Towards a continuum limit from planar algebras.

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A sub factor $N \subseteq M$ has an index $[M : N]$ which is a real number ≥ 1 (or ∞). If the index is between 1 and 4 it is necessarily of the form $4\cos^2 \pi/n$ for some $n \in \mathbb{Z}, \quad n \geq 3$.

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If this induction/restriction graph is finite the sub factor is said to be of finite depth, or "rational".

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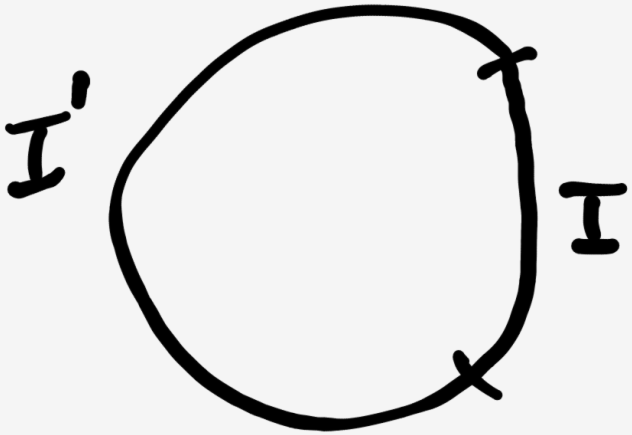
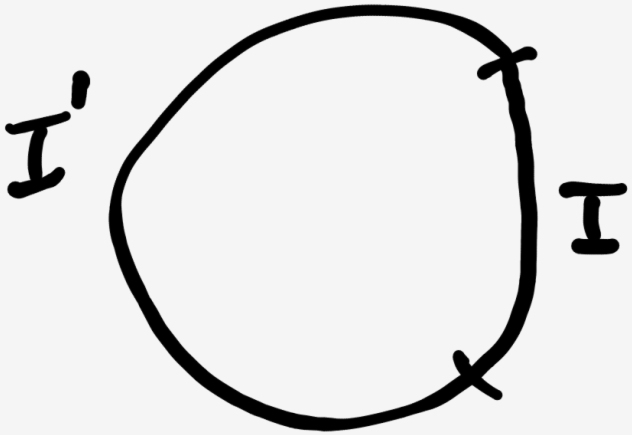


$$\mathfrak{A}(I) \subseteq \mathfrak{A}(I')'$$



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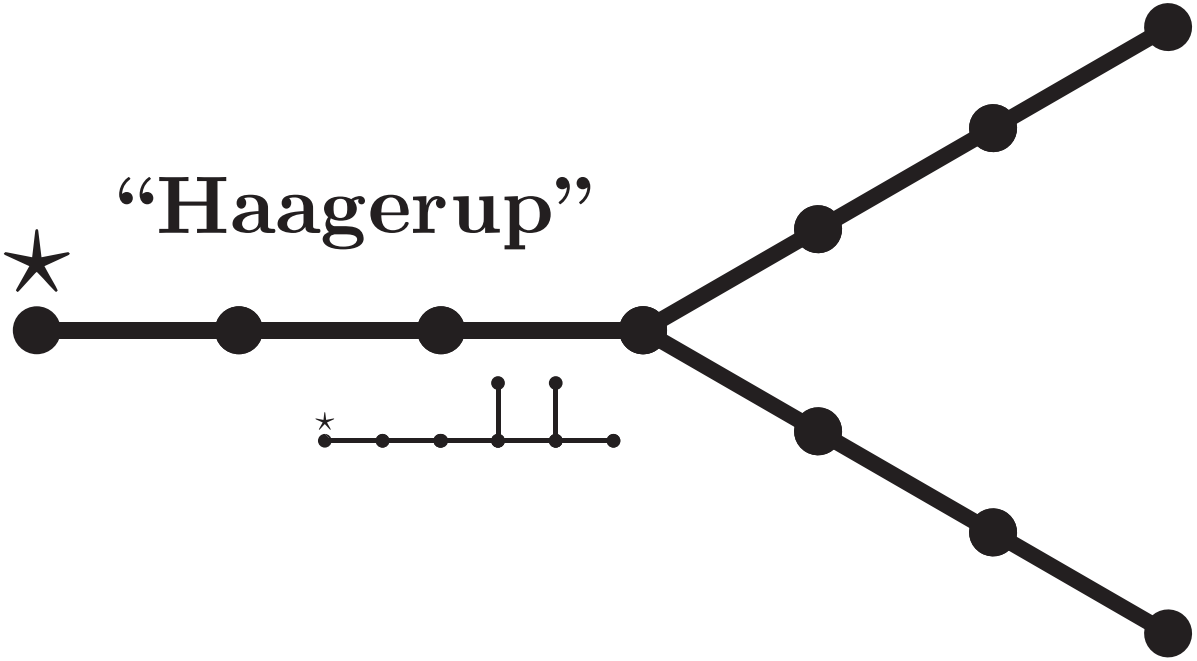
These subfactors were calculated by Wassermann (unpublished) for $SU(N)$ WZW and further extended by Toledano, Loke, Xu, Longo, Kawahigashi, Carpi..... In particular they are all rational and all subfactors of index less than four occur.

