Homologically best modules in classical and quantum functional analysis

A. Ya. Helemskii
Faculty of Mechanics and Mathematics
Moscow State (Lomonosov) University
helemskii@rambler.ru

I

By 'homologically best modules' we mean projective, injective and flat modules; they are three pillars of the whole building of homological algebra. In functional analysis, there exist several different approaches to each one of these notions.

To begin with, let A be an arbitrary normed algebra, supposed, for simplicity, to be unital and contractive. Saying 'normed module' or just 'module', we mean a contractive left normed unital A-module. Morphisms between modules are always supposed to be bounded as operators.

In this talk we shall mostly speak about projective modules.

Let P be a module which is, for some reason, of our main interest. The so-called lifting problem consists of a morphism $\tau : Y \to X$ between two other modules, which is usually surjective, and an arbitrary morphism $\varphi : P \to X$. A **lifting** of φ across τ is a morphism $\psi : P \to Y$, making the diagram

$$\begin{tikzcd} & Y \arrow[d, "\tau"] \\ P \arrow[ur, "\psi"] \arrow[r, "\varphi"] & X \end{tikzcd}$$

commutative.

Various definitions of projectivity appear when we consider different conditions on τ and in some cases also on ψ.

The oldest and most known variety is as follows.

Definition 1. A module P is called **relatively projective** (or just projective, as in the great majority of literature), if, for every module morphism $\tau : Y \to X$, possessing a right inverse bounded operator (not necessarily morphism) and arbitrary morphism $\varphi : P \to X$, there is a lifting of φ across τ.

For example, every closed ideal in $C(\Omega)$, where Ω is a compact set, is relatively projective iff its Gelfand spectrum or, equivalently, the complement to its hull in Ω, is paracompact. On the other hand, the algebra $\mathcal{K}(H)$, being considered as a module over itself, is projective whereas the $\mathcal{K}(H)$-module $\mathcal{B}(H)$ is not only non-projective but has homological dimension 2, that is, speaking informally, $\mathcal{B}(H)$ is two steps worse than projective modules.

The mentioned results have straightforward analogs if we shall consider these modules as operator modules, but, because of time consideration, we have to omit details.

There are many other types of projectivity. But we shall restrict ourself with the following one which, as we shall see, takes into account not only the norm topology, but the exact value of norm of modules in question. We shall say that an operator $\tau : F \to E$ is strictly coisometric (some people say 'exact quotient map') if it maps the closed unit ball of F onto the closed unit ball of E.

Definition 2. A module P is called **metrically projective**, if, for every $\tau : Y \to X$ which is strictly coisometric, and an arbitrary morphism $\varphi : P \to X$, there is a lifting ψ of φ across τ such that $\|\psi\| = \|\varphi\|$.

This notion is interesting even in the case of the simplest algebra $\mathbb{C}$, that is just for normed spaces. Here we answer a quite concrete question of the geometry of normed spaces. What is the structure of metrically projective spaces ?

Theorem 1. *Every metrically projective normed space coincides, up to an isometric isomorphism, to the normed subspace $l_1^0(\Lambda)$ of $l_1(\Lambda)$ for some index set Λ, consisting of all functions with finite supports.*

To run ahead, this theorem shows that for normed spaces the notion of projectivity coincides with that of freeness.

I do not know whether the theorem remains true if one replaces the metric projectivity by the so-called extreme projectivity. (The latter was actually studied by Grothendieck in his old paper of 1955). Our method, based on the study of extreme points of unit balls of normed spaces, does not work.

II

Now we claim that there is a certain categorical-general scheme that contains, as particular cases, all principal versions of projectivity. The main definitions generalize those given by MacLane in his theory of relative Abelian categories. (Note that typical categories of functional analysis, used in this talk, are never Abelian and often even not additive). Apart from general overview, this scheme allows one to study projective objects by means of the so-called free objects.

Let $\mathcal{K}$ be an arbitrary category. A **rig** of $\mathcal{K}$ is a faithful covariant functor $\Box : \mathcal{K} \to \mathcal{L}$, where $\mathcal{L}$ is another category. A pair, consisting of a category and its rig, is called **rigged category**. If a rig is given, we shall call $\mathcal{K}$ the main, and $\mathcal{L}$ the auxiliary category.

