times \mathbb{N}$ and a number $S > 0$ such that

(1) $\frac{\#(A_x \triangle A_y)}{\#(A_x \cap A_y)} < \varepsilon$ whenever $d(x, y) \leq R$,

(2) $A_x \subseteq B(x, S) \times \mathbb{N}$ for every $x \in X$.

Examples:

- amenable groups; hyperbolic groups; discrete linear groups; groups acting on finite dimensional CAT(0) cube complexes, etc.
- metric spaces with finite asymptotic dimension, or finite decomposition complexity, etc.

Property A $\Longrightarrow$ Coarse Embedding

Theorem (G. Yu, 2000)

If X has Property A, then X is coarsely embeddable in Hilbert space.

Corollary:

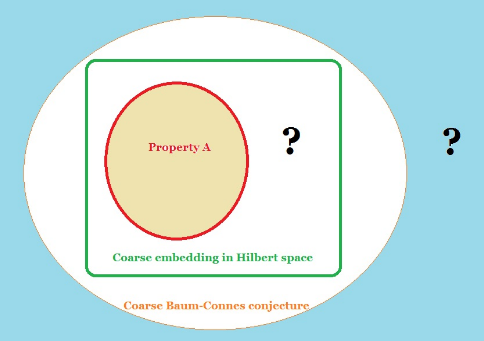
The Novikov conjecture holds for amenable groups; hyperbolic groups; discrete linear groups; groups acting on finite dimensional CAT(0) cube complexes, etc.

Questions since 2000

Are there metric spaces or finitely generated groups that

(1) do not have Property A? or
(2) cannot be coarsely embedded in Hilbert space? or
(3) coarsely embeds in Hilbert space, but do not have Property A?

Questions since 2000



Answers

- **M. Gromov (2000):** The metric space of expander graphs cannot be coarsely embedded in Hilbert space, hence does not have Property A.
- **M. Gromov (2000):** One can construct a finitely generated group which is not coarsely embeddable in Hilbert space.
- **G. Arzhantseva and T. Delzant (2008)** give details of the construction of Gromov's monster groups.
- **G. Arzhantseva, E. Guentner and J. Špakula (2011)** construct a space which is coarsely embeddable in Hilbert space but do not have Property A.

Expander graphs

Definition:

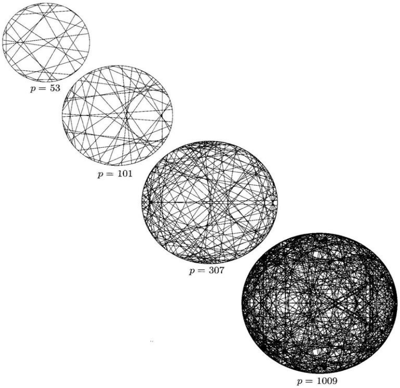
An expander is a sequence $\{(G_n)\}_{n=1}^{\infty}$ of finite connected graphs s.t.

- $\#(G_n) \to \infty$ as $n \to \infty$;
- $\exists\ k > 0$ such that $\deg(G_n) < k$ for all n;
- $\exists c > 0$ such that
$$\frac{\#(\partial A)}{\#(A)} > c$$
for all n and all $A \subset G_n$ with $\#A \leq \frac{1}{2}\#(G_n)$.

Remark: Expander graphs are highly connected sparse graphs.

Expander graphs

EXPANDER GRAPHS AND THEIR APPLICATIONS



A family of explicit 3-regular expander graphs
$x \to \{x-1, x+1, x^{-1}\} \bmod p$

Counterexamples

For an expander $\{(G_n)\}_{n=1}^{\infty}$, let $X = \bigsqcup_{n=1}^{\infty} G_n$ be the disjoint union endowed with a metric d such that d is the graph metric on each graph and $d(G_n, G_m) > n + m$.

Facts: (Gromov, Higson)

- $X = \bigsqcup_{n=1}^{\infty} G_n$ cannot be coarsely embedded into a Hilbert space.

Construction of expander graphs

Definition:

A finitely generated groups Γ is residually finite if there exist normal subgroups $\{\Gamma_i\}_{i=1}^{\infty}$ such that

- $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_i \supset \cdots$.
- $\bigcap_{i=1}^{\infty} \Gamma_i = \{e\}$.
- Γ/Γ_i is a finite group for all i.

