Dynamics of Paranormal Banach Space Operators

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$\mathcal{H}$ = a complex infinite dimensional separable Hilbert space, $B(\mathcal{H})$ = the algebra of operators on $\mathcal{H}$. For a subset $M \subset \mathcal{H}$, let

$$Orb_A(M) = \bigcup_{n \in \mathbb{N}} A^n(M)$$

denote the orbit of M under A. $A \in B(\mathcal{H})$ is *hypercyclic* (resp., *weakly hypercyclic*– $A \in (w-H)$) if there exists a vector $x \in \mathcal{H}$ such that $Orb_A(x)$ is norm (resp., weakly) dense in $\mathcal{X}$. The operator $A \in B(\mathcal{H})$ is *supercyclic* (resp., *n–supercyclic* – $A \in (n-S)$) if there exists a one dimensional subspace M (resp., an n–dimensional subspace M) of $\mathcal{H}$ with (norm) dense orbit under A, and A is *weakly supercyclic,* $A \in (w-S)$, if there exists a one dimensional subspace M of $\mathcal{H}$ with weakly dense orbit.

If $A \in (w-H) \cup (n-S) \cup (w-S)$, then A has dense range, $\mathcal{H}$ is necessarily separable, $\sigma_p(A^*)$ is empty if $A \in (w-H)$, $\sigma_p(A^*)$ has at most n (including multiplicity) elements if $A \in (n-S)$, $\sigma_p(A^*)$ has at most 1 element if $A \in (w-S)$. Thus: A^* has SVEP and $0 \notin \sigma_p(A^*)$ if $A \in (w-H) \cup (n-S) \cup (w-S)$. Recall that $A \in B(\mathcal{H})$ is said to have SVEP – the single–valued extension property – at $\mu \in \mathbb{C}$ if for every neighbourhood $U_\mu \subseteq \mathbb{C}$ of μ the only solution $f : U_\mu \to \mathcal{X}$ to the equation $(A-\lambda)f(\lambda) = 0$ is the function $f \equiv 0$; A has SVEP if A has SVEP at every $\mu \in \mathbb{C}$.

The Problem. $A \in B(\mathcal{H})$ is hyponormal if $AA^* \leq A^*A$ and paranormal if $||Ax||^2 \leq ||A^2x||$ for all unit vectors $x \in \mathcal{H}$. Kitai, [18, Corollary 4.5], proved that hyponormals are not hypercyclic; Sanders [23, Theorem 4.5] proved that hyponormals can not be weakly hypercyclic. Using some interesting estimates of $||A^nx||$, $x \in \mathcal{H}$, Bourdon [6, Theorem 3.1] proved that hyponormals (more generally, paranormals) can not be supercyclic; Bourdon, Feldman and Shapiro proved that a normal operator can not be n-supercyclic [7, Theorem 4.9]; this result has since been extended by Bayart and Matheron [4, Theorem 3.2] to prove that a hyponormal operator can not be n-supercyclic (see also [11, Theorem 7.2] for a more general result). Answering a question raised by Sanders [23], Bayart and Matheron also proved that a weakly supercyclic hyponormal operator must be a scalar multiple of a unitary operator. The argument used by Bayart and Matheron [4] to prove their results for hyponormal operators depends in an essential way upon the Berger–Shaw trace inequality (a deep result not available for Hilbert space operators more general than the class of hyponormal operators). Consequently, their argument does not lend application to classes of operators such as the class of paranormal operators. We develop an alternative argument to prove a more general version of the following result:

Corollary 0.1 *Given a paranormal operator $A \in B(\mathcal{H})$, $A \notin (w-H) \cup (n-S)$, and if $A \in (w-S)$, then A is a scalar multiple of a unitary.*

Recall that $\lambda \in \mathrm{iso}\sigma(A)$ is said to be a pole of (the resolvent of) A, if the ascent and the descent of $A-\lambda$ are finite. It is well known, see [1] or [16], that if $\mathrm{asc}(A-\lambda)$ and $\mathrm{dsc}(A-\lambda)$ are finite then they are equal, and their common value is the order of the pole at λ. A pole of order one is said to be a simple pole. Hyponormal operators, also paranormal operators, $A \in B(\mathcal{H})$ satisfy the properties that:

(i) A has SVEP;

(ii) $\lambda \in \mathrm{iso}\sigma(A_p)$ then λ simple pole (for every part A_p of A);

(iii) $0 \neq \alpha, \beta \in \mathrm{iso}\sigma(A_p)$ then $(A_p-\alpha)^{-1}(0) \perp (A_p-\beta)^{-1}(0)$;

(iv) A_p and A_p^{-1} (whenever it exists) are normaloid (i.e., norm of the operator equals its spectral radius).

A class of operators satisfying the properties above is the so called class of *totally hereditarily normaloid operators* [9, 10].)

The following holds.

Theorem 0.2 *A satisfies (i) and (ii), then $A \notin (w-H)$.*

A satisfies (i), (ii) and (iii), then $A \notin (n-S)$.

A satisfies (i), (ii), (iii) and (iv) and $A \in (w-S)$, then A is a scalar multiple of a unitary.

Sketch Proof. If (i)+(ii) and $A \in (w-H) \cup (n-S) \cup (w-S)$, then :

$P_1(A) = \{\lambda : A-\lambda \text{ semi-Fredholm and } \mathrm{ind}(A-\lambda) \neq 0\} = \emptyset$, $\sigma_{lre}(A) = \sigma_{le}(A) = \sigma_{re}(A) = \sigma_e(A) = \sigma_w(A)$, $\sigma(A) \setminus \sigma_w(A) = \pi_0(A) = E_0(A) = \overline{E_0(A^*)} = \overline{\pi_0(A^*)} = \overline{\sigma(A^*)} \setminus \overline{\sigma_w(A^*)}$ ($\Longrightarrow P_1(A) = \emptyset$, $\sigma(A) = \sigma_{lre}(A) \cup E_0(A)$, A is a point of continuity of σ, $\Longrightarrow \sigma(A)$ is the closure of its isolated points, so totally disconnected).

CONCLUSION: $\mathrm{iso}\sigma(A) \neq \emptyset \Longrightarrow \sigma_p(A^*) \neq \emptyset \Longrightarrow A \notin (w - H)$.

If $A \in (n - S)$, then $\sigma_p(A^*) \neq \emptyset$. If $E_0(A) = \emptyset$, then A^* has an infinite dimensional invariant subspace. If $E_0(A) = \{\lambda_1, ..., \lambda_m\}$, $m \leq n$, then there exist invariant subspaces $M_1 = \cup_{i=1}^{m} (A - \lambda_i)^{-1}(0)$ (of finite dimension) and M_2 (of infinite dimension) of A such that $\mathcal{H} = M_1 \oplus M_2$. This is a contradiction [7, Theorem 3.4]. Hence $A \notin (n - S)$.

Finally, if $A \in (w - S)$, then by the $(w - S)$ circle theorem [4, Proposition 3.5] there exists a real $r > 0$ such that each component of $\sigma(A)$ intersects the boundary of the disc with radius r centered at 0. But then $\sigma(A)$ is contained in this boundary, $(1/r)A$ is an invertible isometry, equivalently unitary.

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