"Conjugacies" between dynamical systems, and their crossed products.

Wei Sun

Research Center for Operator Algebras
Department of Mathematics
East China Normal University

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For compact infinite metric spaces $X$ and $Y$, and for two minimal homeomorphism $\alpha : X \to X$ and $\beta : Y \to Y$, starting from information on crossed products $C(X) \rtimes_{\alpha} \mathbb{Z}$ and $C(Y) \rtimes_{\beta} \mathbb{Z}$, what can we say about the relation between two dynamical systems $(X, \alpha)$ and $(Y, \beta)$?

**Terminology:**

For crossed product $C^*$-algebras:
- Simplesness, isomorphisms, structured isomorphisms, tracial spaces, etc..

For dynamical systems:
- Minimality, Rokhlin dimension, invariant probability measures, induced (co)homology maps, (flip) conjugacy, weak conjugacy, orbit equivalence, etc..

**Spoiler:** The main thing to connect dynamical system side and crossed product side is to find the “right descriptions”.

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**The Goal**

**Terminologies**
- Conjugacy and flip conjugacy
- Weak (approximate) conjugacy
- Approximate $K$-conjugacy
- Orbit equivalence

**General Strategy**
- Augmented isomorphisms

**Good news**
- Minimal Cantor systems
- Base space being $\mathbb{T}^n$ ($n \leq 2$)

**Bad news**

**Our approach**

**Concluding remarks**

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product side is to find the “right descriptions”.
Definition

Let $X$ and $Y$ be two compact metric spaces. Let $(X, \alpha)$ and $(Y, \beta)$ be two dynamical systems. They are **conjugate** if there exists $\sigma \in \text{Homeo}(X, Y)$ such that $\sigma \circ \alpha = \beta \circ \sigma$. That is, the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
Y & \xrightarrow{\beta} & Y
\end{array}
\]

**Definition**

Let $X$ and $Y$ be two compact metric spaces. Let $(X, \alpha)$ and $(Y, \beta)$ be two dynamical systems. They are **flip conjugate** if $(X, \alpha)$ is conjugate to either $(Y, \beta)$ or $(Y, \beta^{-1})$. 
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Definition

Let $X$ and $Y$ be two compact metric spaces. Let $(X, \alpha)$ and $(Y, \beta)$ be two dynamical systems. They are weakly (approximately) conjugate if there exist \{\sigma_n \in Homeo(X, Y)\} and \{\tau_n \in Homeo(Y, X)\}, such that 
\[ \text{dist}(g \circ \beta, g \circ \tau_n^{-1} \circ \alpha \circ \tau_n) \to 0 \] and 
\[ \text{dist}(f \circ \alpha, f \circ \sigma_n^{-1} \circ \beta \circ \sigma_n) \to 0 \] for all $f \in C(X)$ and $g \in C(Y)$. Roughly speaking, the diagrams below “approximately” commute:

![Diagrams](https://via.placeholder.com/150)

Remark: Generally speaking, weak approximate conjugacy might not be an equivalence relation at all.
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\begin{array}{c}
X \xrightarrow{\alpha} X \\
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Y \xrightarrow{\beta} Y
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Remark: Generally speaking, \textit{weak approximate conjugacy} might not be an equivalence relation at all.
Definition (Lin)

Let \((X, \alpha)\) and \((Y, \beta)\) be two minimal dynamical systems. Assume that \(C(X) \rtimes_\alpha \mathbb{Z}\) and \(C(Y) \rtimes_\beta \mathbb{Z}\) both have tracial rank zero. We say that \((X, \alpha)\) and \((Y, \beta)\) are approximately \(K\)-conjugate if there exist homeomorphisms \(\sigma_n : X \to Y\), \(\tau_n : Y \to X\) and unital order isomorphisms \(\rho : K_*(C(Y) \rtimes_\beta \mathbb{Z}) \to K_*(C(X) \rtimes_\alpha \mathbb{Z})\), such that

\[
\sigma_n \circ \alpha \circ \sigma_n^{-1} \to \beta, \quad \tau_n \circ \beta \circ \tau_n^{-1} \to \alpha
\]

and the associated asymptotic morphisms \(\psi_n : C(Y) \rtimes_\beta \mathbb{Z} \to C(X) \rtimes_\alpha \mathbb{Z}\) and \(\varphi_n : C(X) \rtimes_\alpha \mathbb{Z} \to C(Y) \rtimes_\beta \mathbb{Z}\) “induce” the order isomorphisms \(\rho\) and \(\rho^{-1}\) correspondingly.

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Roughly speaking, approximate \(K\)-conjugacy = weak (approximate) conjugacy
+ “\(K\)-theoretic compatibility”.
Definition
Let $X$ be a compact metric space. For two minimal dynamical systems $(X, \alpha)$ and $(X, \beta)$, we say that they are orbit equivalent if there exists a homeomorphism $F: X \to X$ such that $F(\text{orbit}_\alpha(x)) = \text{orbit}_\beta(F(x))$ for all $x \in X$. The map $F$ is called an orbit map.