Fix, for a time, a rigged category, say $(\mathcal{K}, \Box : \mathcal{K} \to \mathcal{L})$. We call a morphism τ in $\mathcal{K}$ **admissible**, if $\Box(\tau)$ is a retraction (that is, has a right inverse) in $\mathcal{L}$. After this, we call an object P in $\mathcal{K}$ **projective** (with respect to the given rig), if, for every admissible epimorphism $\tau : Y \to X$ and an arbitrary morphism $\varphi : P \to X$ in $\mathcal{K}$, there exists a lifting (now in the obvious categorical-general sense) of φ across τ.

Let us denote the category of normed spaces and bounded operators by **Nor**, the category of normed A-modules and their (bounded) morphisms by $\mathbf{A}-\mathbf{mod}$, and the subcategory of the latter with the same objects, but only contractive (as operators) morphisms by $\mathbf{A}\text{-}\mathbf{mod_1}$.

One can easily verify, that

(i) A module is relatively projective iff it is projective with respect to the rig

$$\square : \mathbf{A}\text{-}\mathbf{mod} \to \mathbf{Nor},$$

where $\square$ is the relevant forgetful functor (it forgets about the outer multiplication).

(ii) A module is metrically projective iff it is projective with respect to the rig

$$\odot : \mathbf{A}\text{-}\mathbf{mod_1} \to \mathbf{Set},$$

where $\odot$ takes a given module to its closed unit ball, and a given morphism to the corresponding restriction map.

The suggested frame-work gives the possibility to study projectivity via the so-called freeness. The following definition must be well known, perhaps under different names. Let $\square : \mathcal{K} \to \mathcal{L}$ be a rig. Consider a pair, consisting of an object M in the auxiliary category $\mathcal{L}$ and of an object $F(M)$ in the main category $\mathcal{K}$. A morphism $\rho : M \to \square(F(M))$ in $\mathcal{L}$ is called a **universal arrow** if, for every X in $\mathcal{K}$ and every morphism $\alpha : M \to \square(X)$ in $\mathcal{L}$, there exists a unique morphism $\beta : F(M) \to X$ in $\mathcal{K}$, making the diagram

$$\begin{tikzcd}
 & \square(F(M)) \arrow[d, "\square(\beta)"] \\
M \arrow[ur, "\rho"] \arrow[r, "\alpha"'] & \square(X)
\end{tikzcd}$$

(in $\mathcal{L}$) commutative. If, for our pair, such a universal arrow exists, we call $F(M)$ a **free object with base** M. A rigged category is called **freedom-loving**, if every object in $\mathcal{L}$ is a base of a free object.

It is a good thing to have a lot of free objects. It seems to be also well known that *if a rigged category is freedom-loving, then an object in $\mathcal{K}$ is projective iff it is a retract of a free object* (that is, there is free F and $\sigma : F \to P$ with a right inverse morphism). Thus, if we are given a freedom-loving rigged category and we know its free objects, then we know a lot about its projective objects (and sometimes, like in the above-formulated theorem about normed spaces) we can describe in transparent terms all projective objects.

What are free objects in our two main examples of rigged categories? Denote by '$\otimes_p$' the symbol of the non-completed projective tensor product (of Grothendieck). It is easy to verify that

(i) for the rig $\square : \mathbf{A}\text{-}\mathbf{mod} \to \mathbf{Nor}$, providing, as we remember, the relative projectivity, the free module with base a normed space E is $A \otimes_p E$.

(ii) for the rig $\odot$: $\mathbf{A\text{-}mod_1} \to \mathbf{Set}$, providing, as we remember, the metric projectivity, the free module with base a set M is $A \otimes_p l_1^0(M)$ (or, which is the same, the non-completed l_1-sum of $card(M)$ copies of the A-module A).

In both cases the outer multiplication is well-defined by the equality $a \cdot (b \otimes x) := ab \otimes x$; $a, b \in A, x \in X$.

III

We pass from classical functional analysis to operator space theory, sometimes called quantum functional analysis. For brevity, saying 'quantum norm' and 'quantum space', we mean what is called in the textbook of Effros and Ruan 'abstract operator space structure' and 'abstract operator space', respectively.

