A classification of flows on AFD factors with faithful Connes-Takesaki modules

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Group actions on von Neumann algebras

Let $G$ be a group and $M$ be a von Neumann algebra. Assume $G$ and $M$ are amenable. We want to classify actions of $G$ on $M$, up to cocycle conjugacy.

The first result is the following.

**Theorem (Connes, 1975)**

*There is a unique outer action of $\mathbb{Z}$ on the approximately finite dimensional (AFD) $\text{II}_1$ factor, up to cocycle conjugacy.*

In fact, actions of discrete amenable groups on AFD factors are completely classified (Jones, Ocneanu, Sutherland, Takesaki, Kawahigashi, Katayama, 1998).
How about $\mathbb{R}$-actions?

We want to classify $\mathbb{R}$-actions.

→ Not so easy.

In order to classify $\mathbb{R}$-actions, we need to classify “outer” $\mathbb{R}$-actions.

Here, “outer” means that $\alpha_t \in \text{Aut}(M_\omega)$ is outer for $\forall t \neq 0$ and has hull Connes spectrum.

In order to classify outer $\mathbb{R}$-actions, we take the following two steps.

Step 1. Show that “Outer” actions have the Rohlin property.

Step 2. Classify Rhlin actions (due to Masuda–Tomatsu).
The Rohlin property

Definition (Kishimoto, 1996, Kawamuro, 2000)

An $\mathbb{R}$-action $\alpha$ is said to have the Rohlin property if $\forall p \in \mathbb{R}$, $\exists \{u_n\} \subset U(M)$ s.t.

1. $\alpha_t(u_n) - e^{-ipt}u_n \to 0 \text{ for } \forall t \in \mathbb{R}$,
2. $u_n\phi - \phi u_n \to 0 \text{ for } \forall \phi \in M_*$.

Rohlin $\mathbb{R}$-actions are classified.

Theorem (Masuda–Tomatsu, 2012)

Two $\mathbb{R}$-actions $\alpha^1, \alpha^2$ with the Rohlin property are mutually cocycle conjugate if $\alpha^1_t \circ \alpha^2_{-t} \in \overline{\text{Int}}(M)$ for $\forall t \in \mathbb{R}$. 
What implies the Rohlin property?

In order to complete the classification of outer $\mathbb{R}$-action, we need to show that outerness implies the Rohlin property.

Here, “outer” means $\alpha_t \in \text{Aut}(M_\omega)$ is outer for $\forall t \neq 0$ and has full Connes spectrum.

The condition that $\alpha_t \in \text{Aut}(M_\omega)$ is outer is a kind of pointwise outerness. This does not imply the Rohlin property.

However, stronger pointwise outerness implies the Rohlin property.
Main result

**Theorem (S. 2013)**

Let $\alpha$ be an $\mathbb{R}$-action and $M$ be an AFD factor. Assume that $\alpha_t \notin \text{Int}(M)$ for $\forall t \neq 0$. Then $\alpha$ has the Rohlin property.

This theorem is of the form “very strong pointwise outerness” implies the Rohlin property. This is useful for classification.

By combining with Masuda–Tomatsu’s result, it is possible to see that $\mathbb{R}$-actions with $\alpha_t \notin \text{Int}(M)$ for $\forall t \neq 0$ correspond to certain flows on the standard probability space, up to cocycle conjugacy.
Comparison with compact group actions

For actions of compact groups, there is a similar result due to Izumi.

**Theorem (Izumi, 2003)**

Let $G$ be a compact group and $M$ be an AFD factor. Assume that a $G$-action $\alpha$ satisfies $\alpha_g \notin \overline{\text{Int}(M)}$ for $\forall g \neq e$. Then $\alpha$ is minimal.

However, there are differences between actions of compact groups and actions of $\mathbb{R}$.

He has also shown that for actions of compact groups on AFD factors which are not approximately inner at any non-trivial point, cocycle conjugacy implies the conjugacy.

However, for $\mathbb{R}$-action case, $\exists$ two $\mathbb{R}$-actions which are not approximately inner at any non-trivial point, which are mutually non-conjugate but mutually cocycle conjugate.