From CQs to QSs and vice versa

ICM Satellite on Operator Algebras
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Table of contents

Abstract

Just a survey!?

We survey recent constructions of compact quantum semigroups coming from compact quantum groups. The main motivating example is the quantum semigroup compactification of quantum groups (and vice versa). We give some classical examples of groups coming from (discrete) semigroups and give a report on a work in progress on compact quantum groups coming from compact quantum semigroups.

The non compact case is harder and will not be addressed here.

History

QG and QS

compact quantum groups (CQGs) were introduced by Woronowicz. I do not know who first used the term compact quantum semigroup (QS), but surely the notion is already present in the pioneering work of Woronowicz.

History

Compactification

Quantum Bohr compactification of QSs were initiated by Sołtan (which gives a QG) (reconsidered by Daws). More general compactifications are studied by Salmi (sends a QG to a QS). Other compactifications are considered by Das, Daws.

History

Transitions

There are other ways to make a transition from a QG to a QS and vice versa. These are usually based on taking a subalgebra or quotient (Sołtan and Salmi) and is related to the notion of a quantum sub(semi)group. Quantum subgroups introduced by Woronowicz and Vaes (the two notions are not equivalent, but comparable (Daws, Kasprzak, Skalski, Sołtan).

(Sołtan considered the cases where a C^*-subalgebra is invariant under coproduct of a QG, so restriction gives a QS (as a subalgebra). On the other side, Salmi used the notion of a (left) invariant subspace to make an ideal and going to the quotient C^*-algebra. This latter construction could (and would) be modified (here) to go from a QS to a QG (by taking quotient).

Classical case

Compactification

There is a one-to-one correspondence between semigroup compactifications of a classical group G and m-admissible (unital, left translation invariant, and left m-introverted) C^*-subalgebras X of $C_b(G)$ (see the book by Berglund, Junghenn, Milnes).

Examples

Two examples of X, are weakly almost periodic functions $X = wap(G)$ (Bohr) and the Eberlein algebra (uniform closure of the the Fourier- Stieltjes algebra) $X = \mathcal{E}(G)$ (Spronk, Stokke) whose character spaces are compact semigroups G^{wap} (the largest semitopological semigroup which contains a dense homomorphic image of G) and $G^{\mathcal{E}}$ (the maximal semigroup compactification which arises as a semigroup of contractions on a Hilbert space).

Classical case

Inverse semigroup

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E.

Regular representation

The left regular representation of S (Barnes) is the map $\lambda : S \longrightarrow B(\ell^2(S))$ defined by

$$\lambda(s)f(t) = \begin{cases} f(s^*t), & ss^* \geq tt^* \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

This is a faithful $*$-representation of S (Wordingham).

Classical case

Multiplicative partial isometry

Define the fundamental operator $W : \ell^2(S \times S) \longrightarrow \ell^2(S \times S)$ by

$$W\psi(s,t) = \begin{cases} \psi(s, s^*t), & ss^* \geq tt^* \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

for $\psi \in \ell^2(S)$ and $s, t \in S$. This is a multiplicative partial isometry in the sense of Bohm, Szlachanyi.

Coproduct

With the coproduct $\Gamma : L(S) \longrightarrow L(S \times S)$

$$\lambda(s) \mapsto W(\lambda(s) \otimes 1)W^* = WW^*(\lambda(s) \otimes \lambda(s)),$$

for each $s \in S$, $L(S)$ is a QS (von Neumann algebra setting).

Classical case

Fourier space

Let $L(S)$ be the von Neumann subalgebra of $B(\ell^2(S))$ generated by the range of λ and put $A(S) = L(S)_*$. Let $\omega_{s,t}(\varphi) = \langle \varphi\delta_s \mid \delta_t \rangle$, for each $\varphi \in L(S)$. Then $A(S)$ is generated by the set $\{\omega_{s,t} : s, t \in S\}$ as a Banach space.

