Fibred coarse embeddings of metric spaces and higher index problems

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2014 ICM Satellite Conference on Operator Algebras and Applications
Cheongpung, Korea August 8, 2014
Motivation: Index theory

Let $M$ be a complete Riemannian manifold, and let $D$ be an elliptic differential operator on $M$.

**Case 1. $M$ is compact.**

- $D$ is Fredholm, i.e., invertible modulo compact operators $\mathcal{K}(\mathcal{H})$.
- The **Atiyah-Singer index theorem** computes $\text{Index}(D)$ by topological data:

  
  
  
  "topological data"  $= \text{Index}(D)$

  $= \dim(\ker(D)) - \dim(\ker(D^*))$

  $\in \mathbb{Z} = K_0(\mathcal{K}(\mathcal{H}))$
Index theory on non-compact manifolds

Case 2. $M$ is non-compact.

- $D$ is no longer Fredholm in the usual sense.
- $D$ is ”generalized Fredholm”, i.e., invertible modulo a $C^*$-algebra $C^*(M)$, the Roe algebra of $M$, which is generated by locally compact operators with finite propagation on $M$.

Then, the standard procedure in $C^*$-algebra $K$-theory defines a generalized Fredholm index of $D$ in the $K$-theory groups of $C^*(M)$.

Higher index of $D$

$$\text{Index}(D) \in K_*(- C^*(M))$$

Remark: If $M$ is compact, then $C^*(M) \cong K(H)$. 
Operators on metric spaces: discrete case

Let $X$ be a discrete metric space with **bounded geometry**, i.e.

$$\forall r > 0 \ \exists N > 0 \ s.t. \ \#B(x, r) < N, \ \forall x \in X.$$  

Let $H_0$ be a separable Hilbert space. A bounded linear operator $T$ on $\ell^2(X) \otimes H_0$ has a matrix form

$$T = [T_{x,y}]_{x,y \in X}$$

with entries $T_{x,y} \in \mathcal{B}(H_0)$.

- $T$ is **locally compact** if $T_{x,y} \in \mathcal{K}(H_0)$ for all $x, y \in X$.
- $T$ has **finite propagation** if $\exists R > 0$ such that $T_{x,y} = 0$ whenever $d(x,y) > R$. 
Finite propagation

A locally compact, finite propagation operator $T$ on $\ell^2(X) \otimes H_0$

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T = 
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\end{bmatrix}
$$

with the entries $* \in \mathcal{K}(H_0)$. 
The Roe algebra

Let

$$\mathbb{C}[X] = \left\{ T \in B(\ell^2(X) \otimes H_0) \mid \text{locally compact, finite propagation} \right\}.$$ 

**Definition.** The Roe algebra of $X$ is defined to be

$$C^*(X) = \overline{\mathbb{C}[X]}_{\| \cdot \|}.$$
Suppose $X$ has bounded geometry. The following is well-defined:

**Definition.** The maximal Roe algebra of $X$ is defined to be

$$C^*_\text{max}(X) = \overline{C[X]} \|\cdot\|_{\text{max}},$$

where

$$\| T \|_{\text{max}} = \sup_{\phi} \| \phi(T) \|_{\mathcal{B}(H_\phi)}$$
The Coarse Baum-Connes conjecture

There is an assembly map

\[ \mu : \lim_{d \to \infty} K_*(P_d(X)) \to K_*(C^*(X)). \]

Implications:

- The Novikov conjecture.
- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.
The Coarse Novikov conjecture

There is an assembly map

$$\mu : \lim_{d \to \infty} K_*(P_d(X)) \to K_*(C^*(X)).$$

The Coarse Novikov conjecture

$\mu$ is injective.

Implications:

- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.
Similarly, one can define a maximal higher index map

$$\mu_{\text{max}} : \lim_{d \to \infty} K_\ast(P_d(X)) \to K_\ast(C^*_{\text{max}}(X)).$$

The maximal Coarse Baum-Connes conjecture

$$\mu_{\text{max}} \text{ is an isomorphism.}$$

Same implications as the Coarse Baum-Connes Conjecture:

- The Novikov conjecture.
- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.
The maximal Coarse Novikov conjecture

\[ \mu_{\text{max}} : \lim_{d \to \infty} K_*(P_d(X)) \to K_*(C^*_\text{max}(X)). \]

\textbf{The maximal Coarse Novikov conjecture}

\[ \mu_{\text{max}} \text{ is injective.} \]

