Smooth manifolds in comparison to differential triads (*)

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There is a long tradition of enlarging the framework of Classical Differential Geometry (for sort, CDG), so that singularities and different nonsmooth entities not to be a trouble, but to be faced in the same way as the smooth stuff. In other words, it is unfortunate not to be able to use the powerful tools of CDG to face the preceding phenomena, something that happens also in physics, where many geometrical models of physical phenomena are not smooth.

Around 1990, A. Maliios, using sheaf–theoretic methods, extended the mechanism of the CDG of smooth manifolds to spaces that do not admit the usual smooth structure. In this new setting called Abstract Differential Geometry (for sort, ADG) a large number of notions and results of the CDG have already been extended. What Mallios noticed in his theory of ADG is that calculus, hence smooth functions, are, in fact, not necessary in developing differential geometry. Instead, suitable sheaves of algebras of functions on arbitrary topological spaces can be used. What is essential, in this aspect, is that this theory is applicable to spaces with singularities, as well as to quantum physics. For instance, "particles" can be treated as "geometrical objects" without reference to any space in the usual sense, by just applying the methods of ADG.

In the same spirit, there are also other publications (e.g., the book of J. Nestruev, *Smooth manifolds and observables*, Springer 2002), though in a different setting, that support the algebraic formalism motivated by physical considerations.

What we shall discuss: We consider **differentiability** in the setting of ADG and **investigate conditions of uniqueness of differentials**. In particular, we prove that *a continuous map between smooth manifolds, which is differentiable in the ADG sense, is also smooth in the usual sense and its differential coincides to the ordinary one.* This makes differentials of maps between manifolds unique in both, abstract and classical context. Moreover, the category of manifolds becomes a *full subcategory of the category of differential triads*, where the latter objects are the key tools of ADG.

We proceed now to some definitions and notation needed in what follows.

The smooth manifolds we deal with are always finite dimensional and 2nd countable. All algebras are commutative, over the field $\mathbb{C}$ of complexes (except if otherwise is indicated) and have an identity element. The topological spaces are supposed to be Hausdorff.

Definition 1. Let X be an arbitrary topological space. A *differential triad* over X is a triplet $\delta = (\mathcal{A}, \partial, \Omega)$, where $\mathcal{A}$ is a sheaf of algebras over X, Ω is an $\mathcal{A}-$module and $\partial : \mathcal{A} \to \Omega$ is a Leibniz morphism, i.e., a $\mathbb{C}-$linear sheaf morphism with the property

$$\partial(\alpha\beta) = \alpha\partial(\beta) + \beta\partial(\alpha), \ \forall\ (\alpha, \beta) \in \mathcal{A} \times_X \mathcal{A},$$

where "$\times_X$" means fiber product over X.

There are many examples of differential triads, we refer to the simplest one corresponding to a smooth manifold.

Example 1. Let X be a smooth manifold. Let $\mathcal{C}^\infty_X$ be the structure sheaf of germs of smooth $\mathbb{C}$–valued functions on X and Ω^1_X be the sheaf of germs of smooth $\mathbb{C}$–valued 1–forms on X; namely, Ω^1_X consists of the smooth sections of the complexification of the cotangent bundle of X. Furthermore, let d_X be the standard differential from $\mathcal{C}^\infty_X$ to Ω^1_X, such that with each $\mathbb{C}-$valued (local) smooth function f of X one associates its differential, being by definition, a (local smooth) 1–form of X. Then, the triplet $(\mathcal{C}^\infty_X, d_X, \Omega^1_X)$ is a differential triad, which we shall call *smooth differential triad* of X.

What are morphisms of differential triads

If $\delta_X = (\mathcal{A}_X, \partial_X, \Omega_X)$ is a differential triad over X and $f : X \to Y$ is a continuous map, then *the push-out of δ_X by f*, given by

$$f_*(\delta_X) \equiv (f_*(\mathcal{A}_X), f_*(\partial_X), f_*(\Omega_X))$$

is a differential triad over Y.