In general the passage from a (unitary) VOA to a subfactor is to be achieved by realizing the fields as operator valued distributions on the circle, with z being the circle variable. Smearing over functions supported in I will generate the algebra $\mathfrak{A}(I)$.

In the mid 90's Haagerup discovered the rational subfactor of smallest index >4. Its principal graph is:



- index $\frac{5+\sqrt{13}}{2} \simeq 4.30278$
- constructed in [Asaeda-Haagerup, math.OA/9803044]

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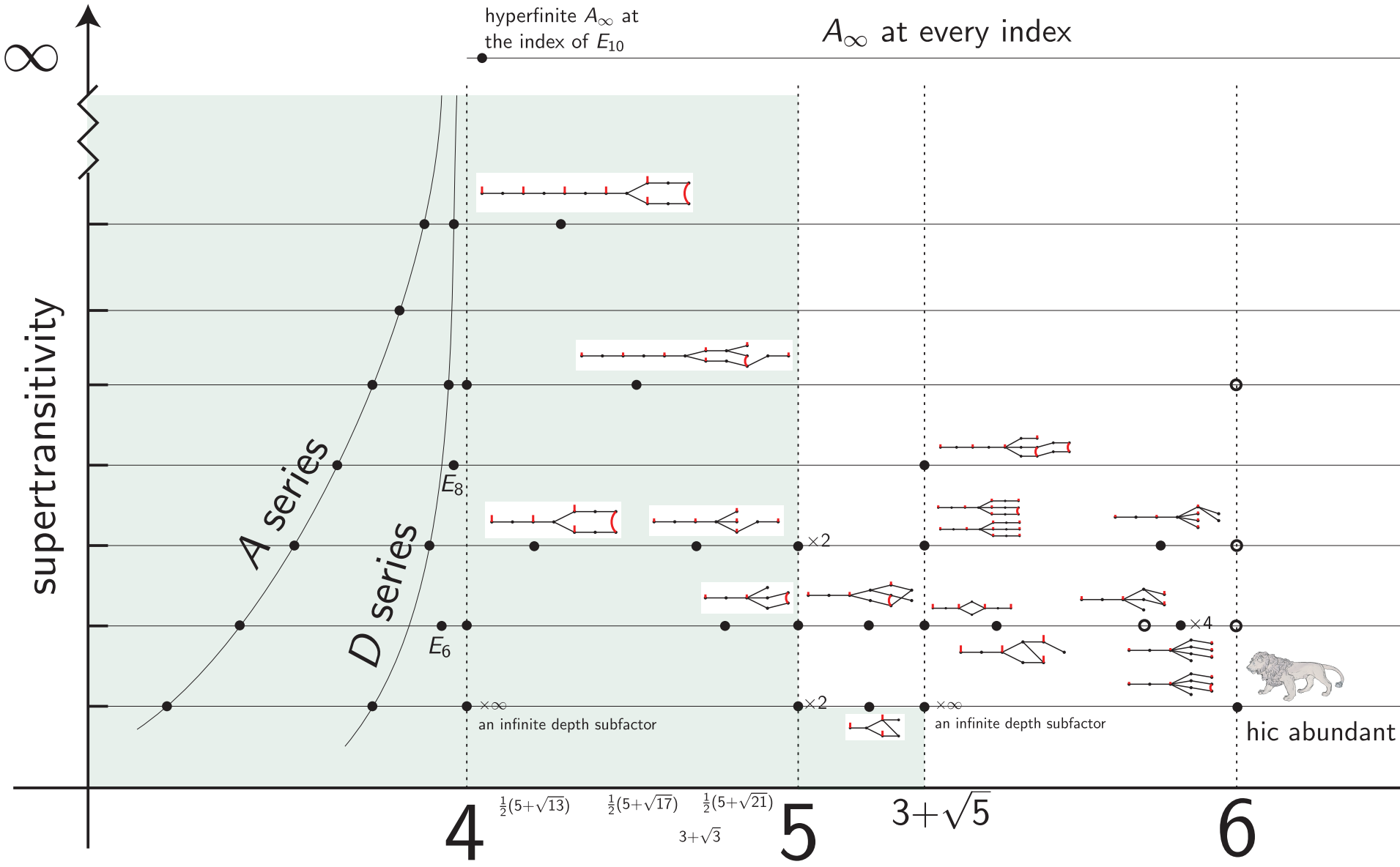
The project (a) has been carried out up to index $3 + \sqrt{5}$. Here is the resulting "map" of subfactors (Morrison):