Examples: Free groups; linear groups.

Box space

Let Γ be a residually finite group.

Definition

The box space of Γ is $X(\Gamma) = \bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$ with a metric d s.t.

- $d(\Gamma/\Gamma_i, \Gamma/\Gamma_j) \to \infty$ $(i, j \to \infty)$.
- d is the quotient metric on each graph Γ/Γ_i:

$$d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2) : \gamma_1, \gamma_2 \in \Gamma_i\}.$$

Kazhdan's Property (T)

Definition:

A finitely generated group Γ has Kazhdan's Property (T) if every continuous isometric action of Γ on an affine Hilbert space has a fixed point.

Fact: (G. Margulis)

For a finitely generated residually finite group Γ with Property (T), the box space $X(\Gamma) = \bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$ is an expander.

Counterexample: (Higson, Higson-Lafforgue-Skandalis)

For a residually finite, Property (T) infinite groups Γ of linear type, the coarse Baum-Connes assembly map for the box space $X(\Gamma)$ fails to be surjective.

The coarse Novikov conjecture for expanders

V. Lafforgue (Duke Math. J. 2008) constructed certain linear groups whose box spaces are expanders and cannot be coarsely embedded in any uniformly convex Banach spaces.

Theorem (G. Gong-W-G. Yu 2008 J. Reine Angew. Math.)

Let Γ be a finitely generated residually finite group such that the classifying space $E\Gamma/\Gamma$ for free Γ-act has compact homotopy type, then the Strong Novikov Conjecture for Γ and all subgroups Γ_i ($i = 1, 2, \ldots$) implies the Maximal Coarse Geometric Novikov Conjecture for the box metric space $X(\Gamma)$.

Corollary: The (maximal) coarse Novikov conjecture holds for Lafforgue's expander.

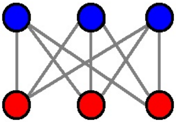
Operator Norm Localization: Chen-Tessera-Wang-Yu (2008), Guentner-Tessera-Yu (2011)

2009 Oyono-Oyono-Yu, JFA

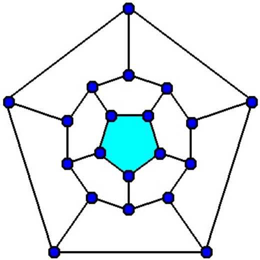
Theorem (H. Oyono-Oyono, G. Yu, 2009 J. Funct. Anal.) If Γ satisfies the Strong Baum-Connes Conjecture, then the maximal Coarse Baum-Connes Conjecture holds for $X(\Gamma)$.

Theorem (H. Oyono-Oyono, G. Yu, 2009 J. Funct. Anal.) If Γ admits a coarse embedding into Hilbert space, then the Coarse Novikov Conjecture holds for $X(\Gamma)$.

Girth of a graph



Girth 4



Girth 5

Definition. The girth of a graph G is the length of the shortest cycle in G. A sequence of graphs $\{G_n\}_{n=1}^{\infty}$ is said to have large girth if $girth(G_n) \to \infty$ as $n \to \infty$.

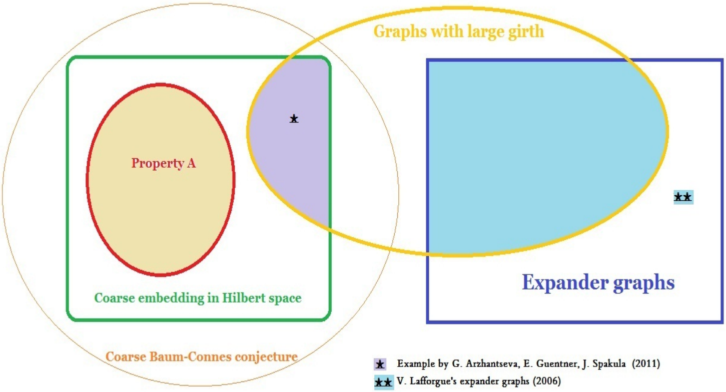
R. Willett and G. Yu

Let $\{G_n\}_{n=1}^{\infty}$ be sequence of graphs with large girth.

