Indecomposable characters of infinite dimensional groups associated with operator algebras

Masaki Izumi
izumi@math.kyoto-u.ac.jp

Graduate School of Science, Kyoto University

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Let $A$ be a separable $C^*$-algebra, and let $S(A)$ be the set of states of $A$.

The set of equivalence classes of irreducible representations of $A$ is identified with $\text{ex } S(A)/U(A)$.

The Borel structure of $\text{ex } S(A)/U(A)$ is countably separated iff $A$ has no non-type I representation.

However, $\text{ex } T(A)$ is always a standard Borel space, where $T(A)$ is the set of tracial states of $A$.

Finite factor representations of $A$ are classifiable!

When $A = C^*(G)$ is the group $C^*$-algebra of a locally compact group $G$, a tracial state of $A$ corresponds to a character of $G$. 
Let $G$ be a (not necessarily locally compact) topological group.

**Definition**

A **character** of $G$ is a continuous function $\chi : G \to \mathbb{C}$ satisfying the following three conditions:

1. $\chi$ is positive definite, i.e. $\forall g_1, g_2, \ldots, g_n \in G$ and $\forall c_1, c_2, \ldots, c_n \in \mathbb{C}$,

   $$\sum_{i,j=1}^{n} \chi(g_j^{-1} g_i) c_i \overline{c_j} \geq 0,$$

2. $\chi$ is central, i.e. $\forall g, h \in G$, $\chi(gh) = \chi(hg)$,

3. $\chi$ is normalized, i.e. $\chi(e) = 1$.

Denote by $\text{Char}(G)$ the set of characters of $G$, which is a convex set.

An extreme point $\chi \in \text{ex Char}(G)$ is called an **indecomposable** character.
Indecomposable characters

Assume that \((\pi, H)\) is a unitary representation of \(G\), and that \(\pi(G)''\) is a finite von Neumann algebra with a tracial state \(\tau\). Then \(\chi = \tau \circ \pi\) is a character of \(G\).

On the other hand, for every \(\chi \in \text{Char}(G)\) there exists a unique unitary representation \((\pi_\chi, H_\chi)\) with a cyclic vector \(\Omega_\chi\) of \(G\) satisfying \(\chi(g) = \langle \pi_\chi(g)\Omega_\chi, \Omega_\chi \rangle\). \(\Omega_\chi\) is a cyclic separating trace vector for \(\pi_\chi(G)''\).

There is a one-to-one correspondence between \(\text{ex Char}(G)\) and the set of quasi-equivalence classes of finite factor representations of \(G\).

Example

When \(G\) is compact, every \(\chi \in \text{ex Char}(G)\) is a normalized character \(\chi(g) = \frac{\text{Tr}(\pi(g))}{\dim \pi}\) of an irreducible representation \(\pi\).
Previous results

Thoma (1964): \( S_\infty = \lim_{n \to \infty} S_n \).

Voiculescu (1976): \( U(\infty) = \lim_{n \to \infty} U(n) \).

Skudlarek (1976): \( GL(\infty, \mathbb{F}_q) = \lim_{n \to \infty} GL(n, \mathbb{F}_q) \),

Vershik-Kerov (1981, 1982): Ergodic method to interpret parameters in \( \mathrm{ex} \, \mathrm{Char}(S_\infty) \) and \( \mathrm{ex} \, \mathrm{Char}(U(\infty)) \).

Boyer (1993): \( U(2^\infty) = \lim_{n \to \infty} U(2^n) \subset U(M_{2^\infty}) \).

