세종대학교
SEJONG UNIVERSITY

Toeplitz operators and their binormality

Ji Eun Lee*
(With Eungil Ko)

Mathematics-Applied Statistics, Sejong University, Seoul, Republic of Korea

2014 ICM Satellite Conference on Operator Algebras and Applications,
August 10, Cheongpung, Korea

Table of contents

1. Motivation
   - Study
2. Introduction
   - Binormal operator
   - Toeplitz operator
3. Main Results
   - Binormal Toeplitz
4. Reference

Study

세종대학교 SEJONG UNIVERSITY

Study

- A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine angew. Math., **213**(1963-1964), 89-102.
- I. Amemiya, T. Ito, and T. K. Wong, *On quasinormal Toeplitz operators*, Proc. Amer. Math. Soc. **50** (1975), 254-258.
- R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
- Eungil Ko and Ji Eun Lee, *Normal truncated Toeplitz operators on finite dimensional spaces*, Linear and Multilinear Algebra, to appear (2014).
- Eungil Ko and Ji Eun Lee, *Characterizations of binormal Toeplitz operators on the Hardy space H^2*, preprint.

Binormal operator

- $\mathcal{H}$: a complex (separable) Hilbert space
- $\mathcal{L}(\mathcal{H})$: the algebra of all bounded linear operators on $\mathcal{H}$.

Definition

- T is *normal* if $T^*T = TT^*$.
- T is *quasinormal* if T^*T and T commute.
- T is *binormal* if T^*T and TT^* commute.
- T is *subnormal* if $\exists$ a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $T = N|_{\mathcal{H}}$.
- T is *hyponormal* if $T^*T - TT^* \geq 0$.

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{binormal}.$$

Binormal operator

- $\mathcal{M}$ is *nontrivial* if it is different from (0) and $\mathcal{H}$.
- A closed subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T if $T\mathcal{M} \subset \mathcal{M}$.

Known fact, Campbell(1972)

- If $T \in \mathcal{L}(\mathcal{H})$ is hyponormal and binormal, then T has a non-trivial invariant subspace.

Hardy space

- $L^2 := L^2(\partial\mathbb{D})$ is the usual Lebesque space on the unit circle $\partial\mathbb{D}$.
- L^∞ is the Banach space consisting of all essentially bounded functions on $\partial\mathbb{D}$.
- $\{z^n : n = 0, \pm 1, \pm 2, \pm 3, \cdots\}$ is an orthonormal basis for L^2.
- $H^2 = \{f \mid f$: analytic functions on $\mathbb{D}$ with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

:*Hilbert Hardy space*.

- $H^2 = \overline{\text{span}\{z^n : n = 0, 1, 2, 3, \cdots\}}$.
- H^∞ is the space of bounded analytic functions on $\mathbb{D}$.

Toeplitz operator

Definition

For any $\varphi \in L^\infty$, the *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by the formula

$$T_\varphi f = P(\varphi f), f \in H^2$$

where P denotes the orthogonal projection of L^2 onto H^2.

- T_φ is bounded if and only if $\varphi \in L^\infty$ and $\|T_\varphi\| = \|\varphi\|_\infty$.
- T_φ is a Toeplitz operator if and only if $S^* T_\varphi S = T_\varphi$ where S is the unilateral shift on H^2, i.e., $Sf(z) = zf(z)$ for $f \in H^2$.

Toeplitz operator

세종대학교 SEJONG UNIVERSITY

Known facts

- (In 1963, A. Brown and P. R. Halmos)
  T_φ is normal if and only if $\varphi = \alpha + \beta\rho$ where ρ is a real valued function in L^∞ and $\alpha, \beta \in \mathbb{C}$.
- (In 1975, I. Amemiya, T. Ito, and T. K. Wong)
  Every quasinormal Toeplitz operator is either normal or analytic.
- (In 1988, C. Cowen)
  For $\varphi \in L^\infty$, let $\varphi = f + \overline{g}$ where f and g are in H^2. Then the Toeplitz operator T_φ is hyponormal if and only if $g = c + T_{\overline{h}} f$ for some constant c and some function $h \in H^\infty$ with $\| h \|_\infty \leq 1$.