Theorem: (R. Willett and G. Yu, Adv. Math. 2012) The maximal coarse Baum-Connes conjecture holds for $X = \bigsqcup_{n=1}^{\infty} G_n$.

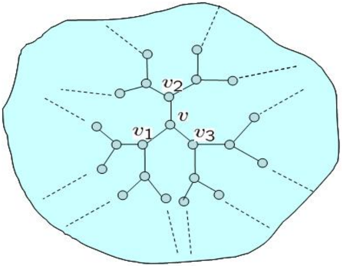
Theorem: (R. Willett, J. Topol. Anal. 2011) If $\deg(G_n) \geq 3$ for all n, then the coarse union $X = \bigsqcup_{n=1}^{\infty} G_n$ does not have Property A.

Expanders with Large girth: Good or Bad ?



Observations:

Large girth implies that local regions C are trees.



There exists coarse embedding $f : C \to H$ such that

$$\left\| f(x) - f(y) \right\| = \left(d(x,y) \right)^{\frac{1}{2}}$$

for all $x, y \in C$.

Notions:

Motivated by graphs with large girth, we consider

- A field of Hilbert spaces over X is a family of Hilbert spaces $\{H_x\}_{x\in X}$.
- A section of the field $\{H_x\}_{x\in X}$ is a map $s : X \to \bigsqcup_{x\in X} H_x$ such that $s(x) \in H_x$.
- For a subset $C \subseteq X$, a trivialization is a map

$$t_C : \bigsqcup_{x\in C} H_x \to C \times H$$

such that $t_C : H_x \to x \times H$ is an affine isometry for all $x \in C$.

Fibred coarse embedding

Definition: (X. Chen-W-G. Yu, 2012/2013, Adv. Math.)
A metric space X is said to admit a fibred coarse embedding into Hilbert space if there exist
(1) a field $\{H_x\}_{x\in X}$ of Hilbert spaces,
(2) a section $s : X \to \bigsqcup_{x\in X} H_x$, and
(3) two maps $\rho_-, \rho_+ : [0,\infty) \to [0,\infty)$ with $\lim_{r\to\infty} \rho_\pm(r) = \infty$
such that, for any $R > 0$ there exists a bounded subset $K \subseteq X$, and for any $C \subset X\backslash K$ of $diameter(C) \leq R$, there exists a trivialization

$$t_C : \bigsqcup_{x\in C} H_x \to C \times H$$

such that

Fibred coarse embedding

(1)

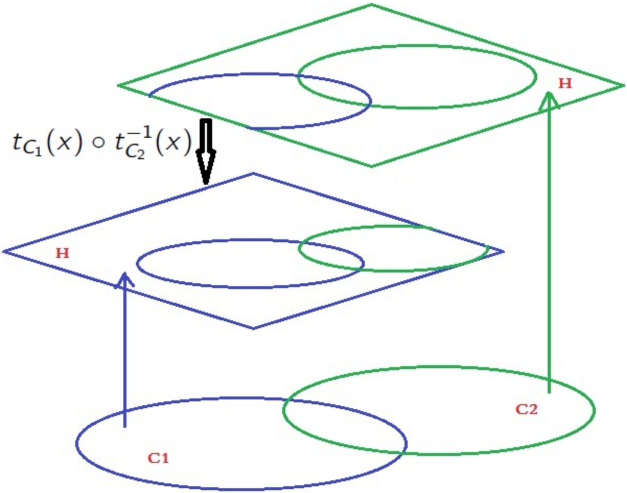
$$\rho_-(d(x,y)) \leq \left\|(t_C \circ s)(x) - (t_C \circ s)(y)\right\| \leq \rho_+(d(x,y))$$

for all $x, y \in C$, and
(2) for $C_1, C_2 \subset X \backslash K$ with $C_1 \cap C_2 \neq \emptyset$,

$$t_{C_1}(x) \circ t_{C_2}^{-1}(x) = t_{C_1}(y) \circ t_{C_2}^{-1}(y)$$

for all $x, y \in C_1 \cap C_2$.

