Finite groups acting on higher dimensional noncommutative tori

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Rotation algebra

Definition

The *Rotation algebra* $\mathcal{A}_\theta$ ($\theta \in \mathbb{R}$) is the universal C^*-algebra generated by two unitaries u_1, u_2 satisfying

$$u_2 u_1 = e^{2\pi i\theta} u_1 u_2.$$

- If $\theta \in \mathbb{Z}$, then $\mathcal{A}_\theta \cong C(\mathbb{T} \times \mathbb{T})$. So rotation algebras are also called 2-dimensional noncommutative tori.
- $\mathcal{A}_\theta$ is simple if and only if θ is irrational.
- For irrational θ and θ', we have $\mathcal{A}_\theta \cong \mathcal{A}_{\theta'}$ if and only if $\theta \equiv \pm\theta' \mod \mathbb{Z}$.

Actions of $SL_2(\mathbb{Z})$ on rotation algebras

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the map

$$u_1 \mapsto e^{\pi i a c \theta} u_1^a u_2^c,$$

$$u_2 \mapsto e^{\pi i b d \theta} u_1^b u_2^d$$

determines an automorphism α_A of $\mathcal{A}_\theta$, and the choice of the scalars enables

$$\alpha : SL_2(\mathbb{Z}) \to Aut(\mathcal{A}_\theta)$$
$$A \mapsto \alpha_A$$

to be a group action.

$\mathcal{A}_\theta \rtimes \mathbb{Z}_n$ for simple $\mathcal{A}_\theta$ and $\mathbb{Z}_n \leq SL_2(\mathbb{Z})$

There are only 4 nontrivial finite subgroups of $SL_2(\mathbb{Z})$ up to conjugate which are necessarily

$$\mathbb{Z}_2 = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle, \quad \mathbb{Z}_3 = \langle \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \rangle,$$

$$\mathbb{Z}_4 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle, \qquad \mathbb{Z}_6 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rangle.$$

Theorem (2010, Echterhoff-Lück-Phillips-Walters)
$\mathcal{A}_\theta \rtimes \mathbb{Z}_n$ are all AF-aglebras if θ is irrational. If θ, θ' are irrational, then $\mathcal{A}_\theta \rtimes \mathbb{Z}_n \cong \mathcal{A}_{\theta'} \rtimes \mathbb{Z}_m$ if and only if $n = m$ and $\theta \equiv \pm\theta'$ mod $\mathbb{Z}$.

Higher dimensional noncommutative torus

Definition

Given $d \times d$ $(d \geq 2)$ real skew symmetric matrix $\Theta = (\theta_{ij})$, the *d dimensional noncommutative torus* $\mathcal{A}_\Theta$ associated with Θ is defined by the universal C^*-algebra generated by d unitaries $u_1, \cdots, u_d$ subject to the relations

$$u_j u_i = e^{2\pi i \theta_{ij}} u_i u_j, \ (i, j = 1, \cdots, d).$$

Rotation algebra $\mathcal{A}_\theta$ is a NC 2-torus $\mathcal{A}_\Theta$ associated with

$$\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.$$

Action of $\mathbb{Z}_2$ on $\mathcal{A}_\Theta$

For every $\mathcal{A}_\Theta$, the *flip* automorphism

$$u_i \mapsto u_i^*$$

determines the *flip* action : $\mathbb{Z}_2 \to Aut(\mathcal{A}_\Theta)$.

Theorem (ELPW)
If $\mathcal{A}_\Theta$ is simple, then $\mathcal{A}_\Theta \rtimes_{flip} \mathbb{Z}_2$ is an AF-algebra.

Note that flip action is coming from the matrix $-I_d$.

Problem

Except the flip action, it is considerably difficult to find group actions by some consistent matrix group which acts on every torus of given dimension $d > 2$. Instead, we want

- to find an action of G_Θ on $\mathcal{A}_\Theta$ which is the action of $SL_2(\mathbb{Z})$ on $\mathcal{A}_\theta$ in case of $d = 2$,
- to find finite subgroups of G_Θ and to study the crossed products of simple $\mathcal{A}_\Theta$ by finite group actions.

$\mathcal{A}_\Theta \cong C_r^*(\mathbb{Z}^d, \omega_\Theta)$

Let Θ be a $d \times d$ real skew symmetrix matrix.

