C*-algebras associated with rings, semigroups and dynamical systems from number theory

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Topic: C*-algebras associated with

- semigroups
- endomorphisms of compact spaces or of C*-algebras
- rings (integral domains)
- boundary quotients of semigroup C*-algebras

The development we describe was triggered by the study of a class of semigroups from number theory that have an intricate, yet tractable structure.

The left regular C*-algebra of a semigroup

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- $P = \mathbb{N}^\times = \mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}$ multiplicative.

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- $P = \mathbb{N} \star \ldots \star \mathbb{N} =$ free semigroup on n generators. $C^*_\lambda P = \mathcal{E}_n =$ standard extension of $\mathcal{O}_n$.

Elementary structure of $C^*_\lambda P$

$C^*_\lambda P$ is generated by the operators $\lambda(s)$, $s \in P$, on $\ell^2 P$. These are isometries. Linear combinations of the partial isometries of the form $\lambda(s_1)^* \lambda(s_2) \lambda(s_3)^* \dots \lambda(s_n)$ are dense in $C^*_\lambda P$.

The range projections of these partial isometries generate a commutative C*-subalgebra $\mathcal{D}$ of $C^*_\lambda P$. Its spectrum is totally disconnected.

In many cases $C^*_\lambda P$ may be described as a crossed product $\mathcal{D} \rtimes P$ and can thus be described by generators and relations.

Semigroups associated with an integral domain

Let R be a commutative ring without zero-divisors. R gives rise to two natural semigroups:

- The multiplicative semigroup $R^\times = R \setminus \{0\}$
- The '$ax + b$'-semigroup, defined as the semidirect product $R \rtimes R^\times$ of the additive group R by the multiplicative semigroup $R^\times$ acting on R.

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More generally we will consider the ring R of algebraic integers in a number field K. In many respects R has properties similar to those of $\mathbb{Z}$ (which is the ring of integers for the field $\mathbb{Q}$), but the associated semigroups are more intricate.

The structure of $C^*_\lambda(R \rtimes R^\times)$

N.B. R is defined explicitly as

$$R = \left\{ x \in K : x \text{ satisfies an equation} \sum_{k=0}^{n} a_k x^k = 0 \right\}$$

where $a_k \in \mathbb{Z}$ and $a_n = 1$.

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Theorem

Let R be as above (ring of algebraic integers). The C-algebra $C^*_\lambda(R \rtimes R^\times)$ is purely infinite and has the ideal property (projections separate ideals) but is not of real rank 0. It can be described by natural generators and relations.*

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There is a natural action (σ_t) of $\mathbb{R}$ on $C^*_\lambda(R \rtimes R^\times)$ defined by

$$\sigma_t(\lambda(a,b)) = N(b)^{it} \lambda(a,b)$$

where $N(b) := |R/bR|$ (this is always finite).

KMS-states

Let $\beta \in [0, \infty)$. A state φ is called a β-KMS state for (σ_t) if

$$\varphi(yx) = \varphi(x\sigma_{i\beta}(y))$$

for analytic elements x, y.

The dynamical system (σ_t) has an interesting structure of KMS-states which is governed for large temperatures by the ideal class group Γ of R. The ideal class group is a finite abelian group defined as the quotient of the semigroup of ideals in R by the subsemigroup of ideals of the form aR (principal ideals).

N.B. The ideals of R form a semigroup for the multiplication $IJ = \{\sum a_i b_i : a_i \in I,\ b_i \in J\}$.

KMS-states on the semigroup algebra

Theorem (Cuntz-Deninger-Laca)

The KMS-states on $C^*_\lambda(R \rtimes R^\times)$ at inverse temperature β can be described (more or less explicitly). One has

- *no KMS-states for $\beta < 1$.*
- *for each $\beta \in [1, 2]$ a unique β-KMS state.*
- *for $\beta \in (2, \infty)$ a bijection between β-KMS states and traces on*

$$\bigoplus_{\gamma \in \Gamma} C^*(J_\gamma) \rtimes R^*$$

where J_γ is any ideal representing γ and R^* denotes the multiplicative group of invertible elements in R (units).

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The proof of the second point uses the fact that the values of a β-KMS state on the standard projections in $\mathcal{D}$ are determined by the values of partial Dedekind ζ-functions and of their asymptotic quotients at $\beta - 1$.

Boundary quotients and ring C*-algebras

The regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$ is obtained from the left action of $R \rtimes R^\times$ on $\ell^2(R \rtimes R^\times)$. However $R \rtimes R^\times$ also acts on the Hilbert space $\ell^2 R$. We denote the C*-algebra generated by this action by $\mathcal{A}[R]$ (the ring C*-algebra).