Fourier algebra

The multiplication map $\Gamma_* : A(S \times S) \longrightarrow A(S)$ is a complete contraction, and $A(S \times S) \cong A(S)\hat{\otimes}A(S)$ (by Effros-Ruan). Therefore $A(S)$ is a completely contractive Banach algebra.

Classical case

Fourier algebra

The pointwise multiplication $\omega_{s,t}\omega_{u,v}$ identifies with the algebra multiplication $\omega_{s,t}.\omega_{u,v}$ defined by

$$\omega_{s,t}.\omega_{u,v}(\varphi) = (\omega_{s,t} \otimes \omega_{u,v})(W(\varphi \otimes 1)W^*) \quad (\varphi \in L(S)).$$

Write $\varepsilon_a := \omega_{a^*a,a}$. The span of ε_a's is dense in $A(S)$. These elements are natural replacements for the point indicator functions δ_a in $A(G)$ for a discrete group G. One important difference between ε_a and δ_a is that ε_a may have a large support.

Classical case

Module structure

Next $A(S)$ is an operator $\ell^1(E)$-module with actions

$$\delta_e.\varepsilon_a = \varepsilon_a\,, \quad \varepsilon_a.\delta_e = \begin{cases} \varepsilon_{ae}, & a^*a \leq e \\ 0, & otherwise. \end{cases} \tag{3}$$

These lift to action of $C^*(E)$ by the universal property.

Define $s \sim t$ if there is $e \in E$ such that $es = et$. The equivalence classes form the maximal group homomorphic image G_S of S (see, books by Howie, Lawson). It is well known that amenability of S is equivalent to that of G_S.

Classical case

Virtual diagonal

The quotient map $\pi : S \to G_S$ lifts to a surjection $\pi : L(S) \to L(G_S)$. When S is amenable, by a result of Ruan, $A(G_S)$ is operator amenable and any extension of the diagonal map on $G_S \otimes G_S$ to a contractive linear functional $\tilde{M}$ on $L(G_S) \otimes L(G_S)$ (which exists by Hahn-Banach) is an operator virtual diagonal for $A(G_S)$.

Put $M = \tilde{M} \circ (\pi \otimes \pi)$,

$$M(\lambda(s) \otimes \lambda(t)) = \tilde{M}(\lambda([s]) \otimes \lambda([t])) = \begin{cases} 1, & s \sim t \\ 0, & s \nsim t. \end{cases}$$

Classical case

Module operator amenability

On the (pre) dual algebra of C^*-algebraic (W^*-algebraic) quantum semigroups (which are completely contractive Banach algebras) with an action of a certain algebra of idempotents, one may define a notion of module operator amenability, by requiring cb derivations which are also module homomorphisms to be inner. This is equivalent to the existence of a module virtual diagonal M, in the sense of M.A.-Rezavand.

Classical case

Theorem (M.A.-Rezavand)

If E has a minimum idempotent, S is amenable if and only if $A(S)$ is module operator amenable.

idea of the proof

If S is amenable then M is a module virtual diagonal, in the sense that

$$\varepsilon_s.M - M.\varepsilon_s \in I^{\perp\perp}, \quad m^{**}(M).\varepsilon_s - \varepsilon_s \in J^{\perp\perp},$$

for appropriate ideals I and J (coming from the action). Conversely, if $A(S)$ is module operator amenable then $A(G_S)$ is operator amenable, therefore G_S is amenable, and so is S.

Compactification

Eberlin compactification

Let $\mathbb{G}$ be a locally compact quantum group. There is an (essentially unique) unital embedded maximal C^*-Eberlein algebra (A, Γ_A, U_A, H_A) (Das, Daws). This is not a QS, but could be, in some cases (as the classical case, see Spronk, Stokke) and could be considered as another candidate for what a quantum semigroup should look like.

Subalgebras (ad hoc)

Restricting coproducts

Let $\mathbb{G} = (B, \Gamma)$ be a CQG and A be a unital C^*-subalgebra of B (with the same unit) such that $\Gamma(A) \subseteq A \otimes A$. Restrict the coproduct Γ of B to Γ_A for A. One could ask when the CQS $\mathbb{S} = (A, \Gamma_A)$ with a structure of a compact quantum group.