Let A be simultaneously an algebra and a quantum space. We say that A is a **quantum algebra** if its multiplication is, in the sense of the same textbook, a completely bounded bilinear operator. Similarly, a quantum space X which is a left module over a quantum algebra A in the algebraic sense, will be called **quantum A-module** if its outer multiplication is a completely bounded bilinear operator.

Both concepts of projectivity, relative and metric, that were considered above for 'classical' modules, have natural 'quantum' versions. And again, like in the 'classical' case, both types of projectivity, as well as various other types, can be studied within the general frame-work of rigged categories. To save time, we shall restrict ourself with the second type.

An operator $\varphi : E \to F$ between quantum spaces will be called completely strictly coisometric, if all its amplifications $\varphi_n : M_n(E) \to M_n(F); n = 1, 2, \dots$ (acting, as we remember, between the respective spaces of vector-valued matrices) are strictly coisometric.

Definition 3. A quantum A-module P is called **metrically projective**, if for an arbitrary completely strictly coisometric operator $\tau\colon Y \to X$ every completely bounded morphism $\varphi\colon P \to X$ has a completely bounded lifting ψ across τ such that $\|\psi\|_{cb} = |\varphi\|_{cb}$.

We pass to the rigged category that provides this type of projectivity. As main category, we take $\mathbf{QA\text{-}mod_1}$, the category of (left) quantum A-modules and their *completely contractive* morphisms, and as auxiliary category just **Set**. Our rig is the functor

$$\odot : \mathbf{QA\text{-}mod_1} \to \mathbf{Set},$$

taking a quantum module X to the cartesian product $\mathsf{X}_{n=1}^{\infty} \bigcirc_{M_n(X)}$ of closed unit balls of the amplifications $M_n(X)$ of X. Thus elements of the set $\odot(X)$ are sequences $(v_1, \dots v_n \dots); v_n \in M_n(X)$, where $\|v_n\| \leq 1$. The action of our functor on morphisms is defined in an obvious way. It is easy to verify that the corresponding admissible morphisms are those that are completely strictly coisomorphic as operators, and projective objects are metrically projective quantum spaces.

What are corresponding free objects? To find them, we shall use the special finite-dimensional quantum spaces $T_n; n = 1, 2, \dots$, distinguished in the known paper 'The standard dual of an operator spaces' by David Blecher (1992). Such a T_n is defined as the quantum dual to the space $\mathcal{B}(\mathbb{C}^n)$, the latter equipped with the standard quantum norm. (Thus, the underlying normed space of T_n is just the matrix space M_n with the trace-class norm). In his paper, among other things, Blecher introduced and studied what he called (just) projective operator spaces. (This important type of projectivity is the 'quantum' version of the 'classical' extreme projectivity, mentioned above. Like its 'classical' prototype, studied long ago by Grothendieck, it has, to speak informally, asymptotic character). For the present talk, Blecher's T_n are valuable because some of their properties can be translated into the categorical language of freeness.

To formulate the relevant assertion, we recall that completely bounded bilinear operators can be linearised with the help of a special type of quantum tensor product, called operator-projective tensor product. (It was discovered by Effros/Ruan and, independently, by Blecher/Paulsen). We shall denote its non-completed version by the symbol '$\otimes_\wedge$'. It is easy to verify that, for every quantum space E, the quantum space $A \otimes_\wedge E$ is a quantum A-module with outer multiplication, well-defined by the equality $a \cdot (b \otimes x) := ab \otimes x;\ a, b \in A, x \in X$.

Proposition. *Consider, for some natural number n, the rig*

$$\odot_n : \mathbf{QA\text{-}mod_1} \to \mathbf{Set},$$

taking a quantum module X to the closed unit ball of (the n-th amplification) $M_n(X)$ and acting on morphisms in an obvious way. Then, with respect to this rig, the quantum module $A \otimes_\wedge T_n$ is a free object with base just one-point.