Definition 2. Let δ_X, δ_Y be differential triads over the topological spaces X and Y, respectively. A *morphism of differential triads* $\hat{f} : \delta_X \to \delta_Y$ is a triplet $\hat{f} = (f, f_\mathcal{A}, f_\Omega)$, where

(i) $f : X \to Y$ is continuous;

(ii) $f_\mathcal{A} : \mathcal{A}_Y \to f_*(\mathcal{A}_X)$ is an identity preserving morphism of sheaves of algebras;

(iii) $f_\Omega : \Omega_Y \to f_*(\Omega_X)$ is an $f_\mathcal{A}-$morphism, i.e., a morphism of sheaves of additive groups, with the property

$$f_\Omega(aw) = f_\mathcal{A}(a) f_\Omega(w), \ \forall\ (a, w) \in \mathcal{A}_Y \times_Y \Omega_Y;$$

(iv) The diagram

$$\begin{array}{ccc}
\mathcal{A}_Y & \xrightarrow{f_{\mathcal{A}}} & f_*(\mathcal{A}_X) \\
{\scriptstyle \partial_Y}\downarrow & & \downarrow{\scriptstyle f_*(\partial_X)} \\
\Omega_Y & \xrightarrow[f_{\Omega}]{} & f_*(\Omega_X)
\end{array}$$

is commutative.

In the above context, we shall say that a continuous map $f : X \to Y$ is *differentiable*, if it is completed into a morphism $\hat{f} = (f, f_{\mathcal{A}}, f_{\Omega})$ of differential triads. Besides, we shall say that f_{Ω} is a *differential* of f.

If $\delta_X, \delta_Y, \delta_Z$ are differential triads over the topological spaces X, Y, Z, respectively, and $\hat{f} = (f, f_{\mathcal{A}}, f_{\Omega}) : \delta_X \to \delta_Y$, $\hat{g} = (g, g_{\mathcal{A}}, g_{\Omega}) : \delta_Y \to \delta_Z$ are morphisms, setting

$$(g \circ f)_{\mathcal{A}} := g_*(f_{\mathcal{A}}) \circ g_{\mathcal{A}} \ \text{ and } \ (g \circ f)_{\Omega} := g_*(f_{\Omega}) \circ g_{\Omega}$$

we obtain a morphism

$$\widehat{g \circ f} = (g \circ f, (g \circ f)_{\mathcal{A}}, (g \circ f)_{\Omega}) : \delta_X \to \delta_Z.$$

The differential triads, their morphisms and the *composition law* defined as before, form a category, denoted by $\mathcal{DT}$. Note that the identity id_{δ} of a differential triad $\delta = (\mathcal{A}, \partial, \Omega)$ over X is the triplet $(\mathrm{id}_X, \mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\Omega})$.

The next example gives the construction of a morphism between differential triads.

Example 2. Consider the smooth manifolds X and Y and their corresponding smooth differential triads $\delta_X^{\infty} = (\mathcal{C}_X^{\infty}, d_X, \Omega_X^1)$ and $\delta_Y^{\infty} = (\mathcal{C}_Y^{\infty}, d_Y, \Omega_Y^1)$. Let $f : X \to Y$ be a smooth map. Then, for every $V \subseteq Y$ open, set

$$\mathcal{C}_Y^{\infty}(V) \equiv \mathcal{C}^{\infty}(V, \mathbb{C}) \ \text{ and}$$
$$f_*(\mathcal{C}_X^{\infty}(V)) := \mathcal{C}_X^{\infty}(f^{-1}(V)) \equiv \mathcal{C}^{\infty}(f^{-1}(V), \mathbb{C}).$$

The map

$$(1) \quad f_{\mathcal{A}V} : \mathcal{C}_Y^{\infty}(V) \longrightarrow \mathcal{C}_X^{\infty}(f^{-1}(V)) : \alpha \longmapsto \alpha \circ f$$

is an identity preserving algebra morphism, while the family $(f_{\mathcal{A}V})_V$ is a presheaf morphism giving rise to an identity preserving algebra sheaf morphism $f_{\mathcal{A}} : \mathcal{C}_Y^{\infty} \to \mathcal{C}_X^{\infty}$. On the other hand, the respective tangent map $Tf : TX \to TY$ defines the so–called *pull–back of the smooth 1–forms by f*

$$f_{\Omega V} : \Omega_Y^1(V) \longrightarrow \Omega_X^1(f^{-1}(V)) : \omega \longmapsto \omega \circ Tf,$$

where

$$(\omega \circ Tf)_x(u) = \omega_{f(x)}(T_x f(u)), \qquad x \in X,\ u \in T_x^{\mathbb{C}} X.$$