\textbf{Same implications as the Coarse Novikov Conjecture:}

- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.
The canonical map $\lambda : C^*_{\text{max}}(X) \to C^*(X)$ induces the following commutative diagram:

$$
\begin{align*}
\lim_{d \to \infty} K_* (P_d(X)) & \xrightarrow{\mu} K_* (C^*(X)) \\
& \xrightarrow{\lambda_*} K_* (C^*_{\text{max}}(X)) \\
& \xrightarrow{\mu_{\text{max}}} K_* (C^*_{\text{max}}(X))
\end{align*}
$$
Fibred coarse embedding into Hilbert space
Coarse embedding into Hilbert space

**Definition (M. Gromov):** A map

\[ f : X \to H \]

from \( X \) to a Hilbert space \( H \) is a *coarse embedding* if there exist non-decreasing functions \( \rho_-, \rho_+ : [0, \infty) \to [0, \infty) \) with \( \lim_{r \to \infty} \rho_{\pm}(r) = \infty \) such that

\[
\rho_-(d(x, y)) \leq \| f(x) - f(y) \| \leq \rho_+(d(x, y))
\]

for all \( x, y \in X \).
Let $\Gamma = \pi_1(M)$ the fundamental group of a closed manifold $M$, equipped with the word-length metric.

M. Gromov (1993):
Coarse embeddability of $\Gamma$ into Hilbert space would be helpful to attack the Novikov conjecture for $M$. 
Let $X$ be a metric space with bounded geometry.

**Theorem:** (G. Yu, Invent. Math. 2000)
If $X$ is coarsely embeddable into Hilbert space, then the coarse Baum-Connes conjecture holds for $X$.

**Applications:**
- The Novikov conjecture.
- The Gromov positive scalar curvature conjecture.
- The zero-in-the-spectrum conjecture.
- ...
Yu’s Property A

**Definition.** A discrete metric space $X$ has Property A if for every $\varepsilon > 0$ and every $R > 0$ there is a family $\{A_x\}_{x \in X}$ of finite subsets of $X \times \mathbb{N}$ and a number $S > 0$ such that

1. $\frac{\#(A_x \triangle A_y)}{\#(A_x \cap A_y)} < \varepsilon$ whenever $d(x, y) \leq R$,

2. $A_x \subseteq B(x, S) \times \mathbb{N}$ for every $x \in X$.

**Examples:**

- amenable groups; hyperbolic groups; discrete linear groups; groups acting on finite dimensional CAT(0) cube complexes, etc.

- metric spaces with finite asymptotic dimension, or finite decomposition complexity, etc.
Property A $\iff$ Coarse Embedding

**Theorem (G. Yu, 2000)**
If $X$ has Property A, then $X$ is coarsely embeddable in Hilbert space.

**Corollary:**
The Novikov conjecture holds for amenable groups; hyperbolic groups; discrete linear groups; groups acting on finite dimensional CAT(0) cube complexes, etc.
Questions since 2000

Are there metric spaces or finitely generated groups that

(1) do not have Property A? or
(2) cannot be coarsely embedded in Hilbert space? or
(3) coarsely embeds in Hilbert space, but do not have Property A?
Questions since 2000

Fibred coarse embeddings of metric spaces and higher index problems

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M. Gromov (2000): The metric space of expander graphs cannot be coarsely embedded in Hilbert space, hence does not have Property A.

M. Gromov (2000): One can construct a finitely generated group which is not coarsely embeddable in Hilbert space.


G. Arzhantseva, E. Guentner and J. Špakula (2011) construct a space which is coarsely embeddable in Hilbert space but do not have Property A.
Expander graphs

**Definition:**
An expander is a sequence \( \{(G_n)\}_{n=1}^{\infty} \) of finite connected graphs s.t.
- \( \#(G_n) \to \infty \) as \( n \to \infty \); 
- \( \exists k > 0 \) such that \( \text{deg}(G_n) < k \) for all \( n \); 
- \( \exists c > 0 \) such that 
  \[
  \frac{\#(\partial A)}{\#(A)} > c
  \]
  for all \( n \) and all \( A \subset G_n \) with \( \#A \leq \frac{1}{2} \#(G_n) \).

**Remark:** Expander graphs are highly connected sparse graphs.
A family of explicit 3-regular expander graphs
\[ x \rightarrow \{x-1, x+1, x^{-1}\} \mod p \]
Counterexamples

For an expander \( \{(G_n)\}_{n=1}^{\infty} \), let \( X = \bigsqcup_{n=1}^{\infty} G_n \) be the disjoint union endowed with a metric \( d \) such that \( d \) is the graph metric on each graph and \( d(G_n, G_m) > n + m \).