Binormal Toeplitz

세종대학교 SEJONG UNIVERSITY

Question

- When are Toeplitz operators binormal?

Notation

(i) $A = T_\varphi^* T_\varphi$, $B = T_\varphi T_\varphi^*$, and S denotes the unilateral shift.
(ii) $\varphi_+ = S^* T_\varphi e_0 = \sum_{n=0}^{\infty} \hat{\varphi}(n+1) e_n$.
(iii) $\varphi_- = S^* T_\varphi^* e_0 = \sum_{n=0}^{\infty} \overline{\hat{\varphi}(-n-1)} e_n$.
(iv) $F = (S^* A e_0 \otimes S^* B e_0) + (\varphi_- \otimes B\varphi_-) + (S^* A S \varphi_+ \otimes \varphi_+)$.
(v) $E_1 = S^* A e_0 \otimes S^* B e_0$, $E_2 = \varphi_- \otimes B\varphi_-$, and
$E_3 = S^* A S \varphi_+ \otimes \varphi_+$.

Binormal Toeplitz

세종대학교 SEJONG UNIVERSITY

Theorem

Let T_φ be a Toeplitz operator on H^2 with a symbol $\varphi \in L^\infty$. Then the following statements are equivalent:

(i) T_φ is binormal.

(ii) The operator

$$F = (S^* A e_0 \otimes S^* B e_0) + (\varphi_- \otimes B\varphi_-) + (S^* A S \varphi_+ \otimes \varphi_+)$$

is a self-adjoint operator.

Binormal Toeplitz

세종대학교 SEJONG UNIVERSITY

Corollary

Let T_φ be Toeplitz operator with a symbol $\varphi \in L^\infty$. Assume that one of the following statements holds:

(i) T_φ is normal.

(ii) T_φ is analytic and $\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$.

(iii) T_φ is coanalytic and $\varphi = \lambda \overline{u}$ for an inner function u and $\lambda \in \mathbb{C}$.

(iv) $B = aA + bI$ for some nonzero real a and real b.

(v) A and B are a linear combination of T_φ and the identity I.

(vi) A and B are a linear combination of T_φ^* and the identity I.

(vii) $A = aT_\varphi + \overline{a}{T_\varphi}^*$ and $B = bT_\varphi + \overline{b}{T_\varphi}^*$ for some $a, b \in \mathbb{C}$.

Then F is self-adjoint. Hence, in this case, T_φ is binormal.

Examples

세종대학교 SEJONG UNIVERSITY

Example

Assume that one of the followings holds.
(i) $\varphi(z) = az^2 + b\overline{z}$ or $\varphi(z) = az + b\overline{z}^2$ for any nonzero $a, b \in \mathbb{C}$.
(ii) $\varphi(z) = z + z^2$ or $\varphi(z) = \overline{z} + \overline{z}^2$.
Hence, by Theorem 1, T_φ is not binormal.

Example

Let $\varphi(z) = az^2 + bz + c + d\overline{z} + e\overline{z}^2$ where a, b, d, e are nonzeros.
If $|a| = |e|$ and $\overline{a}d = \overline{b}e$, then, by [Lee], T_φ is binormal.

Example

Let $\varphi(z) = (1 + 2i)z^7$ or $\varphi(z) = (i - 1)\overline{z}^6$. Then by Theorem 1, T_φ is binormal.

Binormal Toeplitz

세종대학교 SEJONG UNIVERSITY

Corollary

Suppose that $\varphi(z) = az^n + b + c\overline{z}^n$ *for some* $a, b, c \in \mathbb{C}$ *and* $n \in \mathbb{N}$. *Then* T_φ *is binormal if and only if* $|a| = |c|$, $a = c = 0$, $a = b = 0$, *or* $b = c = 0$.

e.g. If $\varphi(z) = z + \frac{1}{2}\overline{z}$, then T_φ is not binormal. But it is hyponormal and not subnormal.

I. One of E_i is nonzero and the others are zero

Theorem

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on H^2 and let E_i be rank-one operators. Assume that one of E_i is nonzero and the others are zero. Then T_φ is binormal if and only if one of the following statements holds:

(1) T_φ is normal.