By a result of Bedos, Murphy, Tuset, if $\mathbb{G}$ is coamenable then $\mathbb{S}$ is a CQG (Sołtan).

Subalgebras (ad hoc)

Restricting coproducts

One could drop the existence of a bounded counit on B and still get the same result if the Haar measure of $\mathbb{G}$ is faithful (this uses the (Kustermans-Vaes non-commutative Weil theorem).

If the Haar measure of $\mathbb{G}$ is not faithful but B is a exact C^*-algebra (in the sense of Kirchberg), then there exists a unital C^*-subalgebra of A such that $\Gamma(A) \subseteq A \otimes A$, but $\mathbb{S} = (A, \Gamma_A)$ is not a CQG a compact quantum group (Sołtan).

Subalgebras (ad hoc)

Example

Let $A = \mathbb{C}^2 = C^*(\mathbb{Z}_2)$ and $B = C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ (the quantum space of all maps from $\{0,1\}$ to $\mathbb{Z}_2$, which is the universal unital C^*-algebra generated by two projections p , q with no relation). The free product construction gives the standard cocommutative coproduct multiplication on the group C^*-algebra B, but its restriction to A is

$$\Gamma(p) = (p-1)\otimes p+1\otimes 1+p\otimes(p-1), \;\; \Gamma(q) = (q-1)\otimes q+1\otimes 1+q\otimes(q-1).$$

which fails to be a QG (Sołtan).

Compactification

Bohr compactification

QSs always have compactifications which are QG (Sołtan, Daws): Given a QS $\mathbb{S} = (A, \Gamma)$, one construct a certain unital C^*-subalgebra $AP(\mathbb{S})$ in $M(A)$ such that (the strict extension of) Γ restricts to a coproduct Γ_{AP} on $AP(\mathbb{S})$ and $b\mathbb{S} = (AP(\mathbb{S}), \Gamma_{AP})$ is a CQG (the quantum Bohr compactification of $\mathbb{S}$) (Daws).

Considering LCQG as a full subcategory of QS, $AP(\mathbb{S})$ gives the Bohr compactification of Sołtan when $A = C_0^u(\mathbb{S})$. But $(AP(\mathbb{S}), \Gamma_{AP})$ might fail to be a universal quantum group (and hence not necessarily in LCQG). This even occurs when $\mathbb{S}$ is cocommutative (Daws). However, one may adapt the same idea to construct a compactification in LCQG (Daws). The price to pay is that the resulting C^*-algebra is more complicated.

Compactification

Classical case

Let S be a discrete semigroup with weak containment property (Milan) and let $\mathbb{S} = (C^*(S), \Gamma)$ be the universal dual of S. Then $b\mathbb{S}$ is the universal algebra of bS (note that bS is a group and $bS = S^{AP}$, when S is a group).

quantum $ax + b$ group

Let $\mathcal{A}$ be the unital $*$-algebra generated by a, a^{-1}, b subject to the relations

$$a^{-1}a = aa^{-1} = 1,\ aa^* = a^*a, bb^* = b^*b,\ ab = q^2ba, ab^* = b^*a,$$

and coproduct $\Gamma(a) = a \otimes a,\ \Gamma(b) = a \otimes b + b \otimes 1$. This could be (but not so easily) turned into a LCQG $\mathbb{G}_q$ (Woronowicz, Sołtan). For q real, $b\mathbb{G}_q$ is the universal algebra of $b\mathbb{Z} \times \mathbb{T}$, and for q non-real, say $q = e^{2\pi i/n}$, $b\mathbb{G}_q$ is the universal algebra of $b\mathbb{R} \times \mathbb{Z}_n$ (Sołtan).