**Facts: (Gromov, Higson)**

- \( X = \bigsqcup_{n=1}^{\infty} G_n \) cannot be coarsely embedded into a Hilbert space.
Construction of expander graphs

Definition:

A finitely generated groups $\Gamma$ is **residually finite** if there exist normal subgroups $\{\Gamma_i\}_{i=1}^{\infty}$ such that

- $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_i \supset \cdots$.
- $\bigcap_{i=1}^{\infty} \Gamma_i = \{e\}$.
- $\Gamma/\Gamma_i$ is a finite group for all $i$.

**Examples:** Free groups; linear groups.
Let $\Gamma$ be a residually finite group.

**Definition**

The box space of $\Gamma$ is $X(\Gamma) = \bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$ with a metric $d$ s.t.
- $d(\Gamma/\Gamma_i, \Gamma/\Gamma_j) \to \infty$ ($i, j \to \infty$).
- $d$ is the quotient metric on each graph $\Gamma/\Gamma_i$:

$$d(a\Gamma_i, b\Gamma_i) = \min\{d(a\gamma_1, b\gamma_2) : \gamma_1, \gamma_2 \in \Gamma_i\}.$$
Kazhdan’s Property (T)

Definition:
A finitely generated group $\Gamma$ has Kazhdan’s Property (T) if every continuous isometric action of $\Gamma$ on an affine Hilbert space has a fixed point.

Fact: (G. Margulis)
For a finitely generated residually finite group $\Gamma$ with Property (T), the box space $X(\Gamma) = \bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$ is an expander.

Counterexample: (Higson, Higson-Lafforgue-Skandalis)
For a residually finite, Property (T) infinite groups $\Gamma$ of linear type, the coarse Baum-Connes assembly map for the box space $X(\Gamma)$ fails to be surjective.
The coarse Novikov conjecture for expanders

V. Lafforgue (Duke Math. J. 2008) constructed certain linear groups whose box spaces are expanders and cannot be coarsely embedded in any uniformly convex Banach spaces.

**Theorem (G. Gong-W-G. Yu 2008 J. Reine Angew. Math.)**

Let $\Gamma$ be a finitely generated residually finite group such that the classifying space $E\Gamma/\Gamma$ for free $\Gamma$-act has compact homotopy type, then the Strong Novikov Conjecture for $\Gamma$ and all subgroups $\Gamma_i$ ($i = 1, 2, \ldots$) implies the *Maximal* Coarse Geometric Novikov Conjecture for the box metric space $X(\Gamma)$.

**Corollary:** The (maximal) coarse Novikov conjecture holds for Lafforgue’s expander.

**Operator Norm Localization:** Chen-Tessera-Wang-Yu (2008), Guentner-Tessera-Yu (2011)
Theorem (H. Oyono-Oyono, G. Yu, 2009 J. Funct. Anal.) If $\Gamma$ satisfies the Strong Baum-Connes Conjecture, then the maximal Coarse Baum-Connes Conjecture holds for $X(\Gamma)$.

Theorem (H. Oyono-Oyono, G. Yu, 2009 J. Funct. Anal.) If $\Gamma$ admits a coarse embedding into Hilbert space, then the Coarse Novikov Conjecture holds for $X(\Gamma)$. 
Girth of a graph

**Definition.** The *girth* of a graph $G$ is the length of the shortest cycle in $G$. A sequence of graphs $\{G_n\}_{n=1}^\infty$ is said to have **large girth** if $\text{girth}(G_n) \to \infty$ as $n \to \infty$. 
Let $\{G_n\}_{n=1}^{\infty}$ be sequence of graphs with large girth.

**Theorem:** (R. Willett and G. Yu, Adv. Math. 2012) The maximal coarse Baum-Connes conjecture holds for $X = \bigsqcup_{n=1}^{\infty} G_n$.

**Theorem:** (R. Willett, J. Topol. Anal. 2011) If $\text{deg}(G_n) \geq 3$ for all $n$, then the coarse union $X = \bigsqcup_{n=1}^{\infty} G_n$ does not have Property A.
Expanders with Large girth: Good or Bad?

Fibred coarse embeddings of metric spaces and higher index problems

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- Property A
- Coarse embedding in Hilbert space
- Coarse Baum-Connes conjecture
- Graphs with large girth
- Expander graphs

Example by G. Arzhantseva, E. Guentner, J. Spakula (2011)
V. Lafforgue's expander graphs (2006)
Observations:

Large girth implies that local regions $C$ are trees.