After this, we withdraw from functional analysis into category theory and work with the so-called (abstract) composite rigs. We omit their general definition and only say that it generalizes the way the rig $\odot$ is 'composed' by its 'components' $\odot_n$. Here we obtain a certain general assertion, connecting free objects of a composite rigged category with those of its components. Finally, applying this assertion, together with the previous proposition, to our concrete rigged category $(\mathbf{QA\text{-}mod_1}, \odot)$, we obtain the following description of its free objects.

Blecher in his mentioned paper discovered an important kind of sum of quantum spaces which he denoted by the symbol $\oplus_1$; we shall use this sign for the non-completed version of this operation. In fact, one can observe that it is just categorical coproduct in $\mathbf{QA\text{-}mod_1}$. It is this categorical property, and not the explicit construction of the operation $\oplus_1$ what we actually need in our argument.

Introduce the quantum A-module

$$\mathcal{F} := (A \otimes_\wedge T_1) \oplus_1 (A \otimes_\wedge T_2) \oplus_1 \cdots \oplus_1 (A \otimes_\wedge T_n) \oplus_1 \cdots .$$

Theorem 2. *The rigged category $(\mathbf{QA\text{-}mod_1}, \odot)$ is freedom-loving. Moreover, the metrically free quantum A-module with the one-point base is $\mathcal{F}$ whereas the* *metrically free quantum A-module with a base set M is $\oplus_1\{\mathcal{F}_t; t \in M\}$, that is the $\oplus_1$-sum of the family of copies of the module $\mathcal{F}$, indexed by points of M.*

This theorem, taken together with the categorical-general characterization of projective objects in terms of free objects, gives the description of metrically quantum modules. For brevity, the modules of the form $A \otimes_\wedge T_n$ for some n will be referred as **bricks**.

Corollary. *A quantum A-module P is metrically projective iff for some quantum A-module Q, which is the $\oplus_1$-sum of some family of bricks, P is a retract of Q in* $\mathbf{QA}$-$\mathbf{mod_1}$*, or, in the explicit form, P is a submodule of Q such that there is a projection σ of Q onto P which is a morphism with $\|\sigma\|_{cb} = 1$.*

Concluding remarks

1. We would like to emphasize that the latter corollary was strongly influenced by a certain theorem of Blecher. Namely, he characterized extremely projective (in our terminology) quantum spaces as the so-called almost direct summands of $\oplus_1$-sums of the spaces T_n. This kind of projectivity also can be included in a general-categorical frame-work, however after some elaboration, reflecting the asymptotic character of the relevant definition. (In a certain sense, speaking informally, we can interpret the extreme projectivity as 'asymptotic metric projectivity'.) In this way the extreme projectivity becomes a concrete example of the general-categorical notion of an asymptotically projective object, and this leads to an alternative proof of the mentioned theorem of Blecher.

2. We spoke about projectivity. The same categorical-general scheme works for the dual concept of injectivity. Particularly, there are rigs that provide such notions as relative and metric injectivity in its classical, as well as quantum setting. The role of free objects passes to the so-called cofree objects. In particular, a cofree object of the rigged category, providing metrically injective quantum A-modules, is the so-called $\oplus_\infty$-sum (= categorical *product* in $\mathbf{QA}$-$\mathbf{mod_1}$) of a family of copies of the quantum module which, in its turn, is the $\oplus_\infty$-sum of the quantum modules $\mathcal{CB}(A, \mathcal{B}(\mathbb{C}^n)); n = 1, 2, \ldots$. The outer multiplication in the latter modules is given by the equality $[a \cdot \varphi](b) = \varphi(ba)$.

References

[1] S. Mac Lane. *Homology.* Springer-Verlag, Berlin, 1967.

[2] S. Mac Lane. *Categories for the working mathematician.* Springer-Verlag, Berlin, 1971.

[3] D. P. Blecher. The standard dual of an operator space, Pacific J. of Math. v. 153, No. 1 (1992) 15-30.

[4] A. Ya. Helemskii. Metric freeness and projectivity for classical and quantum normed modules. Mat. Sbornik, 204:7 (2013) 3-33.

[5] A. Ya. Helemskii. Projectivity for operator modules: approach based on freedom. Rev. Roum. Math. Pures Appl, 53:2 (2014) 219-236.