Finite groups acting on higher dimensional noncommutative tori

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The **Rotation algebra** $A_\theta$ ($\theta \in \mathbb{R}$) is the universal $C^*$-algebra generated by two unitaries $u_1, u_2$ satisfying

$$u_2 u_1 = e^{2\pi i \theta} u_1 u_2.$$ 

- If $\theta \in \mathbb{Z}$, then $A_\theta \cong C(\mathbb{T} \times \mathbb{T})$. So rotation algebras are also called 2-dimensional noncommutative tori.

- $A_\theta$ is simple if and only if $\theta$ is irrational.

- For irrational $\theta$ and $\theta'$, we have $A_\theta \cong A_{\theta'}$ if and only if $\theta \equiv \pm \theta' \mod \mathbb{Z}$. 

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Actions of $SL_2(\mathbb{Z})$ on rotation algebras

Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the map

$$u_1 \mapsto e^{\pi i ac \theta} u_1^a u_2^c,$$

$$u_2 \mapsto e^{\pi ibd \theta} u_1^b u_2^d$$

determines an automorphism $\alpha_A$ of $A_\theta$, and the choice of the scalars enables

$$\alpha : SL_2(\mathbb{Z}) \to Aut(A_\theta)$$

$$A \mapsto \alpha_A$$

to be a group action.
\( A_\theta \rtimes \mathbb{Z}_n \) for simple \( A_\theta \) and \( \mathbb{Z}_n \leq SL_2(\mathbb{Z}) \)

There are only 4 nontrivial finite subgroups of \( SL_2(\mathbb{Z}) \) up to conjugate which are necessarily

\[
\mathbb{Z}_2 = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle, \quad \mathbb{Z}_3 = \langle \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \rangle, \\
\mathbb{Z}_4 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle, \quad \mathbb{Z}_6 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \rangle.
\]

**Theorem** (2010, Echterhoff-Lück-Phillips-Walters)
\( A_\theta \rtimes \mathbb{Z}_n \) are all AF-aglebras if \( \theta \) is irrational. If \( \theta, \theta' \) are irrational, then \( A_\theta \rtimes \mathbb{Z}_n \cong A_{\theta'} \rtimes \mathbb{Z}_m \) if and only if \( n = m \) and \( \theta \equiv \pm \theta' \mod \mathbb{Z} \).
Given $d \times d$ ($d \geq 2$) real skew symmetric matrix $\Theta = (\theta_{ij})$, the $d$ dimensional noncommutative torus $A_\Theta$ associated with $\Theta$ is defined by the universal $C^*$-algebra generated by $d$ unitaries $u_1, \cdots, u_d$ subject to the relations

$$u_j u_i = e^{2\pi i \theta_{ij}} u_i u_j, \quad (i, j = 1, \cdots, d).$$

Rotation algebra $A_\theta$ is a NC 2-torus $A_\Theta$ associated with

$$\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.$$
For every $\mathcal{A}_\Theta$, the *flip* automorphism

$$u_i \mapsto u_i^*$$

determines the *flip* action: $\mathbb{Z}_2 \to \text{Aut}(\mathcal{A}_\Theta)$.

**Theorem (ELPW)**

If $\mathcal{A}_\Theta$ is simple, then $\mathcal{A}_\Theta \rtimes_{\text{flip}} \mathbb{Z}_2$ is an AF-algebra.

Note that flip action is coming from the matrix $-I_d$. 
Except the flip action, it is considerably difficult to find group actions by some consistent matrix group which acts on every torus of given dimension $d > 2$. Instead, we want

- to find an action of $G_\Theta$ on $\mathcal{A}_\Theta$ which is the action of $SL_2(\mathbb{Z})$ on $\mathcal{A}_\theta$ in case of $d = 2$,

- to find finite subgroups of $G_\Theta$ and to study the crossed products of simple $\mathcal{A}_\Theta$ by finite group actions.
Let $\Theta$ be a $d \times d$ real skew symmetric matrix.