Theorem

$\mathcal{A}[R]$ *is simple purely infinite, in fact a Kirchberg algebra and can be described naturally by generators and relations.*

The ring C*-algebra as a boundary quotient

Recall that $C^*_\lambda(R \rtimes R^\times)$ is naturally a crossed product of the commutative subalgebra $\mathcal{D}$ by $R \rtimes R^\times$. The spectrum of $\mathcal{D}$ is totally disconnected and can be described as a natural completion of the set

$$\bigsqcup_{I \subset R \text{ ideal}} R/I$$

$X = \mathrm{Spec}\, \mathcal{D}$ contains a natural invariant closed subset consisting of limit points, the 'boundary' ∂X.

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Theorem

The ring C-algebra $\mathcal{A}[X]$ is isomorphic to the boundary quotient $C(\partial X) \rtimes (R \rtimes R^\times)$.*

K-theory

The computation of the K-groups for both $\mathcal{A}[R]$ and $C^*_\lambda(R \rtimes R^\times)$ is a non-trivial task and needs in both cases unconventional new methods.

Theorem (Cuntz-Li)

Let K be a number field and μ its group of roots of unity. Choose a free abelian subgroup Γ of $K^\times$ such that $K^\times = \mu \times \Gamma$. We obtain for the K-theory of the ring C-algebra $\mathcal{A}[R]$ attached to the ring of integers R of K:*

$$K_*(\mathcal{A}[R]) \cong \begin{cases} K_0(C^*(\mu)) \otimes_{\mathbb{Z}} \Lambda^*(\Gamma) & K \textit{ has no real embeddings} \\ \Lambda^*(\Gamma) & K \textit{ has real embeddings.} \end{cases}$$

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In the case $R = \mathbb{Z}$ this means that each prime number in $\mathbb{Z}$ gives an element of $K_1(\mathcal{A}[R])$ and that general elements in K_* arise as 'Bott elements' over these generators.

The proof of the theorem uses a result establishing a duality between the finite and infinite adele spaces for R and, based on this, a completely different description (up to Morita equivalence) of $\mathcal{A}[R]$ as a crossed product of $C_0(\mathbb{R}^n)$ by an action of $K \rtimes K^\times$.

The K-theory of the semigroup algebra $C^*_\lambda(R \rtimes R^\times)$ looks different and its computation relies on entirely different methods.

Theorem (Cuntz-Echterhoff-Li)

The K-theory of $C^*_\lambda(R \rtimes R^\times)$ is isomorphic to the K-theory of the algebra

$$\bigoplus_{\gamma\in\Gamma} C^*(J_\gamma) \rtimes R^*$$

where Γ is the ideal class group for R, J_γ is any ideal representing $\gamma \in \Gamma$ and R^* denotes the multiplicative group of invertible elements in R (units).

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N.B. One always has $C^*(J_\gamma) \cong C^*(\mathbb{Z}^n)$. The group R^* is the product of the finite group of roots of unity and a group isomorphic to $\mathbb{Z}^k$.

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The proof of the theorem uses a new general method to compute the K-theory of the crossed products for an action of a group on a totally disconnected space - which works if the action satisfies a certain regularity property.

Theorem (Cuntz-Echterhoff-Li)

Assume that the group G (satisfying the Baum-Connes conjecture) acts on the totally disconnected locally compact space X in such a way that the topology of X admits a G-invariant basis B of compact open sets, closed under finite intersections and without non-trivial unions. Then the K-theory of the crossed product $C_0(X) \rtimes G$ is isomorphic to the K-theory of the C-algebra*

$$\bigoplus_{\gamma \in G \backslash B} C_r^*(G_\gamma)$$

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The proof uses techniques developed for treating the Baum-Connes conjecture.

Crossed products for algebraic actions by endomorphisms

Let H be a compact abelian group and let α be an endomorphism of H. It induces an action on the dual group $\hat{H}$ and an isometry s_α on $L^2 H \cong \ell^2 \hat{H}$.

We can consider the regular C*-algebra $C^*_\lambda(\hat{H} \rtimes_{\hat{\alpha}} \mathbb{N})$ and the C*-algebra $\mathcal{A}[\alpha]$ generated by the action of $C(H)$ by multiplication operators on $L^2 H$ together with the isometry s_α (the 'boundary quotient').

Theorem (Cuntz-Vershik)

Assume that α is surjective with finite kernel and exact (i.e. the union of the kernels of the α^n is dense. Then $\mathcal{A}[\alpha]$ is simple purely infinite.