(2) T_φ is analytic and $\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$.

(3) T_φ is coanalytic and $\varphi = \lambda \overline{u}$ for an inner function u and $\lambda \in \mathbb{C}$.

(4) e_0 is an eigenvector of A with respect to real λ, φ_+ is an eigenvector of S^*AS with respect to nonzero real μ, and φ_- is an eigenvector of B with respect to 0.

I. One of E_i is nonzero and the others are zero

Theorem

(5) e_0 *and* φ_- *are eigenvectors of* B *with respect to real* λ_1 *and* 0*, respectively, and* φ_+ *is an eigenvector of* S^*AS *with respect to nonzero real* μ*.*

(6) e_0 *is an eigenvector of* A *with respect to real* λ*,* φ_+ *is an eigenvector of* S^*AS *with respect to* 0*, and* φ_- *is an eigenvector of* B *with respect to nonzero real* μ*.*

(7) e_0 *and* φ_- *are eigenvectors of* B *with respect to real* λ_1 *and* μ*, respectively, and* φ_+ *is an eigenvector of* S^*AS *with respect to* 0*.*

(8) e_0 *is an eigenvector of* $A - \mu B$ *with respect to real* λ*,* φ_+ *is an eigenvector of* S^*AS *with respect to* 0*, and* φ_- *is an eigenvector of* B *with respect to* 0*.*

II. Two of E_i are nonzero

세종대학교 SEJONG UNIVERSITY

Theorem

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on H^2. Suppose that two of E_i are nonzero for $i = 1, 2, 3$.
(a) If $E_1 + E_2$ is rank one, then T_φ is binormal if and only if one of the following statements holds:
(1) T_φ is coanalytic and $\varphi = \lambda \overline{u}$ for an inner function u and $\lambda \in \mathbb{C}$.
(2) φ_+ is an eigenvector of S^*AS corresponding to the eigenvalue 0 and one of the following assertions holds:

$$\begin{cases} (i\ )\ B\varphi_- = a\varphi_- \text{ for some } a \in \mathbb{C}. \\ (ii\ )\ \varphi_- = aS^*Be_0 + bB\varphi_- \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}. \\ (iii\ )\ \varphi_- + aS^*Ae_0 = bB\varphi_- \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}. \end{cases}$$

II. Two of E_i are nonzero

세종대학교 SEJONG UNIVERSITY

Theorem

(b) If $E_1 + E_3$ is rank one, then T_φ is binormal if and only if one of the following arguments holds:

(1) T_φ is analytic and $\varphi = \lambda u$ for an inner function u and $\lambda \in \mathbb{C}$.

(2) φ_- is an eigenvector of B corresponding to the eigenvalue 0 and one of the following assertions holds:

$$\begin{cases} (i)\ \varphi_+ = aS^*AS\varphi_+ \text{ for some } a \in \mathbb{C}. \\ (ii)\ S^*AS\varphi_+ = aS^*Be_0 + b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}. \\ (iii)\ aS^*Ae_0 + S^*AS\varphi_+ = b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and } b \in \mathbb{R} \setminus \{0\}. \end{cases}$$

II. Two of E_i are nonzero

세종대학교 SEJONG UNIVERSITY

Theorem

(c) If $E_2 + E_3$ is rank one, then T_φ is binormal if and only if e_0 is an eigenvector of A with respect to the real eigenvalue λ or e_0 is an eigenvector of B with respect to the real eigenvalue λ_1 and one of the following statements holds:

$$\begin{cases} (i)\ \varphi_+ = aS^*AS\varphi_+ \text{ for some } a \in \mathbb{C}. \\ (ii)\ S^*AS\varphi_+ = aB\varphi_- + b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and nonzero } b \in \mathbb{R}. \\ (iii)\ a\varphi_- + S^*AS\varphi_+ = b\varphi_+ \text{ for some } a \in \mathbb{C} \text{ and nonzero } b \in \mathbb{R}. \end{cases}$$

II. Two of E_i are nonzero

세종대학교 SEJONG UNIVERSITY

Theorem

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on H^2 with a symbol $\varphi \in L^\infty$ and let two of E_i be nonzero for $i = 1, 2, 3$.
(a) If $E_1 + E_2$ is rank two, then T_φ is binormal if and only if φ_+ is an eigenvector of S^*AS corresponding to the eigenvalue 0 and

$$S^*Be_0 = aS^*Ae_0 + b\varphi_- \text{ and } B\varphi_- = \overline{b}S^*Ae_0 + c\varphi_-$$

for some nonzero constants $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$.

