Bost-Connes system for local fields of characteristic zero

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9 August, 2014

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- G_K acts by right multiplication on Y_K and trivially on I_K.

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A_K^{arith} is called an arithmetic subalgebra.

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Definition 1 (T '14)

(A_K, σ_t) is called the Bost-Connes system for K.

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We have "$\mathbb{R}$-equivariant" $(A_K, A_{K_{\mathfrak{p}}})$-correspondence $E_{\mathfrak{p}}$

$\mathbb{R}$-equivariance means that there is one-parameter group of isometries U_t on $E_{\mathfrak{p}}$ such that

- $U_t a\xi = \sigma_t(a) U_t \xi$
- $\langle U_t\xi, U_t\eta\rangle = \sigma_t(\langle \xi, \eta\rangle)$

for any $a \in A_K$, $\xi, \eta \in E_{\mathfrak{p}}$ and $t \in \mathbb{R}$.

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We have $X_{\mathfrak{p}} \hookrightarrow X_K$ by $\quad [\rho, g] \mapsto [(\rho, 1), g|_K]$

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Define $E_{\mathfrak{p}} = 1_{Y_K} \tilde{E}_{\mathfrak{p}} 1_{Y_{\mathfrak{p}}}$
One-parameter group of isometries on $E_{\mathfrak{p}}$ is defined by

$$U_t(u_{\mathfrak{a}} \otimes f) = N(\mathfrak{a})^{it} u_{\mathfrak{a}} \otimes f.$$

Thank you for the attention !