- Let $\omega_\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ be the 2 cocycle on $\mathbb{Z}^d$ defined by
$$\omega_\Theta(x, y) = e^{\pi i \langle \Theta x, y \rangle}$$
for $x, y \in \mathbb{Z}^d$.
- The *regular ω_Θ-representation* $\ell_\Theta : \mathbb{Z}^d \to U(\ell^2(\mathbb{Z}^d))$ is defined by
$$(\ell_\Theta(y)\xi)(x) = \omega_\Theta(y, x - y)\xi(x - y)$$
for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x, y \in \mathbb{Z}^d$.
- The reduced *twisted group C^*-algebra* $C_r^*(\mathbb{Z}^d, \omega_\Theta)$ is defined by the C^*-subalgebra generated by $\ell_\Theta(\mathbb{Z}^d)$ in $B(\ell^2(\mathbb{Z}^d))$.
$$C_r^*(\mathbb{Z}^d, \omega_\Theta) := C^*\{\ell_\Theta(\mathbb{Z}^d)\}.$$

(Continued)

We have the following isomorphism

$$\begin{aligned} \mathcal{A}_\Theta &\xrightarrow{\cong} C_r^*(\mathbb{Z}^d, \omega_\Theta) \subset B(\ell^2(\mathbb{Z}^d)) \\ u_i &\longmapsto \ell_\Theta(e_i), \end{aligned}$$

So we consider every d dimensional torus as a C^*-subalgebra of $B(\ell^2(\mathbb{Z}^d))$.

$\mathrm{Ad}\, U_A : \mathcal{A}_\Theta \cong \mathcal{A}_{(A^{-1})^t \Theta A^{-1}}$

- Let $A \mapsto U_A : GL_d(\mathbb{Z}) \to U(\ell^2(\mathbb{Z}^d))$ be a group homomorphism where

$$(U_A \xi)(x) = \xi(A^{-1}x)$$

for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$.

- We have an isomorphism

$$\begin{aligned} \mathrm{Ad}\, U_A : \mathcal{A}_\Theta &\xrightarrow{\cong} A_{(A^{-1})^t \Theta A^{-1}} \\ \ell_\Theta(x) &\longmapsto \ell_{(A^{-1})^t \Theta A^{-1}}(Ax), \end{aligned}$$

since

$$\begin{aligned} \mathrm{Ad}\, U_A(\ell_\Theta(x)) &= U_A \circ \ell_\Theta(x) \circ U_{A^{-1}} \\ &= \ell_{(A^{-1})^t \Theta A^{-1}}(Ax). \end{aligned}$$

Canonical action

- If $(A^{-1})^t \Theta A^{-1} = \Theta$, then $\mathrm{Ad}\, U_A$ is an automorphism of $\mathcal{A}_\Theta$ such that $\mathrm{Ad}\, U_A(\ell_\Theta(x)) = \ell_\Theta(Ax)$. Moreover,

$$G_\Theta := \{A \in GL_d(\mathbb{Z}) : \Theta = (A^{-1})^t \Theta A^{-1}\}$$

  forms a subgroup of $GL_d(\mathbb{Z})$.

- So we have a group action

$$\begin{aligned} G_\Theta &\to Aut(\mathcal{A}_\Theta) \\ A &\mapsto \mathrm{Ad}\, U_A. \end{aligned}$$

  The restriction of this action to a subgroup G of G_Θ will be called the *canonical action* of G on $\mathcal{A}_\Theta$.

Canonical action$=SL_2(\mathbb{Z})$ action if $d = 2$

If $d = 2$ and $\Theta \neq 0$, then the canonical action of G_Θ is same as the $SL_2(\mathbb{Z})$ action on the rotation algebra since $G_\Theta = SL_2(\mathbb{Z})$ and

$$\mathrm{Ad}\, U_A(u_1) = e^{\pi i \theta ac} u_1^a u_2^c,$$
$$\mathrm{Ad}\, U_A(u_2) = e^{\pi i \theta bd} u_1^b u_2^d,$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Simple torus

- A $d \times d$ real skew symmetric matrix Θ is said to be degenerate if there exists nonzero $x \in \mathbb{Z}^d \setminus \{0\}$ satisfying

$$e^{2\pi i \langle x, \Theta y \rangle} = 1$$

  for all $y \in \mathbb{Z}^d$. Otherwise, Θ is said to be nondegenerate.