Quotients (ad hoc)

Example

For $0 \leq q \leq 1$, the C^*-algebra $C(SU_q(2))$ is the universal C^*-algebra generated by a and b, subject to:

$$a^*a + b^*b = I,\ ab = qba,\ aa^* + q^2b^*b = 1,\ ab^* = qb^*a,\ b^*b = bb^*,$$

and has a counit $\varepsilon(a) = 1$ and $\varepsilon(b) = 0$.

For $0 \leq q < 1$, $C(SU_q(2))$ is the universal C^*-algebra generated by operators T and S, subject to $T^*T = I,\ S^*S = SS^*,\ TT^* + S^*S = I$. If J is the closed 2-sided ideal of generated by S, then J is isomorphic to $C(\mathbb{T}) \otimes \mathcal{B}_0(\ell^2)$ and A/J is isomorphic to $C(\mathbb{T})$. Note that $SU_0(2)$ is only a QS, but its isomorphism to $SU_(2)$ is implemented by a cocycle (Szymański).

Quotients (ad hoc)

Left invariant subalgebras

One may construct a CQsubG from a left invariant C^*-subalgebra X of $C_0(\mathbb{G})$ of a coamenable QG $\mathbb{G}$ with counit ε (Salmi).

A non-degenerate representation $\rho : C_0(\mathbb{G}) \to M(A)$, A a C^*-algebra, is X-trivial if $\rho(x) = \varepsilon(x)1$ for $x \in X$. The intersection J_X of kernels of X-trivial representations is a closed ideal in $C_0(\mathbb{G})$.

Theorem (Salmi)

There is a CQsubG $(\mathbb{H}, \pi)$ of $\mathbb{G}$ such that $C(\mathbb{H}) = C_0(\mathbb{G})/J_X$ and $\pi : C_0(\mathbb{G}) \to C_0(\mathbb{H})$ is the quotient map.

Quotients (ad hoc)

Universality

The construction almost works if $\mathbb{G}$ is not necessarily co-amenable (apply it to the universal C^*-algebra $C_0^u(\mathbb{G})$). The resulting quotient is a C^*-algebra of a CQG, but not a closed QsubG (universality may fail).

Uniqueness

Different X's may induce the same $\mathbb{H}$ through this construction. If X is (co-action) symmetric, that is $\Gamma(X) \subseteq M(X \otimes \mathcal{B}_0(H))$ (Salmi), we get a unique QG. This is some sort of normality of the subQG (compare to Vaes, Vainerman). Under this condition one characterizes the dual space of $C(\mathbb{H})$ as the space of those states that have X-trivial GNS-representations.

QsubGs of QSs

Left invariant subalgebras

One may adapt Salmi's construct to get a CQsubG from a left (or right) invariant C^*-subalgebra X of $C_0(\mathbb{S})$ of a coamenable QS $\mathbb{S}$ (Universal duals of discrete semigroups always have a counit: the augmentation character on the semigroup algebra extends by universality).

A X-trivial non-degenerate representation $\rho : C_0(\mathbb{S}) \to M(A)$ and the corresponding ideal J_X are defined similarly.
Put $\mathcal{L} = \Gamma(C_0(\mathbb{S}))(C_0(\mathbb{S}) \otimes 1)$ and $\mathcal{R} = \Gamma(C_0(\mathbb{S}))(1 \otimes C_0(\mathbb{S}))$.

Theorem (M.A.-M.S.M. Moakhar)

There is a CQsubS $(\mathbb{H}, \pi)$ of $\mathbb{S}$ such that $C(\mathbb{H}) = C_0(\mathbb{S})/J_X$ and $\pi : C_0(\mathbb{S}) \to C_0(\mathbb{H})$ is the quotient map.
If $\ker(\pi \otimes \pi)^\perp \cap \mathcal{L}^\perp = \ker(\pi \otimes \pi)^\perp \cap \mathcal{R}^\perp = 0$, $(\mathbb{H}, \pi)$ is a CQsubG.

QsubGs of QSs

The zero intersection holds in many natural examples (if X is properly chosen).

Universality

Again the construction works if $\mathbb{S}$ is not co-amenable but the universality may be lost.