II. Two of E_i are nonzero

세종대학교 SEJONG UNIVERSITY

Theorem

(b) If $E_1 + E_3$ is rank two, then T_φ is binormal if and only if φ_- is an eigenvector of B corresponding to the eigenvalue 0 and $S^*Ae_0 = aS^*Be_0 + b\varphi_+$ and $S^*AS\varphi_+ = \overline{b}S^*Be_0 + c\varphi_+$ for some nonzero constants $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$.

(c) If $E_2 + E_3$ is rank two, then T_φ is binormal if and only if e_0 is an eigenvector of A with respect to the real eigenvalue λ or e_0 is an eigenvector of B with respect to the real eigenvalue λ_1 and

$$B\varphi_- = a\varphi_- + bS^*AS\varphi_+ \text{ and } \varphi_+ = \overline{b}\varphi_- + cS^*AS\varphi_+$$

for some nonzero constants $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$.

III. All of E_i are nonzero

Theorem

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on H^2 and let E_i be nonzero rank-one operators.
(a) If F is rank one, then T_φ is binormal if and only if $S^*Ae_0 = a\varphi_+, S^*Be_0 = b\varphi_+, S^*AS\varphi_+ = c\varphi_+, \varphi_- = d\varphi_+$, and $B\varphi_- = e\varphi_+$ for some $a, b, c, d, e \in \mathbb{C}$.

III. All of E_i are nonzero

세종대학교 SEJONG UNIVERSITY

Theorem

(b) If F is rank two, then T_φ is binormal if and only if one of the following statements holds:

$$\begin{cases} (i)\ S^*Ae_0 = a\varphi_+ + b\varphi_-,\ S^*Be_0 = c\varphi_+, \\ S^*AS\varphi_+ = dS^*Ae_0,\ \text{and}\ B\varphi_- = e\varphi_+ + f\varphi_-. \\ (ii)\ S^*Ae_0 = a\varphi_-,\ S^*Be_0 = b\varphi_+ + c\varphi_-, \\ B\varphi_- = dS^*Be_0,\ \text{and}\ S^*AS\varphi_+ = e\varphi_+ + f\varphi_-. \\ (iii)\ S^*Ae_0 = a\varphi_+ + b\varphi_-,\ S^*Be_0 = c\varphi_+ + d\varphi_-, \\ S^*AS\varphi_+ = e\varphi_-,\ \text{and}\ B\varphi_- = f\varphi_+ \end{cases}$$

for some $a, b, c, d, e, f \in \mathbb{C}$.

III. All of E_i are nonzero

세종대학교 SEJONG UNIVERSITY

Theorem

(c) If F is rank three, then T_φ is binormal if and only if

$$\begin{cases} S^*Be_0 = aS^*Ae_0 + b\varphi_- + cS^*AS\varphi_+, \\ B\varphi_- = \overline{b}S^*Ae_0 + d\varphi_- + eS^*AS\varphi_+, \text{ and} \\ \varphi_+ = \overline{c}S^*Ae_0 + \overline{e}\varphi_- + fS^*AS\varphi_+ \end{cases}$$

for some nonzero $a, d, f \in \mathbb{R}$ and $b, c, e \in \mathbb{C}$.

IV. All of E_i are zero

세종대학교 SEJONG UNIVERSITY

Theorem

Let $T_\varphi \in \mathcal{L}(H^2)$ be a Toeplitz operator on H^2 and let $E_1 = 0, E_2 = 0$, and $E_3 = 0$. Then T_φ is binormal if and only if one of the following statements holds:

(1) T_φ is normal, i.e, $T_\varphi = T_{\varphi_0}$ where φ_0 is the 0-th Fourier coefficient of φ.

(2) e_0 is an eigenvector of A with respect to real eigenvalue λ, φ_+ is an eigenvector of S^*AS with respect to the eigenvalue 0, and φ_- is an eigenvector of B with respect to 0.