More recently, Hirai-Hirai, Hirai-Hirai-Hora, Dudko, Dudko-Medynets, Creutz-Peterson,...
Voiculescu characters

Let $U(\infty) = \lim_{\rightarrow} U(n)$ be the inductive limit group with embedding

$$U(n) \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in U(n+1).$$

Voiculescu (1976) gave the following indecomposable characters of $U(\infty)$:

$$\chi(u) = e^{\gamma^+ \text{Tr}(u-1) + \gamma^{-1} \text{Tr}(u^{-1}-1)}$$

$$\times \prod_{i=1}^{\infty} \det \left( \frac{1 + \beta_i^+(u-1)}{1 - \alpha_i^+(u-1)} \frac{1 + \beta_i^-(u^{-1}-1)}{1 - \alpha_i^-(u^{-1}-1)} \right)$$

$$\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0, \ \beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0, \ 1 \geq \beta_1^+ + \beta_1^-, \ \gamma^\pm \geq 0.$$ 

Vershik-Kerov (1982) and Boyer (1983) observed their completeness based on Edrei’s classical result in 1953 on a doubly-infinite totally positive sequence.
The unitary group of the CAR algebra

The inductive limit group $U(2^\infty) = \lim_{n \to \infty} U(2^n)$ with embedding

$$U(2^n) \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \in U(2^{n+1}).$$

is identified with a dense subgroup of the unitary group $U(M_{2^\infty})$ of the CAR algebra $M_{2^\infty}$.

Boyer (1993) showed

$$\text{ex Char}(U(2^\infty)) = \text{ex Char}(U(M_{2^\infty})) = \{ \tau^p \tau^q; \; p, q \in \mathbb{Z}_{\geq 0} \},$$

where $\tau$ is the unique tracial state of $M_{2^\infty}$ using Stratila-Voiculescu AF algebra. But his proof is not very convincing...
Groups associated with AF algebras

An AF algebra $A = \lim_{\longrightarrow} A_n$ is an inductive limit of finite dimensional $C^*$-algebras.

When $A$ is not unital, set $U(A) = \{ u \in U(A + \mathbb{C}1) ; u - 1 \in A \}$.

Denote $U_{\rightarrow}(A) = \lim_{\longrightarrow} U(A_n)$, equipped with the inductive limit topology.

Then $U_{\rightarrow}(A)$ is a dense subgroup of $U(A)$ though the topology of $U_{\rightarrow}(A)$ is strictly stronger than the relative topology of $U(A)$.

The difference between $U_{\rightarrow}(A)$ and $U(A)$ could be drastic, e.g.

- Voiculescu showed that $U_{\rightarrow}(\mathbb{K}(\ell^2)) = U(\infty)$ has continuously many type II$_1$ factor representations,
- Kirillov (1973) showed that $U(\mathbb{K}(\ell^2))$ is a type I group having only countably many irreducible representations.
Let $\varphi \in \text{Hom}(K_0(A), \mathbb{Z})$.

Let $u \in U(A_n)$ with spectral decomposition $u = \sum_{i=1}^{m} z_i p_i$.

We set

$$\det \varphi u = \prod_{i=1}^{m} \bar{z}_i^{\varphi([p_i])}.$$

Then $\det \varphi : U(A) \to \mathbb{T}$ is a well-defined continuous homomorphism.

$\det \varphi$ does not extend to $U(A)$. 
When $\tau \in \text{ex } T(A)$, we have $\tau, \bar{\tau} \in \text{ex Char}(U(A))$.

**Theorem (Schur-Weyl duality)**

Let $\mathcal{R}_0$ be the hyperfinite $II_1$ factor acting on $L^2(\mathcal{R}_0)$, and let $J$ be the canonical conjugation of $\mathcal{R}_0$. For non-negative integers $p, q$ with $(p, q) \neq (0, 0)$,

$$\{u^\otimes p \otimes (JuJ)^\otimes q \in \mathcal{R}_0^\otimes p \otimes \mathcal{R}_0'^\otimes q; \ u \in U(\mathcal{R}_0)\}'' = (\mathcal{R}_0^\otimes p)^{S_p} \otimes (\mathcal{R}_0'^\otimes q)^{S_q},$$

where the symmetric group $S_p$ (resp. $S_q$) acts on $\mathcal{R}_0^\otimes p$ (resp. $\mathcal{R}_0'^\otimes q$) as the permutations of tensor components. In particular it is isomorphic to $\mathcal{R}_0$.

The above theorem shows that if $\tau \in \text{ex } T(A)$, $\tau^p \bar{\tau}^q \in \text{ex Char}(U(A))$. 
Let $A$ be an infinite dimensional unital simple AF algebra.