Local trivialization and embedding



Main Result

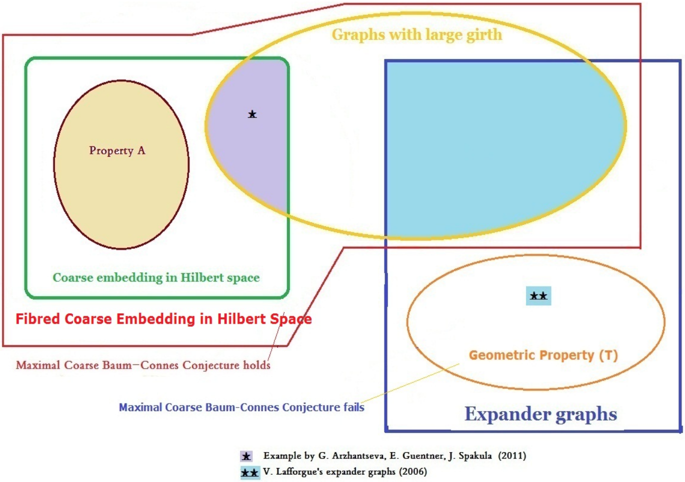
Theorem. (X. Chen-W-G. Yu, 2013 Adv. Math.)

If X admits a fibred coarse embedding into Hilbert space, then the maximal coarse Baum-Connes conjecture holds for X.

Related works in recent years:

- R. Willett and G. Yu's work (2012 Adv. Math.)
- H. Oyono-Oyono and G. Yu's work (2009 J. Funct. Anal.)
- R. Willett and J. Spakula's work (2010), together with G. Yu's work (2000).

Status


Graphs with large girth
Property A
Coarse embedding in Hilbert space
Fibred Coarse Embedding in Hilbert Space
Maximal Coarse Baum−Connes Conjecture holds
Maximal Coarse Baum-Connes Conjecture fails
Geometric Property (T)
Expander graphs
★ Example by G. Arzhantseva, E. Guentner, J. Spakula (2011)
★★ V. Lafforgue's expander graphs (2006)

Examples

If a metric space X is coarsely embeddable into Hilbert space, then clearly it admits a fibred coarse embedding into Hilbert space.

Examples

Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of graphs with large girth. Then $X = \bigsqcup_{n=1}^{\infty} G_n$ admits a fibred coarse embedding in a Hilbert space.

Analytic v.s. Geometric

For a residually finite group Γ, the following hold (by E. Guentner ?, J. Roe 93, G. Margulis 73)

$$\begin{array}{rcll} \Gamma \text{ amenable} & \Longleftrightarrow & \mathrm{Box}_{\{\Gamma_n\}}\Gamma & \text{Yu's property A} \\ \Gamma \text{ Haagerup} & \Longleftarrow & \mathrm{Box}_{\{\Gamma_n\}}\Gamma & \text{coarsely embeds into Hilbert} \\ \Gamma \text{ property (T)} & \Longrightarrow & \mathrm{Box}_{\{\Gamma_n\}}\Gamma & \text{expander.} \end{array}$$

Analytic v.s. Geometric

X. Chen-W-X. Wang, 2013, Bull. London Math. Soc.

Γ has the Haagerup property if and only if any $\mathrm{Box}_{\{\Gamma_n\}}\Gamma$ admits a fibred coarse embedding into Hilbert space.

R. Willett, G. Yu, 2012, Adv. Math.

Γ has property (T) if and only if any $\mathrm{Box}_{\{\Gamma_n\}}\Gamma$ has geometric property (T).

Hence,

$$\begin{aligned} \Gamma \text{ amenable} \quad &\Longleftrightarrow \quad \mathrm{Box}_{\{\Gamma_n\}}\Gamma \text{ Yu's property A}, \\ \Gamma \text{ Haagerup} \quad &\Longleftrightarrow \quad \mathrm{Box}_{\{\Gamma_n\}}\Gamma \text{ fibred coarsely embeddable into} \\ \Gamma \text{ property (T)} \quad &\Longleftrightarrow \quad \mathrm{Box}_{\{\Gamma_n\}}\Gamma \text{ geometric property (T)}, \end{aligned}$$

where $\{\Gamma_n\}$ is any nested sequence of finite index normal subgroups of Γ with trivial intersection.

Thank you !