There exists coarse embedding $f : C \to H$ such that

$$\|f(x) - f(y)\| = \left( d(x, y) \right)^{\frac{1}{2}}$$

for all $x, y \in C$. 
Notions:

Motivated by graphs with large girth, we consider

- A field of Hilbert spaces over $X$ is a family of Hilbert spaces $\{H_x\}_{x \in X}$.
- A section of the field $\{H_x\}_{x \in X}$ is a map $s : X \to \bigsqcup_{x \in X} H_x$ such that $s(x) \in H_x$.
- For a subset $C \subseteq X$, a trivialization is a map

$$t_C : \bigsqcup_{x \in C} H_x \to C \times H$$

such that $t_C : H_x \to x \times H$ is an affine isometry for all $x \in C$. 
**Definition:** (X. Chen-W-G. Yu, 2012/2013, Adv. Math.)
A metric space $X$ is said to admit a fibred coarse embedding into Hilbert space if there exist
(1) a field $\{H_x\}_{x \in X}$ of Hilbert spaces,
(2) a section $s : X \to \bigsqcup_{x \in X} H_x$, and
(3) two maps $\rho_- : [0, \infty) \to [0, \infty)$ with $\lim_{r \to \infty} \rho_+(r) = \infty$
such that, for any $R > 0$ there exists a bounded subset $K \subseteq X$, and for any $C \subset X \setminus K$ of diameter $(C) \leq R$, there exists a trivialization
\[
t_C : \bigsqcup_{x \in C} H_x \to C \times H
\]
such that
Fibred coarse embedding

(1) \[ \rho_-(d(x, y)) \leq \left\| (t_C \circ s)(x) - (t_C \circ s)(y) \right\| \leq \rho_+(d(x, y)) \]

for all \( x, y \in C \), and

(2) for \( C_1, C_2 \subset X \setminus K \) with \( C_1 \cap C_2 \neq \emptyset \),

\[ t_{C_1}(x) \circ t_{C_2}^{-1}(x) = t_{C_1}(y) \circ t_{C_2}^{-1}(y) \]

for all \( x, y \in C_1 \cap C_2 \).
Local trivialization and embedding

\[ t_{C_1}(x) \circ t_{C_2}^{-1}(x) \]
Main Result

If $X$ admits a fibred coarse embedding into Hilbert space, then the maximal coarse Baum-Connes conjecture holds for $X$.

Related works in recent years:

Status

Fibred coarse embeddings of metric spaces and higher index problems

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- Fibred Coarse Embedding in Hilbert Space
- Coarse embedding in Hilbert space
- Maximal Coarse Baum–Connes Conjecture holds
- Maximal Coarse Baum–Connes Conjecture fails

Graphs with large girth

Expander graphs

- V. Lafforgue's expander graphs (2006)
Examples

If a metric space $X$ is coarsely embeddable into Hilbert space, then clearly it admits a fibred coarse embedding into Hilbert space.
Examples

Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of graphs with large girth. Then $X = \bigsqcup_{n=1}^{\infty} G_n$ admits a fibred coarse embedding in a Hilbert space.
Analytic v.s. Geometric

For a residually finite group $\Gamma$, the following hold (by E. Guentner ?, J. Roe 93, G. Margulis 73)

\[ \Gamma \text{ amenable} \iff \text{Box}\{\Gamma_n\}\Gamma \quad \text{Yu’s property A} \]
\[ \Gamma \text{ Haagerup} \iff \text{Box}\{\Gamma_n\}\Gamma \quad \text{coarsely embeds into Hilbert} \]
\[ \Gamma \text{ property (T)} \implies \text{Box}\{\Gamma_n\}\Gamma \quad \text{expander.} \]
Analytic v.s. Geometric

Γ has the Haagerup property if and only if any $\Box\{\Gamma_n\} \Gamma$ admits a fibred coarse embedding into Hilbert space.

Γ has property (T) if and only if any $\Box\{\Gamma_n\} \Gamma$ has geometric property (T).

Hence,

Γ amenable $\iff$ $\Box\{\Gamma_n\} \Gamma$ Yu’s property A,
Γ Haagerup $\iff$ $\Box\{\Gamma_n\} \Gamma$ fibred coarsely embeddable into
Γ property (T) $\iff$ $\Box\{\Gamma_n\} \Gamma$ geometric property (T),

where $\{\Gamma_n\}$ is any nested sequence of finite index normal subgroups of Γ with trivial intersection.
Fibred coarse embeddings of metric spaces and higher index problems

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Thank you!