- $\mathcal{A}_\Theta$ is simple if and only if Θ is nondegenerate.

Canonical actions on simple NC 3-torus by finite cyclic groups

Theorem (Jeong-L.)

There are no canonical actions on 3 dimensional simple tori by finite cyclic groups except the flip action.

(Proof) We use the complete list of elements of non-trivial finite orders in $GL_3(\mathbb{Z})$ up to conjugate which consists of only 15 elements due to [1971, K. Tahara]. For $\mathcal{A}_\Theta$ on where elements in that list can act canonically, we get only degenerate Θ except $-I_3$.

$\mathcal{A}_\Theta \rtimes G$ for simple $\mathcal{A}_\Theta$ and finite $G \leq G_\Theta$

Let $\alpha : G \to Aut(\mathcal{A}_\Theta)$ be a canonical action of a finite group G on a simple $\mathcal{A}_\Theta$. Then

- α has the tracial Rokhlin property [ELPW]. So $\mathcal{A}_\Theta \rtimes_\alpha G$ is simple and has tracial rank zero [N. Phillips].
- $\mathcal{A}_\Theta \rtimes_\alpha G$ satisfies the UCT [ELPW].

So in this case, $\mathcal{A}_\Theta \rtimes G$ belongs to the class of simple unital separable nuclear C^*-algebras with tracial rank zero which satisfy the UCT.

Theorem (H. Lin)
Algebras in this class can be classified by K-groups.

(Continued)

Theorem (N. Phillips)
Let $\mathcal{A}$ be a simple infinite dimensional separable unital nuclear C^*-algebra with tracial rank zero and which satisfies the UCT. Then $\mathcal{A}$ is a simple AH-algebra with real rank zero and no dimension growth. If $K_*(\mathcal{A})$ is torsion free, then $\mathcal{A}$ is an AT-algebra. If, in addition, $K_1(\mathcal{A}) = 0$, then $\mathcal{A}$ is an AF-algebra.

Theorem (ELPW)
If $\mathcal{A}_\Theta$ is a NC d-torus and G is a finite subgroup of G_Θ, then $K_i(\mathcal{A}_\Theta \rtimes G) = K_i(C_r^*(\mathbb{Z}^d \rtimes G))$ $(i = 0, 1)$.

(**Proof**) $\mathcal{A}_\Theta \rtimes G \cong C_r^*(\mathbb{Z}^d, \omega_\Theta) \rtimes G \cong C_r^*(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta)$ which has the same K-groups with $C_r^*(\mathbb{Z}^d \rtimes G)$.

$K_i(C_r^*(\mathbb{Z}^d \rtimes \mathbb{Z}_n))$

Theorem (2012, Langer, Lück)
Let $n, d \in \mathbb{N}$. Consider the extension of groups $1 \to \mathbb{Z}^d \to \mathbb{Z}^d \rtimes \mathbb{Z}_n \to \mathbb{Z}_n \to 1$ such that conjugation action of $\mathbb{Z}_n$ on $\mathbb{Z}^d$ is free outside the origin. Then $K_i(C_r^*(\mathbb{Z}^d \rtimes \mathbb{Z}_n)) = \mathbb{Z}^{s_i}$ where

$$s_0 = \sum_{l \geq 0} \mathrm{rk}_{\mathbb{Z}}((\Lambda^{2l}\mathbb{Z}^d)^{\mathbb{Z}_n}) + \sum_{(M) \in \mathcal{M}} (|M| - 1),$$

$$s_1 = \sum_{l \geq 0} \mathrm{rk}_{\mathbb{Z}}((\Lambda^{2l+1}\mathbb{Z}^d)^{\mathbb{Z}_n}).$$

If n is even, then $s_1 = 0$.
If n is a odd prime number and $d = n - 1$, then $s_1 = \frac{2^{n-1} - (n-1)^2}{2n}$.

(**Remark**) If n is a prime number and $d = n - 1$, then $s_1 = 0$ if and only if $n = 2, 3, 5$.