(3) e_0 is an eigenvector of B with respect to real eigenvalue λ_1, φ_+ is an eigenvector of S^*AS with respect to the eigenvalue 0, and φ_- is an eigenvector of B with respect to 0.

Binormal Toeplitz

세종대학교 SEJONG UNIVERSITY

Corollary

If $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is a trigonometric polynomial where one of a_{-m} and a_N is nonzero, then T_φ is binormal if and only if it has one of the following symbols:

(i) $\varphi(z) = \sum_{n=-m}^{m} a_n z^n$ where $|a_{-m}| = |a_m|$ with $a_m \neq 0$ and

$$\overline{a_m} \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \vdots \\ \overline{a_m} \end{pmatrix}, \tag{1}$$

(ii) $\varphi(z) = a_N z^N$, (iii) $\varphi(z) = a_{-m}\overline{z}^m$, (iv) $\varphi(z) = a_0$.

Binormal Toeplitz

세종대학교 SEJONG UNIVERSITY

Corollary

Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ *be a trigonometric polynomial where one of* a_{-m} *and* a_N *is nonzero. If* T_φ *is binormal with* $T_\varphi \neq \lambda I$*, then it has a nontrivial invariant subspace.*

Conditions of binormality

세종대학교 SEJONG UNIVERSITY

Theorem

Let $\varphi = \varphi_+ + \overline{\varphi_-}$ *where* φ_+ *and* φ_- *are in* zH^2 *and let* $\varphi_- = \lambda\varphi_+$ *for some* $\lambda \in \mathbb{C}$ *with* $\varphi_+ \neq 0$. *Suppose that one of the following assertions holds:*

(i) $\varphi_- = \lambda\varphi_+$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$.

(ii) $S^*Be_0 = \mu S^*Ce_0$ *for some* $\mu \in \mathbb{R} \setminus \{0\}$ *and* $S^*BS - |\lambda|^2 B = (|\lambda|^2 - 1)S^*CS$.

(iii) $S^*Be_0 = \mu S^*Ce_0$ *and* $[S^*BS - |\lambda|^2 B + (1 - |\lambda|^2)S^*CS]\varphi_+ = \mu_1\varphi_+$ *for some* $\mu, \mu_1 \in \mathbb{R} \setminus \{0\}$, *where* $B = T_\varphi T_\varphi^*$ *and* $C = [T_{\varphi_+}^*, T_{\varphi_+}]$ *is self-commutator of* T_{φ_+}. *Then* T_φ *is binormal.*

Conditions of binormality

세종대학교 SEJONG UNIVERSITY

Corollary

Let $\varphi(z) = \varphi_+ + \lambda\overline{\varphi_+}$ *where* $\varphi_+ \in H^\infty$ *and* $\lambda \in \mathbb{C}$. *If* T_φ *is binormal and* $[T_\varphi^*, T_\varphi]$ *has a non-trivial kernel, then* T_φ *is normal.*

Corollary

Let $\varphi(z) = \varphi_+ + \overline{\varphi_-}$ *where* φ_+ *and* φ_- *are in* zH^2. *If* $\varphi_- = \lambda\varphi_+$ *for some complex number* λ *with* $|\lambda| < 1$ *and the conditions (ii) or (iii) in Theorem 8 hold, then* T_φ *has a nontrivial invariant subspace.*

Nontrivial invariant subspace

Theorem

Let $\varphi(z) = \varphi_+ + \overline{\varphi_-}$ *where* $\varphi_+, \varphi_- \in zH^2$ *with* $\|\varphi_+\| = \|\varphi_-\|$. *Suppose that* $\varphi_+ = u\varphi_-$ *for some inner function* u. *If an operator*

$$(\varphi_- \otimes B\varphi_-) + (S^* ASu\varphi_- \otimes u\varphi_-)$$

is self-adjoint, then T_φ *has a nontrivial invariant subspace.*

Nontrivial invariant subspace

Corollary

Let $\varphi(z) = \varphi_+ + \overline{\varphi_-}$ *where* $\varphi_+, \varphi_- \in zH^2$ *with* $\|\varphi_+\| = \|\varphi_-\|$. *Suppose that* $\varphi_+ = \frac{a-z}{1-\bar{a}z}\varphi_-$ *for every* $a \in \mathbb{D}$. *If an operator*