- Let $\omega_\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ be the 2 cocycle on $\mathbb{Z}^d$ defined by
  \[ \omega_\Theta(x, y) = e^{\pi i \langle \Theta x, y \rangle} \]
  for $x, y \in \mathbb{Z}^d$.

- The regular $\omega_\Theta$-representation $\ell_\Theta : \mathbb{Z}^d \to U(\ell^2(\mathbb{Z}^d))$ is defined by
  \[ (\ell_\Theta(y)\xi)(x) = \omega_\Theta(y, x - y)\xi(x - y) \]
  for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x, y \in \mathbb{Z}^d$.

- The reduced twisted group C*-algebra $C^*_r(\mathbb{Z}^d, \omega_\Theta)$ is defined by the C*-subalgebra generated by $\ell_\Theta(\mathbb{Z}^d)$ in $B(\ell^2(\mathbb{Z}^d))$.
  \[ C^*_r(\mathbb{Z}^d, \omega_\Theta) := C^*\{\ell_\Theta(\mathbb{Z}^d)\}. \]
We have the following isomorphism

\[ A_\Theta \xrightarrow{\cong} C_\ast^*(\mathbb{Z}^d, \omega_\Theta) \subset B(\ell^2(\mathbb{Z}^d)) \]

\[ u_i \mapsto \ell_\Theta(e_i), \]

So we consider every \( d \) dimensional torus as a \( C^\ast \)-subalgebra of \( B(\ell^2(\mathbb{Z}^d)) \).
Let $A \mapsto U_A : GL_d(\mathbb{Z}) \to U(\ell^2(\mathbb{Z}^d))$ be a group homomorphism where

$$(U_A \xi)(x) = \xi(A^{-1}x)$$

for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$.

We have an isomorphism

$$\text{Ad } U_A : A_\Theta \cong A(A^{-1})^t \Theta A^{-1}$$

$$\ell_\Theta(x) \mapsto \ell_{(A^{-1})^t \Theta A^{-1}}(Ax),$$

since

$$\text{Ad } U_A(\ell_\Theta(x)) = U_A \circ \ell_\Theta(x) \circ U_{A^{-1}} = \ell_{(A^{-1})^t \Theta A^{-1}}(Ax).$$
If \((A^{-1})^t\Theta A^{-1} = \Theta\), then \(\text{Ad } U_A\) is an automorphism of \(A_\Theta\) such that \(\text{Ad } U_A(\ell_\Theta(x)) = \ell_\Theta(Ax)\). Moreover,

\[ G_\Theta := \{A \in GL_d(\mathbb{Z}) : \Theta = (A^{-1})^t\Theta A^{-1}\} \]

forms a subgroup of \(GL_d(\mathbb{Z})\).

So we have a group action

\[ G_\Theta \to \text{Aut}(A_\Theta) \]

\[ A \mapsto \text{Ad } U_A. \]

The restriction of this action to a subgroup \(G\) of \(G_\Theta\) will be called the **canonical action** of \(G\) on \(A_\Theta\).
If $d = 2$ and $\Theta \neq 0$, then the canonical action of $G_\Theta$ is same as the $SL_2(\mathbb{Z})$ action on the rotation algebra since $G_\Theta = SL_2(\mathbb{Z})$ and

$$\text{Ad } U_A(u_1) = e^{\pi i \theta ac} u_1^a u_2^c,$$
$$\text{Ad } U_A(u_2) = e^{\pi i \theta bd} u_1^b u_2^d,$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. 
A $d \times d$ real skew symmetric matrix $\Theta$ is said to be degenerate if there exists nonzero $x \in \mathbb{Z}^d \setminus \{0\}$ satisfying

$$e^{2\pi i \langle x, \Theta y \rangle} = 1$$

for all $y \in \mathbb{Z}^d$. Otherwise, $\Theta$ is said to be nondegenerate.

$\mathcal{A}_\Theta$ is simple if and only if $\Theta$ is nondegenerate.
Theorem (Jeong-L.)

There are no canonical actions on 3 dimensional simple tori by finite cyclic groups except the flip action.