Uniqueness

If $\Gamma(X) \subseteq M(X \otimes \mathcal{B}_0(H))$ the dual space of $C(\mathbb{H})$ is the space of those states that have X-trivial GNS-representations.

QsubGs of QSs

Example

Let $B = C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ (the quantum space of all maps from $\{0,1\}$ to $\mathbb{Z}_2$). The free product construction gives the standard cocommutative coproduct multiplication on the group C^*-algebra B, but also there is a co-amenable QS structure on B with $\varepsilon(p) = 1, \varepsilon(q) = 0$ and

$$\Gamma(p) = p \otimes p + (1-p) \otimes q, \quad \Gamma(q) = q \otimes p + (1-q) \otimes q.$$

The C^*-subalgebra X generated by p and 1 is right invariant and J_X is the ideal generated by q. The quotient A/J_X is the universal dual algebra of $\mathbb{Z}_2$.

QsubGs of QSs

Example

Let ω_q be the state $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a+q^2 d}{1+q^2}$ on $M_2(\mathbb{C}) =: C(\mathbb{M})$. The CQS $\mathbb{S}$ of all maps $\mathbb{M} \to \mathbb{M}$ preserving ω_q has $C(\mathbb{S})$ generated by three elements β, γ and δ subject to:

$q^4\delta^*\delta + \gamma^*\gamma + q^4\delta\delta^* + \beta\beta^* = 1, \quad \beta\gamma = -q^4\delta^2, \quad \gamma\beta = -\delta^2,$

$\beta^*\beta + \delta^*\delta + \gamma\gamma^* + \delta\delta^* = 1, \quad \gamma^*\delta - q^2\delta^*\beta + \beta\delta^* - q^2\delta\gamma^* = 0,$

$\beta\delta = q^2\delta\beta, \quad q^4\delta\delta^* + \beta\beta^* + q^2\gamma\gamma^* + q^2\delta\delta^* = 1, \quad \delta\gamma = q^2\gamma\delta,$

$q^4\delta^*\delta + \gamma^*\gamma + q^2\beta^*\beta + q^2\delta^*\delta = q^2 1,$

with coproduct

$\Gamma(\beta) = q^4\delta\gamma^* \otimes \delta - q^2\beta\delta^* \otimes \delta + \beta \otimes \beta + \gamma^* \otimes \gamma - q^2\delta^*\beta \otimes \delta + \gamma^*\delta \otimes \delta,$

$\Gamma(\gamma) = q^4\gamma\delta^* \otimes \delta - q^2\delta\beta^* \otimes \delta + \gamma \otimes \beta + \beta^* \otimes \gamma - q^2\beta^*\delta \otimes \delta + \delta^*\gamma \otimes \delta,$

$\Gamma(\delta) = -q^2\gamma^*\gamma \otimes \delta - q^2\delta\delta^* \otimes \delta + \delta \otimes \beta + \delta^* \otimes \gamma + \beta^*\beta \otimes \delta + \delta^*\delta \otimes \delta,$

and counit $\varepsilon(\gamma) = \varepsilon(\delta) = 0$, $\varepsilon(\beta) = 1$.

QsubGs of QSs

Example(continued)

If we mod-out an appropriate closed ideal J, and use the change of variables $a = \beta + q^2\gamma^*$ and $b = \delta + q^2\delta^*$, then by a universality argument,

$$C^u(\mathbb{S})/J \simeq C^u(SU_{q^2}(2)).$$

QsubGs of QSs

List of Open Problems

It would be desirable to give answers to the following:

- Find new classes of QSs and their distinguished QsubGs coming from the action (of certain algebras generated by projections).
- Find appropriate module structure on the C^*-algebra of the QS.
- Relate weak (operator) amenability and approximation properties of algebras(+modules) on QS to that of their quotients on QsubGs.

M.A., R. Rezavand, Operator amenability of the Fourier algebras of an inverse semigroups, preprint.

-M.A., M.S.M. Moakhar, Compact quantum subgroups of quantum semigroups, in preparation.