$\langle C_n \rangle \cong \mathbb{Z}_n \leq GL_{\phi(n)}(\mathbb{Z})$

For a monic polynomial $p(x) = a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + x^d$, the *companion matrix* $C(p)$ is the $d \times d$ matrix given by

$$C(p) := \begin{pmatrix} & & & & -a_0 \\ 1 & & & & -a_1 \\ & 1 & & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & -a_{d-1} \end{pmatrix}.$$

Given $n \in \mathbb{N}$, the *nth cyclotomic polynomial* $\Phi_n(x)$ is the polynomial given by

$$\Phi_n(x) = \prod_{1 \leq k \leq n,\ g.c.d(k,n)=1} (x - e^{2\pi i \frac{k}{n}}).$$

Then $C_n := C(\Phi_n)$ is a $\phi(n) \times \phi(n)$ integral matrix of order n. (Here, ϕ denotes the Euler phi-function.)

$$\langle C_n \rangle \cong \mathbb{Z}_n \leq GL_{\phi(n)}(\mathbb{Z}).$$

Simple $\mathcal{A}_\Theta$ on which $\langle C_n \rangle \cong \mathbb{Z}_n$ acts canonically

Theorem (Jeong-L.)

Let $n \geq 3$ and $d := \phi(n)$. Then there exists simple d-dimensional tori $\mathcal{A}_\Theta$ on which the group $\mathbb{Z}_n = \langle C_n \rangle$ acts canonically.

(Proof) Choose any irrational number θ and put $\Theta := \theta \sum_{k=0}^{n-1} (C_n^k)^t (C_n^t - C_n) C_n^k$. Then it is easy to see that Θ is nondegenerate and $C_n \in G_\Theta$.

$\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ for simple $\mathcal{A}_\Theta$

Let $d = \phi(n)$. Then the conjugation action of $\mathbb{Z}_n \cong \langle C_n \rangle$ on $\mathbb{Z}^d$ is free outside the origin.

Theorem (Jeong-L.)

Let $\alpha : \mathbb{Z}_n \to Aut(\mathcal{A}_\Theta)$ be a canonical action of $\mathbb{Z}_n \cong \langle C_n \rangle$ on simple $\mathcal{A}_\Theta$. If n is even then $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ ia an AF-algebra. If n is a prime number, then $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is an AF-algebra if and only if $n = 2, 3, 5$.

Examples

Example (Jeong-L.)

- The map given by

$$u_1 \mapsto u_2 u_4^*, \quad u_2 \mapsto e^{\pi i \theta} u_1^* u_2^*,$$
$$u_3 \mapsto u_4, \quad u_4 \mapsto e^{\pi i \theta} u_1^* u_2^* u_3^*$$

  determines an action of $\mathbb{Z}_5$ on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$.

- The map given by

$$u_1 \mapsto u_2, \quad u_2 \mapsto u_1^* u_3, \quad u_3 \mapsto u_2 u_4,$$
$$u_4 \mapsto u_3^* u_5, \quad u_5 \mapsto u_2 u_4 u_6, \quad u_6 \mapsto e^{\pi i \theta} u_2^* u_4^* u_5^* u_6^*$$

  determines an action of $\mathbb{Z}_7$ on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta \otimes \mathcal{A}_\theta$.

These actions are conjugate to the canonical actions of $\langle C_5 \rangle$ and $\langle C_7 \rangle$ on some 4 and 6 dimensional tori, resp. If θ is irrational, then $(\mathcal{A}_\theta \otimes \mathcal{A}_\theta) \rtimes \mathbb{Z}_5$ is AF, $(\mathcal{A}_\theta \otimes \mathcal{A}_\theta \otimes \mathcal{A}_\theta) \rtimes \mathbb{Z}_7$ is not AF.

$\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ for simple $\mathcal{A}_\Theta$ and odd n

Theorem (Jeong-L.)

Let $n \geq 3$ be an odd integer with $2\phi(n) \geq n+5$. If $\alpha : \mathbb{Z}_n \to Aut(\mathcal{A}_\Theta)$ is a canonical action of $\mathbb{Z}_n \cong \langle C_n \rangle$ on simple $\mathcal{A}_\Theta$, then $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is not an AF-algebra.

(**Remark**) The odd n's for which $2\phi(n) \geq n+5$ do not hold are $9, 15, 21, 45, \ldots$ etc. Davis and Lück also provided a simple formula for a prime power n. According to them, K_1-group is trivial for $n = 9$. We don't know yet for the rest.

Thank you for listening.