$$(\varphi_- \otimes B\varphi_-) + (S^* AS \frac{a-z}{1-\bar{a}z}\varphi_- \otimes \frac{a-z}{1-\bar{a}z}\varphi_-)$$

is self-adjoint, then T_φ *has a nontrivial invariant subspace.*

Nontrivial invariant subspace

Example

If $\varphi_+ = z\varphi_-$ and an operator

$$(\varphi_- \otimes B\varphi_-) + (S^* ASz\varphi_- \otimes z\varphi_-)$$

is self-adjoint, then T_φ has a nontrivial invariant subspace by Corollary.

Even if $\varphi(z) = \varphi_+ + \overline{\varphi_-}$ with $\varphi_+(z) = z^2$ and $\varphi_-(z) = z$ in the above example, then, since $\varphi_+ = z\varphi_-$, T_φ is hyponormal by [GS]. But T_φ is not binormal by Example.

Binormal Toeplitz and CSO

- A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$.
- An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$.

Corollary

Let $\varphi(z) = az^n + b\bar{z}^n$ for some nonzero $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$. Then the following assertions are equivalent.
(i) T_φ is binormal.
(ii) T_φ is complex symmetric.
(iii) T_φ is normal.
(iv) $|a| = |b|$.

Binormal Toeplitz and CSO

Proposition

Let $\varphi \in L^\infty$ and let $C(\sum_{k=0}^{\infty} a_k z^k) = \sum_{k=0}^{\infty}(-1)^k \overline{a}_k z^k$ for $\sum_{k=0}^{\infty} a_k z^k \in H^2$. Then the following statements are equivalent.

(i) T_φ is complex symmetric with the conjugation C.

(ii) T_φ is normal.

(iii) $\hat{\varphi}(-n) = \hat{\varphi}(n)(-1)^n$ for all n.

Binormal Toeplitz and CSO

Proposition

Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ where $N \geq m > 0$ and $a_n \in \mathbb{C}$ with nonzero a_{-m}, a_N and let $C(\sum_{k=0}^{\infty} a_k z^k) = \sum_{k=0}^{\infty} (-1)^k \overline{a}_k z^k$ for $\sum_{k=0}^{\infty} a_k z^k \in H^2$. Then T_φ is complex symmetric with the conjugation C if and only if $m = N$ and $a_{-n} = a_n(-1)^n$ for all n.

Let $\varphi(z) = z^2 + z - \overline{z} + \overline{z}^2$. Then T_φ is complex symmetric with the same conjugation C by Proposition or [GZ]. However, T_φ is not binormal by Example 7.

Reference

- C. C. Cowen, *More subnormal Toeplitz operators*, J. Reine Angew. Math. **367**(1986), 215-219.
- C. C. Cowen and J. J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351**(1984), 216-220.
- D. R. Farenick and W. Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **384**(1996), 4153-4174.
- S. R. Foguel, *A counterexample to a problem of Sz.-Nagy*, Proc. Amer. Math. Soc. **15**(1964), 788-790.
- C. Gu and J. E. Shapiro, *Kernels of Hankel operators and hyponormality of Toeplitz operators*, Math. Ann. **319**(2001), 553-572.

Reference

- P. R. Halmos, *A Hilbert space problem book*, seconded., Springer-Verlag, New York, 1982.
- Sami M. Hamid, *On the structure of a class of operators*, Texas A M University, Dissertation, (2005).
- T. Ito and T. K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34**(1972), 157-164.
- B. B. Morrel and P. S. Muhly, *Centered operators*, Studia Math., **51**(1974), 251-263.
- E. M. Klein, *The numerical range of a Toeplitz operator*, Proc. Amer. Math. Soc. **35**(1972), 101-103.
- V. Paulsen, C. Pearcy, and S. Petrovic, *On centered and weakly centered operators*, J. Func. Anal., **128**(1995), 87-101.

Thank you

- Thank you for your attention!

(jieunlee7@sejong.ac.kr; jieun7@ewhain.net)