Note that $T_x^{\mathbb{C}} X$ is the complexification of the tangent space of X at x and $T_x f$ stands also for the extension of the tangent map $T_x f$ on $T_x^{\mathbb{C}} X$. Then $f_{\Omega V}$ is an $f_{\mathcal{A}V}$−morphism and the family $(f_{\Omega V})_V$ is a presheaf morphism yielding an $f_{\mathcal{A}}$−morphism $f_\Omega : \Omega_Y^1 \to f_*(\Omega_X^1)$. If (V, ψ) is a chart of Y with coordinates $(y_1, \ldots, y_n)$, and $\omega \in \Omega_Y^1(V)$, then there are $\alpha_i \in \mathcal{C}^\infty(V, \mathbb{C})$, $i = 1, \ldots, n$, with $\omega = \sum_{i=1}^n \alpha_i \cdot d_Y y_i$. In this case the pull–back of ω by f is given by

$$(2) \quad f_{\Omega V}(\omega) = \sum_{i=1}^{n} (\alpha_i \circ f) \cdot (d_Y y_i \circ Tf).$$

The commutativity of the diagram, in Definition 2, is equivalent to the chain rule, therefore

$$(f, f_{\mathcal{A}}, f_\Omega) : \delta_X^\infty \to \delta_Y^\infty$$

is a morphism in $\mathcal{DT}$.

Thus, if $\mathcal{M}an$ denotes the category of smooth manifolds, the following functor is defined

$$F : \mathcal{M}an \to \mathcal{DT},$$

where $F(X)$, $X \in \mathcal{M}an$, is the smooth differential triad δ_X^∞ and $F(f)$, $f : X \to Y$ a smooth map, is the triplet $(f, f_{\mathcal{A}}, f_\Omega)$, as before. It is clear that the functor F is "faithful", so that the category $\mathcal{M}an$ is embedded in the category $\mathcal{DT}$.

In the abstract setting of differentiability, the following problems arise:

(1) For arbitrary algebra sheaves, the existence of a morphism extending a map is not assured, even for very simple maps, like e.g., the constant map, and

(2) the uniqueness of a morphism over a fixed map f (the analogue of the uniqueness of differentials in the classical case) is not assured either.

However, we can prove the following:

Proposition 1. *If $\delta_I = (\mathcal{A}_I, \partial_I, \Omega_I)$ are differential triads over the spaces $I = X, Y$ and $\mathcal{A}_Y$ is functional* (in the sense that $\mathcal{A}_Y$ is a subsheaf of the sheaf $\mathcal{C}_Y$ of germs of all continuous $\mathbb{C}$−valued functions on Y), *then every constant map $c : X \to Y$ is differentiable.*

Proposition 2. *If $(f, f_{\mathcal{A}}, f_\Omega)$ is a morphism in $\mathcal{DT}$, then f_Ω* (the differential of f) *is uniquely determined by $f_{\mathcal{A}}$ on the image* $\operatorname{Im} \partial_Y$ *of ∂_Y.*

Conversely, if ∂_X vanishes only on the constant subsheaf $X \times \mathbb{C} \subseteq \mathcal{A}_X$, then $f_{\mathcal{A}}$ is uniquely determined by f_Ω.

We have seen that every smooth manifold X gives rise to a differential triad δ_X^∞ and according to Example 2, every smooth map $f : X \to Y$ is completed into a morphism $\hat{f} = F(f) = (f, f_{\mathcal{A}}, f_{\Omega})$ of differential triads, i.e., f is differentiable in the setting of the ADG. So, **a natural question** now is: *Whether a continuous map* $f : X \to Y$ (between smooth manifolds) *can be differentiable in the abstract setting, without being smooth in the classical sense.* **The answer is No!**

We prove that differentiability in the abstract setting yields smoothness in the classical environment and that the abstract differential coincides with the classical one.

How we achieve this result: We reach to the latter conclusion by applying "Gelfand theory on commutative non–normed topological algebras". In fact, a key tool to the whole process is that every character of a topological $Q-$algebra is continuous and that the continuous characters of the algebras of smooth functions are uniquely determined by the point evaluations of the elements of their domain. We note that a *topological* $Q-$*algebra* is a topological algebra (with identity) whose the group of invertible elements is open. Take for instance, the algebra $\mathcal{C}^\infty[0,1]$ of all ($\mathbb{C}-$valued) smooth functions on the unit interval $[0,1]$. We remind that *every Banach algebra is a* $Q-$*algebra, but* $\mathcal{C}^\infty[0,1]$ *is an example of a* $Q-$*algebra, which under its usual topology cannot be Banach.*

If Y is a compact manifold, then the algebra of ($\mathbb{C}-$valued) smooth functions $\mathcal{C}^\infty(Y)$ is a $Q-$algebra and every algebra morphism $h : \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$, X a smooth manifold, takes the form (1) (as in Example (2)), for a suitable $f : X \to Y$ (between the algebra spectra of $\mathcal{C}^\infty(X)$ and $\mathcal{C}^\infty(Y)$, respectively). In the case that the smooth manifold Y is not compact, the nice way that sheaf morphisms localize does the trick!