(1)\[
ex \text{Char}(U \rightarrow (A)) = \{ \det_\varphi \left( \prod_{i=1}^{p} \tau_i \right) \left( \prod_{j=1}^{q} \overline{\tau'_j} \right); \ \tau_i, \tau'_j \in \text{ex} T(A), \ p, q \geq 0, \ \varphi \in \text{Hom}(K_0(A), \mathbb{Z}) \}.
\]

(2)\[
ex \text{Char}(U(A)) = \{ \left( \prod_{i=1}^{p} \tau_i \right) \left( \prod_{j=1}^{q} \overline{\tau'_j} \right); \ \tau_i, \tau'_j \in \text{ex} T(A), \ p, q \geq 0 \}.
\]
Let $A$ be a separable infinite dimensional unital simple exact $C^*$-algebra with tracial topological rank 0 and torsion-free $K_0(A)$.

1. Let $U(A)_0$ be the connected component of $1_A$ in $U(A)$. Then

\[
\text{ex Char}(U(A)_0) = \{(\prod_{i=1}^{p} \tau_i)(\prod_{j=1}^{q} \tau'_j); \ \tau_i, \tau'_j \in \text{ex T}(A), \ p, q \geq 0\}.
\]

2. Let $\widehat{K_1(A)} = \text{Hom}(K_1(A), \mathbb{T})$ be the dual group of $K_1(A)$. Then

\[
\text{ex Char}(U(A)) = \{\psi(\prod_{i=1}^{p} \tau_i)(\prod_{j=1}^{q} \tau'_j); \ \tau_i, \tau'_j \in \text{ex T}(A), \ p, q \geq 0, \ \psi \in \widehat{K_1(A)}\},
\]
Let $R$ be a type $\text{II}_1$ factor with a unique tracial state $\tau$. We equip the unitary group $U(R)$ of $R$ with the strong operator topology. Then

$$\text{ex Char}(U(R)) = \{\tau^p\tau^q; \ p \geq 0, \ q \geq 0\}.$$

The above theorem shows that $\{u \otimes (JuJ) \otimes q \in R^\otimes p \otimes R'^\otimes q; \ u \in U(R)\}''$ is a $\text{II}_1$ factor, where $J$ is the canonical conjugation of $R$ acting on $L^2(R)$.

Does it coincide with $(R^\otimes p)^\mathcal{S}_p \otimes (R'^\otimes q)^\mathcal{S}_q$ for general $R$?

In particular, does $\{u \otimes JuJ \in R \otimes R'; \ u \in U(R)\}$ generate $R \otimes R'$?

**Note after the conference:** It turned out during the conference that the statements above are true for arbitrary factors. I would like to thank Reiji Tomatsu for useful discussions.
Stable AF case

For a stable simple AF algebra $A$, we denote by $TW(A)$ the set of densely defined lower semi-continuous semifinite traces on $A$.

The function $U_\to(A) \ni u \mapsto \tau(u - 1)$ is continuous for any $\tau \in TW(A)$.

For $\tau, \tau' \in TW(A)$ and $u \in U_\to(A)$, we set $\chi_{\tau,\tau'}(u) = e^{\tau(u-1)+\tau'(u^{-1}-1)}$.

The function $\tau(u-1)+\tau'(u^{-1}-1)$ is conditionally positive definite and $\chi_{\tau,\tau'}$ is a character of $U_\to(A)$ thanks to the well-known Schoenberg theorem.

**Theorem**

Let $A$ be a stable simple AF algebra not isomorphic to $K$. If $TW(A)$ is finite dimensional,

$$\text{ex Char}(U_\to(A)) = \{\det_\varphi \chi_{\tau,\tau'}; \; \tau, \tau' \in TW(A), \; \varphi \in \text{Hom}(K_0(A), \mathbb{Z})\}.$$
Let $M$ be a type $\text{II}_\infty$ factor with separable predual, and let $\tau_\infty$ be a normal semifinite trace of $M$.