Definition (Giordano, Putnam, Skau)
Let $(X, \alpha)$ and $(Y, \beta)$ be two minimal Cantor dynamical systems that are orbit equivalent. Two integer-valued functions $m, n: X \to \mathbb{Z}$ are called orbit cocycles associated with the orbit map $F$ if $F \circ \alpha(x) = \beta^n(x) \circ F(x)$ and $F \circ \alpha^m(x)(x) = \beta \circ F(x)$ for all $x \in X$. We say that $(X, \alpha)$ and $(Y, \beta)$ are strongly orbit equivalent if they are orbit equivalent and the orbit cocycles have at most one point of discontinuity.
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As for the crossed product $C^*$-algebras side, we first check whether they are classifiable. If so, we use the Elliott invariants to replace the original crossed products. Isomorphism of crossed products gives rise to isomorphism of the Elliott invariants, and we check how that is related to the dynamical system properties.

For example, for two irrational rotation algebras $A_{\theta_1}$ and $A_{\theta_2}$, if they are isomorphic, we simply consider the following isomorphism:

$$(\mathbb{Z} + \theta_1 \mathbb{Z}, (\mathbb{Z} + \theta_1 \mathbb{Z})_+, 1) \longrightarrow (\mathbb{Z} + \theta_2 \mathbb{Z}, (\mathbb{Z} + \theta_2 \mathbb{Z})_+, 1).$$
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Theorem (Giordano, Putnam, Skau)

For minimal Cantor dynamical systems $(X, \alpha)$ and $(Y, \beta)$, $C(X) \rtimes_\alpha \mathbb{Z}$ and $C(Y) \rtimes_\beta \mathbb{Z}$ are isomorphic if and only if $(X, \alpha)$ and $(Y, \beta)$ are strongly orbit equivalent.

Remark: The proof uses the ordered Bratteli-Vershik model for the Cantor dynamics.

Fact: If the base space $X$ is connected, then strong orbit equivalence is not a “good” definition. Besides, in case the base space is connected, orbit equivalence alone will simply imply flip conjugacy.
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Remark: The proof essentially follows the above mentioned “general strategy”.

Remark: Rokhlin tower construction and the Berg technique are used to show the existence of the weak (approximate) conjugacies.

Remark: In case the base space is connected, weak (approximate) conjugacy + “K-theoretic compatibility” might still be found.
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The base space is $\mathbb{T}^1$

Isomorphism of crossed products implies that the two dynamical systems $(\mathbb{T}, \alpha)$ and $(\mathbb{T}, \beta)$ are weakly approximately conjugate (in fact, they are just flip conjugate). This comes from the Poincare classification theorem.

The base space is $\mathbb{T}^2$

(Result of Lin) Two Furstenberg transformations $\alpha$ and $\beta$ on $\mathbb{T}^2$ are approximately $K$-conjugate if and only if the crossed product $C^*$-algebras are isomorphic.

During the proof of this result, the weak (approximate) conjugacy maps are constructed using “brutal force”.
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Weak (approximate) conjugacy
Approximate $K$-conjugacy
(Strong) orbit equivalence
General Strategy
Good news
Minimal Cantor systems
Base space being $\mathbb{T}^n$ ($n \leq 2$)
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Augmented isomorphisms
The base space is $\mathbb{T}^n$, $n \geq 3$

Isomorphism of crossed products might not imply the existence of weak (approximate) conjugacies.

Example (see Chris’s 2002 arXiv paper): Minimal Furstenberg dynamical systems $(\mathbb{T}^3, \alpha)$ and $(\mathbb{T}^3, \beta)$, where

$$\alpha: (z_1, z_2, z_3) \mapsto (e^{2\pi i \theta} z_1, z_1^m z_2, z_2^n z_3)$$

and $\beta: (z_1, z_2, z_3) \mapsto (e^{2\pi i \theta} z_1, z_1^n z_2, z_2^m z_3)$.

Classification result ensures that the two crossed product $C^*$-algebras are isomorphic. The induced maps (from $\alpha$ and $\beta$) on singular cohomology groups $H^1(\mathbb{T}^3; \mathbb{Z}) \cong \mathbb{Z}^3$ can be denoted as $egin{pmatrix} 1 & m & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$ and $egin{pmatrix} 1 & n & 0 \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}$. Choose $m, n \in \mathbb{N} \setminus \{0\}$ such that these two matrices are not similar in $M_3(\mathbb{Z})$, which indicates that for all $\gamma \in \text{Homeo}(\mathbb{T}^3)$, $\alpha$ and $\gamma \circ \beta \circ \gamma^{-1}$ cannot be very close.

The base space is $S^{2n+1}$, $n \geq 1$

For uniquely ergodic homeomorphism on $S^{2n+1}$ ($n \geq 1$), by classification results of Winter, Lin and Niu, and due to Strung and Winter, we know that the crossed product $C^*$-algebras are classifiable. But the Elliott invariants do not contain much information.
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For uniquely ergodic homeomorphism on $S^{2n+1}$ ($n \geq 1$), by classification results of Winter, Lin and Niu, and due to Strung and Winter, we know that the crossed product $C^*$-algebras are classifiable. But the Elliott invariants do not contain much information.
The base space is $\mathbb{T}^n$, $n \geq 3$

Isomorphism of crossed products might not imply the existence of weak (approximate) conjugacies.