Theorem (Cuntz-Vershik)

Let α be as above. There is an exact sequence of Pimsner-Voiculescu type

$$\begin{tikzcd} K_*C(H) \arrow[r, "1-N(\alpha)"] & K_*C(H) \arrow[r] & K_*\mathcal{A}[\alpha] \arrow[ll, bend left] \end{tikzcd}$$

where $N(\alpha)$ is not the map α_* induced by α, but related to this map by the equation $N(\alpha)\,\alpha_* =$ (number of elements in Ker α) id.

On the other hand we always have $K_*(C^*_\lambda(\hat{H} \rtimes_{\hat{\alpha}} \mathbb{N})) \cong K_*(C(H))$.

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Note: $\mathcal{A}[\alpha]$ contains a natural commutative C*-algebra $\mathcal{D}$ with totally disconnected spectrum (the image of the canonical commutative subalgebra $\mathcal{D}$ of $C^*_\lambda(\hat{H} \rtimes_\alpha \mathbb{N})$) and the crossed product $B_\alpha = \mathcal{D} \rtimes \hat{H}$. This is a simple algebra of UHF- or Bunce-Deddens type.

Examples

Let $H = \prod_{k\in\mathbb{N}} \mathbb{Z}/n$, $\hat{H} = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}/n$ and α the one-sided shift on H defined by $\alpha((a_k)) = (a_{k+1})$. In this case $\mathcal{A}[\alpha] \cong \mathcal{O}_n$ and B_α is the canonical UHF-subalgebra.

We have $K_0C(H) \cong C(H,\mathbb{Z}),\ K_1C(H) = 0$. If we describe the elements of H by sequences $(x_0, x_1, \ldots)$ with $x_i \in \mathbb{Z}/n$, then on $f \in C(H,\mathbb{Z}) \cong K_0C(H)$, the map $N(\alpha)$ is given by

$$N(\alpha)(f)(x_0, x_1, \ldots) = \sum_{k=0}^{n-1} f(k, x_1, x_2, \ldots)$$

We obtain the well known formulas for the K-theory of B_α and $\mathcal{O}_n$, i.e

$K_0(B_\alpha) = \mathbb{Z}[\frac{1}{n}]$, $K_1(B_\alpha) = 0$, $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)$, $K_1(\mathcal{O}_n) = 0$.

Examples

Let α be an endomorphism of $H = \mathbb{T}^n$ with finite kernel and $\hat{\alpha}$ the dual endomorphism of $\hat{H} = \mathbb{Z}^n$. We assume that the intersection of all $\hat{\alpha}^k(\mathbb{Z}^n)$ is $\{0\}$.
$K_*(C(\mathbb{T}^n))$ is isomorphic to the exterior algebra $\Lambda^*\mathbb{Z}^n = \bigoplus_{p=0}^n \Lambda^p\mathbb{Z}$. The endomorphism α_* of $K_*(C(\mathbb{T}^n))$ induced by α corresponds to the endomorphism $\Lambda\hat{\alpha}$ of $\Lambda^*\mathbb{Z}^n$.

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$$K_*\mathcal{A}[\alpha] \cong \Lambda\hat{H}/(1 - D(\Lambda\hat{\alpha}')D^{-1})\Lambda\hat{H} \;\oplus\; \mathrm{Ker}\,(1 - D(\Lambda\hat{\alpha}')D^{-1})$$

where the first term has the natural even/odd grading. The second term $\mathrm{Ker}\,(1 - D\Lambda\hat{\alpha}'D^{-1})$ is $\Lambda^n\mathbb{Z}^n \cong \mathbb{Z}$ if $\det\hat{\alpha} > 0$ and $\{0\}$ if $\det\hat{\alpha} < 0$. It contributes to K_0 if n is odd and to K_1 if n is even.

Examples

Consider the solenoid group H

$$H = \varprojlim_p \mathbb{T} \qquad \hat{H} = \mathbb{Z}[\frac{1}{p}]$$

with the endomorphism α_q determined on $\hat{H}$ by $\hat{\alpha}_q(x) = qx$ (q coprime to p). The description of $\hat{H}$ as an inductive limit of groups of the form $\mathbb{Z}$ immediately leads to the formulas

$$K_0(C(H)) = K_0(C^*\hat{H}) = \mathbb{Z}[\frac{1}{p}] \qquad K_1(C(H)) = K_1(C^*\hat{H}) = \mathbb{Z}$$

Now α_q acts as id on $K_0(C(H))$ and by multiplication by q on $K_1(C(H))$. We find that $N(\alpha) = q$ id on K_0 and $N(\alpha) =$ id on K_1. The exact sequence then shows that

$$K_0(\mathcal{A}[\alpha]) = \mathbb{Z}/(q-1) + \mathbb{Z}[\frac{1}{p}] \qquad K_1(\mathcal{A}[\alpha]) = \mathbb{Z}[\frac{1}{p}]$$

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