(Proof) We use the complete list of elements of non-trivial finite orders in $GL_3(\mathbb{Z})$ up to conjugate which consists of only 15 elements due to [1971, K. Tahara]. For $A_\Theta$ on where elements in that list can act canonically, we get only degenerate $\Theta$ except $-I_3$. 
Let \( \alpha : G \to Aut(A_\Theta) \) be a canonical action of a finite group \( G \) on a simple \( A_\Theta \). Then

- \( \alpha \) has the tracial Rokhlin property [ELPW]. So \( A_\Theta \rtimes_\alpha G \) is simple and has tracial rank zero [N. Phillips].
- \( A_\Theta \rtimes_\alpha G \) satisfies the UCT [ELPW].

So in this case, \( A_\Theta \rtimes G \) belongs to the class of simple unital separable nuclear \( C^* \)-algebras with tracial rank zero which satisfy the UCT.

**Theorem** (H. Lin)

*Algebras in this class can be classified by K-groups.*
Theorem (N. Phillips)
Let $\mathcal{A}$ be a simple infinite dimensional separable unital nuclear $C^*$-algebra with tracial rank zero and which satisfies the UCT. Then $\mathcal{A}$ is a simple AH-algebra with real rank zero and no dimension growth. If $K_*(\mathcal{A})$ is torsion free, then $\mathcal{A}$ is an AT-algebra. If, in addition, $K_1(\mathcal{A}) = 0$, then $\mathcal{A}$ is an AF-algebra.

Theorem (ELPW)
If $\mathcal{A}_\Theta$ is a NC $d$-torus and $G$ is a finite subgroup of $G_\Theta$, then $K_i(\mathcal{A}_\Theta \rtimes G) = K_i(C^*_r(\mathbb{Z}^d \rtimes G))$ ($i = 0, 1$).

(Proof) $\mathcal{A}_\Theta \rtimes G \cong C^*_r(\mathbb{Z}^d, \omega_\Theta) \rtimes G \cong C^*_r(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta)$ which has the same $K$-groups with $C^*_r(\mathbb{Z}^d \rtimes G)$. 

(Continued)
Theorem (2012, Langer, Lück)
Let \( n, d \in \mathbb{N} \). Consider the extension of groups
\[ 1 \to \mathbb{Z}^d \to \mathbb{Z}^d \rtimes \mathbb{Z}_n \to \mathbb{Z}_n \to 1 \]
such that conjugation action of \( \mathbb{Z}_n \) on \( \mathbb{Z}^d \) is free outside the origin. Then \( K_i(C^*_{r}(\mathbb{Z}^d \rtimes \mathbb{Z}_n)) = \mathbb{Z}^{s_i} \)
where
\[
\begin{align*}
s_0 &= \sum_{l \geq 0} \text{rk}_\mathbb{Z}((\Lambda^{2l} \mathbb{Z}^d)^{\mathbb{Z}_n}) + \sum_{(M) \in \mathcal{M}} (|M| - 1), \\
s_1 &= \sum_{l \geq 0} \text{rk}_\mathbb{Z}((\Lambda^{2l+1} \mathbb{Z}^d)^{\mathbb{Z}_n}).
\end{align*}
\]
If \( n \) is even, then \( s_1 = 0 \).
If \( n \) is an odd prime number and \( d = n - 1 \), then
\[
s_1 = 2^{n-1} - \frac{(n-1)^2}{2n}.
\]
(Remark) If \( n \) is a prime number and \( d = n - 1 \), then \( s_1 = 0 \) if and only if \( n = 2, 3, 5 \).
For a monic polynomial $p(x) = a_0 + a_1x + \cdots + a_{d-1}x^{d-1} + x^d$, the companion matrix $C(p)$ is the $d \times d$ matrix given by

$$C(p) := \begin{pmatrix}
1 & -a_0 \\
1 & -a_1 \\
& \ddots \\
1 & -a_{d-1}
\end{pmatrix}.$$ 

Given $n \in \mathbb{N}$, the $n$th cyclotomic polynomial $\Phi_n(x)$ is the polynomial given by

$$\Phi_n(x) = \prod_{1 \leq k \leq n, \ g.c.d(k,n)=1} (x - e^{2\pi i \frac{k}{n}}).$$