Example (see Chris’s 2002 arXiv paper): Minimal Furstenberg dynamical systems $(\mathbb{T}^3, \alpha)$ and $(\mathbb{T}^3, \beta)$, where

\[
\alpha: (z_1, z_2, z_3) \mapsto (e^{2\pi i \theta} z_1, z_1^m z_2, z_2^n z_3) \quad \text{and} \quad \beta: (z_1, z_2, z_3) \mapsto (e^{2\pi i \theta} z_1, z_1^n z_2, z_2^m z_3).
\]

Classification result ensures that the two crossed product $C^*$-algebras are isomorphic. The induced maps (from $\alpha$ and $\beta$) on singular cohomology groups $H^1(\mathbb{T}^3; \mathbb{Z})$ ($\cong \mathbb{Z}^3$) can be denoted as

\[
\begin{pmatrix}
1 & m & 0 \\
0 & 1 & n \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & n & 0 \\
0 & 1 & m \\
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\end{pmatrix}.
\]

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For bad cases with base space $D$, consider a new dynamical system with base $X \times D$, where $X$ is the Cantor set. Due to the fact that $X$ is totally disconnected, we might be able to recover weak (approximate) conjugacies on the new dynamical system.

For example, take base space to be $X \times \mathbb{T}^2$ (or $X \times \mathbb{T}^n$ in general), and consider the homeomorphisms such as

$$\alpha \times \varphi: (x, (t_1, t_2)) \mapsto (\alpha(x), \varphi_x((t_1, t_2))),$$

where $\alpha \in \text{Homeo}(X)$ and each $\varphi_x$ is a Furstenberg transformation on $\mathbb{T}^2$.

**Theorem (S)**

Let $(X \times \mathbb{T}^2, \alpha \times \varphi)$ and $(X \times \mathbb{T}^2, \beta \times \psi)$ be two minimal dynamical systems such that all cocycle actions are Furstenberg transformations. Use $A$ and $B$ to denote these corresponding crossed product $C^*$-algebras. Suppose that $A \cong B$ and there exist $\{\gamma_n\}_{n \in \mathbb{N}}$ and $\{\sigma_n\}_{n \in \mathbb{N}}$ in $\text{Homeo}(X)$ satisfying

1) $\deg(\varphi) = \deg(\psi) \circ \gamma_n$ for all $n \in \mathbb{N}$,
2) $\gamma_n \circ \alpha \circ \gamma_n^{-1} \to \beta, \sigma \circ \beta \circ \sigma_n^{-1} \to \alpha$.

Then $(X \times \mathbb{T}^2, \alpha \times \varphi)$ and $(X \times \mathbb{T}^2, \beta \times \psi)$ are weakly approximately conjugate.
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As a given $C^*$-algebra might be realized as crossed product of minimal dynamical systems on different base spaces, $C^*$-algebra alone might be missing information on the base space. Instead of considering isomorphism on crossed product only, we assume the base space $X$ is given and require one extra commutative diagram in $K$-theory:

\[
\begin{CD}
K_*(A) @>\varphi>> K_*(B) \\
@VV\rho_A V @VV\rho_B V \\
K_*(C(X)) @>\psi>> K_*(C(X))
\end{CD}
\]

This is the idea of augmented isomorphisms (by Lin & Matui).

**Remark:** For all the cases in the “Good news” part, isomorphism of crossed products automatically implies “augmented isomorphism”. For example, for two Cantor crossed products $C(X) \rtimes_{\alpha} \mathbb{Z}$ and $C(X) \rtimes_{\beta} \mathbb{Z}$, if there is an ordered isomorphism between $C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1}: f \in C(X, \mathbb{Z})\}$ and $C(X, \mathbb{Z})/\{f - f \circ \beta^{-1}: f \in C(X, \mathbb{Z})\}$, we can always find an “ordered” lift such that the following diagram commutes

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Start with “augmented isomorphisms”, we have the following result:

**Theorem (S)**

Let \((X \times \mathbb{T}^2, \alpha \times \varphi)\) and \((X \times \mathbb{T}^2, \beta \times \psi)\) be two minimal dynamical systems such that all cocyle actions are Furstenberg transformations. Use \(A\) and \(B\) to denote these corresponding crossed product \(C^*\)-algebras. Suppose that there is an “augmented isomorphism” between these two systems. Then \((X \times \mathbb{T}^2, \alpha \times \varphi)\) and \((X \times \mathbb{T}^2, \beta \times \psi)\) are weakly approximately conjugate.

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"Conjugacies" between dynamical systems, and their crossed products.

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The Goal

Terminologies

Conjugacy and flip conjugacy
Weak (approximate) conjugacy
Approximate K-conjugacy (Strong)
Orbit equivalence

General Strategy

Good news

Minimal Cantor systems
Base space being $\mathbb{T}^n$ ($n \leq 2$)

Bad news

Our approach

Concluding remarks

Augmented isomorphisms

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Base space being \( T^n \) \((n \leq 2)\)

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Thank you!