For $1 \leq p < \infty$, we set $U(M)_p = \{u \in U(M); \|u - 1\|_p < \infty\}$, where $\|x\|_p = \tau_\infty(|x|^p)^{1/p}$.

For $a, b \geq 0$ and $u \in U(M)_1$, we set $\chi_{a,b}(u) = e^{a\tau_\infty(u-1)+b\tau_\infty(u^{-1}-1)}$.

For $u \in U(M)_2$, we set $\chi_a(u) = e^{-a\|u-1\|^2_2}$, which is $\chi_{a,a}(u)$ on $U_1(M)$.

**Theorem**

Let the notation be as above.

1. $\text{ex Char}(U(M)_1) = \{\chi_{a,b}; \ a, b \geq 0\}$.
2. $\text{ex Char}(U(M)_p) = \{\chi_a; \ a \geq 0\}$ for $1 < p \leq 2$.
3. $\text{ex Char}(U(M)_p) = \{1\}$ for $2 < p$. 
Questions/Problems

(1) Let $A$ be a unital separable simple $C^*$-algebra. Then

$$\{\psi(\prod_{i=1}^{p} \tau_i)(\prod_{j=1}^{q} \overline{\tau_j}'); \tau_i, \tau_j' \in \text{ex} \, T(A), \, p, q \geq 0, \, \psi \in \text{Hom}(U(A), \mathbb{T})\}$$

is a subset of $\text{ex \, Char}(U(A))$.

Does the above set exhaust $\text{ex \, Char}(U(A))$ ?

The first test case is the Jiang-Su algebra $\mathcal{Z}$.

(2) Compute $\text{ex \, Char}(U_{\rightarrow}(A))$ for more general AF algebras such as the gauge invariant CAR algebra (GICAR).

GICAR has $\mathbb{K} + \mathbb{C}1$ as a quotient, and $\text{ex \, Char}(U_{\rightarrow}(A))$ is expected to be considerably complicated.

(3) What is the relationship between a II$_1$ factor $R$ and the II$_1$ factors arising from representations of $U(R \otimes B(\ell^2))_p$ with $1 \leq p \leq 2$. 
Theorem (Vershik-Kerov)

Let \( \{G_n\}_{n=1}^{\infty} \) be an inductive system of topological groups with countable \( \text{ex Char}(G_n) \) for any \( n \), and let \( G = \lim_{\rightarrow} G_n \) be the inductive limit group. We assume that any \( \omega \in \text{Char}(G_n) \) is uniquely decomposed as

\[
\omega = \sum_{\chi \in \text{ex Char}(G_n)} c_{\chi} \chi
\]

with non-negative \( c_{\chi} \).

Then for any \( \chi \in \text{ex Char}(G) \), there exists a sequence of indecomposable characters \( \{\chi_n\}_{n=1}^{\infty} \) with \( \chi_n \in \text{ex Char}(G_n) \) such that \( \{\chi_n\}_{n=m}^{\infty} \) converges to \( \chi \) uniformly on \( G_m \) for any \( m \).

We may choose \( \{\chi_n\} \) so that the restriction of \( \chi_{n+1} \) to \( G_n \) contains \( \chi_n \) for any \( n \).
A signature $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_d)$ is a tuple of integers satisfying $\Lambda_i \geq \Lambda_{i+1}$ for any $1 \leq i \leq d - 1$.

It is well known that the set of the equivalence classes $\widehat{U(d)}$ of the irreducible representations of $U(d)$ is naturally in one-to-one correspondence with the set of signatures.

A signature $\Lambda$ is characterized by a pair of partitions, or Yong diagrams, as follows. We choose $p, q \in \mathbb{N}$ satisfying $\Lambda_p > 0 \geq \Lambda_{p+1}$ and $\Lambda_{q-1} \geq 0 > \Lambda_q$.

Then $\lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_p)$ and $\mu = (-\Lambda_d, -\Lambda_{d-1}, \ldots, -\Lambda_q)$ are partitions, and $\Lambda$ is uniquely determined by the pair $(\lambda, \mu)$ and $d$.

We denote by $\chi_{\{\mu;\lambda\}}^{(d)}(u)$ the corresponding normalized character of $U(d)$.