∞
supertransitivity
hyperfinite A_∞ at the index of E_{10}
A_∞ at every index
A series
D series
E_8
E_6
×2
×4
×∞
an infinite depth subfactor
×2
×∞
an infinite depth subfactor
hic abundant
4
$\frac{1}{2}(5+\sqrt{13})$
$\frac{1}{2}(5+\sqrt{17})$
$\frac{1}{2}(5+\sqrt{21})$
$3+\sqrt{3}$
5
$3+\sqrt{5}$
6

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It is a fascinating open question as to whether supertransitivity is bounded for index >4 . If this is the case the situation is quite analogous to that of group actions where high transitivity implies either the symmetric or alternating groups, or the finite family of Mathieu groups.

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Izumi's Cuntz algebra method generalizes and several of the subfactors in the map can be obtained in this way. The speculations of Evans and Gannon seem to apply to these generalizations as well.

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The "royal road" would be to use some elements of the planar algebra as Boltzmann weights for a critical stat mech model, take the continuum limit and get a subfactor from one of the chiral halves. Then identify that subfactor with the original one. The royal road is paved with mathematical difficulties. Even in the cases where it "should" work this would amount to solving major mathematical difficulties of the continuum limit.

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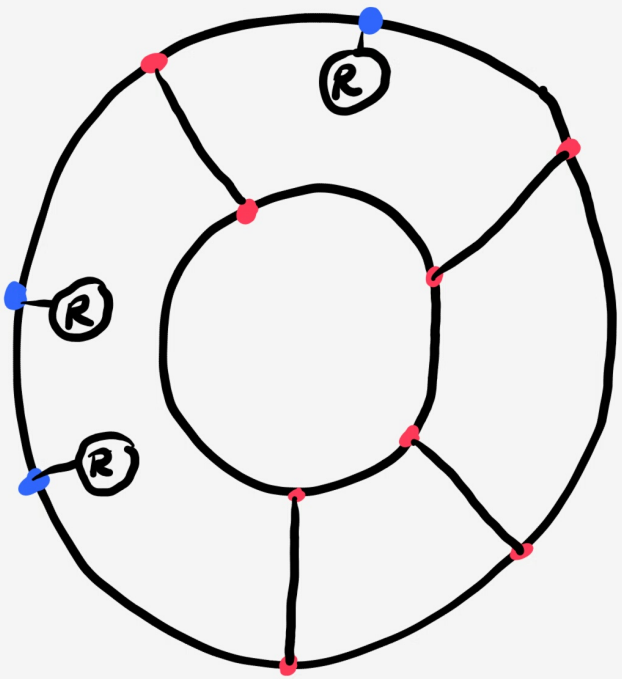

Red points = F

Blue points = points in G not in F

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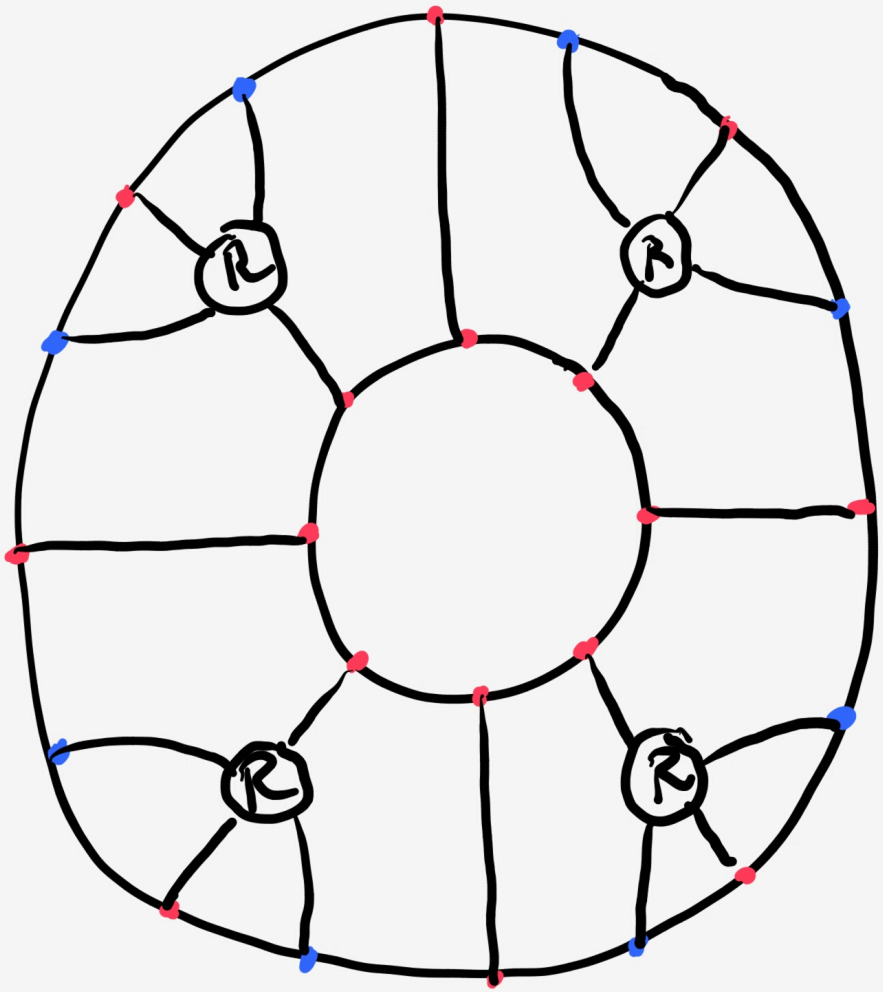
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R could be chosen to be a crossing from knot theory. In this way one obtains knots and links from the Thompson group, and, by varying R, sequences of unitary erepresentations tending to the trivial and tending to the left regular representation.