More precisely, if X is a smooth manifold and K a compact subset of X, consider the inductive system $\{\mathcal{C}^\infty_X(V)\}_V$, $V \subseteq X$ open, with $K \subseteq V$ (where the connecting maps are the obvious ones) and put

$$\mathcal{C}^\infty_X(K) := \varinjlim_{K \subseteq V} \mathcal{C}^\infty_X(V).$$

Then, we can prove that:

(i) $\mathcal{C}^\infty_X(K)$ *is a* $Q-$*algebra, whose spectrum coincides with* K.

(ii) *For every* $x \in X$*, the stalk* $\mathcal{C}^\infty_{X,x}$ (of the structure sheaf $\mathcal{C}^\infty_X$), *under the inductive limit topology, is a* $Q-$*algebra whose the only character is the evaluation map* ev_x*, at* x*, i.e.,*

$$ev_x := \varinjlim_{x \in V} ev_x^V : \mathcal{C}^\infty_{X,x} \longrightarrow \mathbb{C}, \quad \text{with} \quad ev_x^V : \mathcal{C}^\infty_X(V) \longrightarrow \mathbb{C} : \alpha \longmapsto \alpha(x).$$

The proof of (i), as well as the proofs of the results that follow are not at all immediate.

Theorem 1. *Let X, Y be smooth manifolds and let $f : X \to Y$ be a continuous map. If there is an identity preserving morphism of algebra sheaves $f_{\mathcal{A}} : \mathcal{C}^\infty_Y \to f_*(\mathcal{C}^\infty_X)$, then f is smooth and*

$$f_{\mathcal{A}V}(\alpha) = \alpha \circ f, \ \forall \ \ V \subseteq Y \ \text{ open and } \ \forall \ \ \alpha \in \mathcal{C}^\infty_Y(V).$$

Using repeatedly (ii) for the stalks $\mathcal{C}^\infty_{Y,f(x)}$, $x \in X$, we conclude the preceding equality, which, in fact, shows smoothness of f, since $\alpha \circ f = f_{\mathcal{A}V}(\alpha) \in \mathcal{C}^\infty_X(f^{-1}(V))$, for every $V \subseteq Y$ open and every $\alpha \in \mathcal{C}^\infty_Y(V)$.

Theorem 2. *Let X, Y be smooth manifolds and δ^∞_X, δ^∞_Y their corresponding smooth differential triads. Let $\hat{f} = (f, f_{\mathcal{A}}, f_\Omega) : \delta^\infty_X \to \delta^\infty_Y$ be a morphism in $\mathcal{DT}$. Then, f is smooth in the ordinary sense and $\hat{f} = F(f)$, where F is the faithful functor between the categories $\mathcal{M}an$ and $\mathcal{DT}$.*

From Theorem 1 we obtain smoothness of f and that the presheaf morphisms $(f_{\mathcal{A}V})_V$, $V \subseteq Y$ open, are exactly those of Example 2. So, it remains to show that for every chart (V, ψ) from the maximal atlas defining the structure of Y, each map $f_{\Omega V}$, at every element $\omega \in \Omega^1_Y(V)$, has the representation given by the relation (2) of Example 2. This follows from the commutativity of the diagram:

$$\begin{array}{ccc}
\mathcal{C}^\infty(V, \mathbb{C}) & \xrightarrow{f_{\mathcal{A}V}} & \mathcal{C}^\infty(f^{-1}(V), \mathbb{C}) \\
\Big\downarrow d_{YV} & & \Big\downarrow d_{Xf^{-1}(V)} \\
\Omega^1_Y(V) & \xrightarrow[f_{\Omega V}]{} & \Omega^1_X(f^{-1}(V))
\end{array}$$

and so the proof is completed.

Now an application of Theorems 1 and 2 gives

Theorem 3. *$\mathcal{M}an$ is a full subcategory of $\mathcal{DT}$. In other words, when smooth manifolds X and Y are considered, the sets of morphisms between them in the categories $\mathcal{M}an$ and $\mathcal{DT}$ coincide; that is,*

$$Hom_{\mathcal{M}an}(X, Y) \cong Hom_{\mathcal{DT}}(X, Y).$$

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