Then $C_n := C(\Phi_n)$ is a $\phi(n) \times \phi(n)$ integral matrix of order $n$. (Here, $\phi$ denotes the Euler phi-function.)

$$\langle C_n \rangle \cong \mathbb{Z}_n \leq GL_{\phi(n)}(\mathbb{Z}).$$
Simple $\mathcal{A}_{\Theta}$ on which $\langle C_n \rangle \cong \mathbb{Z}_n$ acts canonically

**Theorem (Jeong-L.)**

Let $n \geq 3$ and $d := \phi(n)$. Then there exists simple $d$-dimensional tori $\mathcal{A}_{\Theta}$ on which the group $\mathbb{Z}_n = \langle C_n \rangle$ acts canonically.

*(Proof)* Choose any irrational number $\theta$ and put

$$\Theta := \theta \sum_{k=0}^{n-1} (C_n^k)^t (C_n^t - C_n) C_n^k.$$  

Then it is easy to see that $\Theta$ is nondegenerate and $C_n \in G_{\Theta}$. 

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Let \( d = \phi(n) \). Then the conjugation action of \( \mathbb{Z}_n \cong \langle C_n \rangle \) on \( \mathbb{Z}^d \) is free outside the origin.

**Theorem (Jeong-L.)**

Let \( \alpha : \mathbb{Z}_n \rightarrow \text{Aut}(A_\Theta) \) be a canonical action of \( \mathbb{Z}_n \cong \langle C_n \rangle \) on simple \( A_\Theta \). If \( n \) is even then \( A_\Theta \rtimes_\alpha \mathbb{Z}_n \) ia an AF-algebra. If \( n \) is a prime number, then \( A_\Theta \rtimes_\alpha \mathbb{Z}_n \) is an AF-algebra if and only if \( n = 2, 3, 5 \).
Examples

Example (Jeong-L.)

The map given by

\[ u_1 \mapsto u_2 u_4^*, \quad u_2 \mapsto e^{\pi i \theta} u_1^* u_2^*, \]
\[ u_3 \mapsto u_4, \quad u_4 \mapsto e^{\pi i \theta} u_1^* u_2^* u_3^* \]

determines an action of \( \mathbb{Z}_5 \) on \( A_\theta \otimes A_\theta \).

The map given by

\[ u_1 \mapsto u_2, \quad u_2 \mapsto u_1^* u_3, \quad u_3 \mapsto u_2 u_4, \]
\[ u_4 \mapsto u_3^* u_5, \quad u_5 \mapsto u_2 u_4 u_6, \quad u_6 \mapsto e^{\pi i \theta} u_2^* u_4^* u_5^* u_6^* \]

determines an action of \( \mathbb{Z}_7 \) on \( A_\theta \otimes A_\theta \otimes A_\theta \).

These actions are conjugate to the canonical actions of \( \langle C_5 \rangle \) and \( \langle C_7 \rangle \) on some 4 and 6 dimensional tori, resp. If \( \theta \) is irrational, then \( (A_\theta \otimes A_\theta) \rtimes \mathbb{Z}_5 \) is AF, \( (A_\theta \otimes A_\theta \otimes A_\theta) \rtimes \mathbb{Z}_7 \) is not AF.
Theorem (Jeong-L.)

Let \( n \geq 3 \) be an odd integer with \( 2\phi(n) \geq n + 5 \). If \( \alpha : \mathbb{Z}_n \to \text{Aut}(A_\Theta) \) is a canonical action of \( \mathbb{Z}_n \cong \langle C_n \rangle \) on simple \( A_\Theta \), then \( A_\Theta \rtimes \alpha \mathbb{Z}_n \) is not an AF-algebra.

(Remark) The odd \( n \)'s for which \( 2\phi(n) \geq n + 5 \) do not hold are 9, 15, 21, 45, \ldots etc. Davis and Lück also provided a simple formula for a prime power \( n \). According to them, \( K_1 \)-group is trivial for \( n = 9 \). We don’t know yet for the rest.
Thank you for listening.