When the negative part $\mu$ is empty, we have $\chi_{\{\emptyset;\lambda\}}(u) = s_\lambda(u)/s_\lambda(1_d)$ for $u \in U(d)$, where $s_\lambda$ is the Schur polynomial.
Lemma

For two partitions $\lambda$ and $\mu$, there exists a positive constant $C_{\lambda,\mu}$ depending only on $\lambda$ and $\mu$ such that for any $u \in U(d)$ we have

$$\left| \chi_{\{\mu;\lambda\}}^{(d)}(u) - \left( \frac{\text{Tr}(u)}{d} \right)^{|\lambda|} \left( \frac{\text{Tr}(u)}{d} \right)^{|\mu|} \right| \leq \frac{C_{\lambda,\mu}}{d}.$$

To apply the Vershik-Kerov ergodic method to $U_{\rightarrow}(A)$ for a unital simple AF algebra $A$, we need to bound the size of $\lambda, \mu$ appearing in the characters of $U(A_n) = U(d_{n,1}) \times U(d_{n,2}) \times \cdots \times U(d_{n,m_n})$. 

Asymptotic factorization
Okounkov-Olshanski’s observation

When \( U(1) \subset G \), we may assume \( G_1 = U(1) \).

Then the Fourier expansion \( \chi_n(e^{it}) = \sum_{k \in \mathbb{Z}} M^{(n)}_k e^{ikt} \) of the restriction of \( \chi_n \) to \( U(1) \) gives a tight family \( \{M^{(n)}\}_{n \in \mathbb{N}} \) of probability measures on \( \mathbb{Z} \).

Lemma (Okounkov-Olshanski)

Let \( \{M^{(n)}\}_{n=1}^{\infty} \) be a tight family of probability measures on \( \mathbb{Z} \) having 4-th moments \( \langle k^4 \rangle_{M^{(n)}} \).

If the second moment sequence \( \{\langle k^2 \rangle_{M^{(n)}}\}_{n=1}^{\infty} \) diverges, then the sequence \( \{\langle k^4 \rangle_{M^{(n)}} / \langle k^2 \rangle_{M^{(n)}}^2\}_{n=1}^{\infty} \) diverges too.

When we have an estimate \( \langle k^4 \rangle_{M^{(n)}} = O(\langle k^2 \rangle_{M^{(n)}}^2) \), the second moment sequence \( \{\langle k^2 \rangle_{M^{(n)}}\}_{n=1}^{\infty} \) should be bounded.

To obtain explicit formulae of the above moments in the case of \( U(\infty) \), Okounkov-Olshanski used shifted Schur polynomial.

We use Harish-Chandara-Itzykson-Zuber integral instead.
Lemma

Let $A, B \in M_d(\mathbb{C})$ be Hermitian matrices with eigenvalues $\{\alpha_i\}_{i=1}^d$ and $\{\beta_i\}_{i=1}^d$ respectively, and let $dU$ be the normalized Haar measure of $U(d)$.

(1) \[
\int_{U(d)} e^{\sqrt{-1} \text{Tr}(UAU^{-1}B)} \, dU = \frac{\prod_{i=1}^{d-1} i!}{\sqrt{-1}^{d(d-1)/2}} \frac{\det(e^{\sqrt{-1} \alpha_i \beta_j})}{\Delta(A)\Delta(B)}.
\]
where
\[
\Delta(A) = \prod_{1 \leq i < j \leq d} (\alpha_i - \alpha_j).
\]

(2) Let $n \in \mathbb{N}$.

\[
\int_{U(d)} \text{Tr}(UAU^{-1}B)^n \, dU = \sum_{\lambda \vdash n} \frac{\dim \Pi_{\lambda}s_\lambda(A)s_\lambda(B)}{s_\lambda(1_d)},
\]
where $\lambda$ runs over all the partitions of $n$, $s_\lambda$ is the Schur polynomial, and $\Pi_{\lambda}$ is the irreducible representation of the symmetric group $\mathfrak{S}_n$ corresponding to the partition $\lambda$. 