# Coactions of Hopf $C^*$-algebras on Cuntz-Pimsner algebras

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# Motivation and Aim

$C^*$-correspondence $(X, A) \rightsquigarrow$ Cuntz-Pimsner algebra $\mathcal{O}_X = C^*\langle X, A\rangle$.

Related to group actions and coactions on $\mathcal{O}_X$ we have:

- $G \overset{(\gamma,\alpha)}{\curvearrowright} (X, A) \rightsquigarrow \begin{cases} G \overset{\beta}{\curvearrowright} \mathcal{O}_X & \text{(Hao and Ng, 2008)} \\ (X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G) & \text{(Echterhoff et al., 2005)} \end{cases}$

  If $G$ is amenable, $\mathcal{O}_X \rtimes_\beta G \cong \mathcal{O}_{X \rtimes_\gamma G}$ (Hao and Ng, 2008).

- The same is true for a group coaction on $(X, A)$ under a mild condition (Kaliszewski et al., 2013).

We aim to unify the above two works into a single framework of Hopf $C^*$-algebra and show that *essentially* the same result can be obtained if $G$ is replaced by a Hopf $C^*$-algebra $S$.

# $C^*$-correspondence; representation

A $C^*$-*correspondence* is a (right) Hilbert module $X$ over a $C^*$-algebra $A$ equipped with a homomorphism $\varphi_A : A \to \mathcal{L}(X)$, called the *left action*. We write $(X, A)$ to refer to a $C^*$-correspondence.

A *representation* of $(X, A)$ on a $C^*$-algebra $B$ is a pair

$$(\psi, \pi) : (X, A) \to B$$

of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

(i) $(\psi, \pi)$ preserves the left action: $\psi(\varphi_A(a)\xi) = \pi(a)\psi(\xi)$,

(ii) $(\psi, \pi)$ preserves the inner product: $\pi(\langle \xi, \eta \rangle_A) = \psi(\xi)^*\psi(\eta)$.

(It is automatic that $\psi(\xi \cdot a) = \psi(\xi)\pi(a)$.)

# The ideal $J_X$; universal covariant representation

Every representation $(\psi, \pi) : (X, A) \to B$ determines a homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ such that $\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$.

For $(X, A)$, $J_X$ denotes the largest ideal of $A$ which is mapped injectively into $\mathcal{K}(X)$ by $\varphi_A$:

$$J_X := \{a \in A : \varphi_A(a) \in \mathcal{K}(X),\ a \ker \varphi_A = 0\}.$$

A representation $(\psi, \pi) : (X, A) \to B$ is said to be *covariant* if $\psi^{(1)} \circ \varphi_A = \pi$ on $J_X$ (Katsura, 2004).

We denote by $(k_X, k_A)$ the universal covariant representation of $(X, A)$.

# Cuntz-Pimsner algebra

The *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the $C^*$-algebra generated by $k_X(X)$ and $k_A(A)$ (Pimsner, 1997; Katsura, 2004). Hence by universality, we have the commutative diagram for a covariant representation $(\psi, \pi) : (X, A) \to B$:

$$\begin{tikzcd}
 & \mathcal{O}_X \arrow[d, dashed, "\exists!\ \psi\times\pi"] \\
(X, A) \arrow[ur, "(k_X, k_A)"] \arrow[r, "(\psi,\pi)"'] & B.
\end{tikzcd}$$

Basic examples:

- $\varphi \in Aut(A) \rightsquigarrow (X, A) = (A(\varphi), A)$, where $A(\varphi) = A$ as Hilbert $A$-modules and $\varphi_A(a)$ is the left multiplication by $\varphi(a)$. Then $\mathcal{O}_X = A \rtimes_\varphi \mathbb{Z}$.

- If $(X, A) = (\mathbb{C}^n, \mathbb{C})$ with $\varphi_A(1) = 1$ $(n \geq 2)$, then $\mathcal{O}_X = \mathcal{O}_n$.

# Multiplier correspondence

From now on, $(X, A)$ is assumed to be *nondegenerate*, i.e., $\varphi_A : A \to \mathcal{L}(X)$ is nondegenerate.

For $(X, A)$, let $M(X) := \mathcal{L}_A(A, X)$. The *multiplier correspondence* is the $C^*$-correspondence $(M(X), M(A))$ such that

$$m \cdot a := ma, \quad \langle m, n \rangle_{M(A)} := m^* n, \quad \varphi_{M(A)} := \overline{\varphi_A}$$

for $m, n \in M(X)$. Note that $X \subseteq M(X)$ via $X \cong \mathcal{K}_A(A, X)$.

The *strict topology* on $M(X)$ is the locally convex topology generated by the family of seminorms

$$m \mapsto ||Tm|| + ||m \cdot a||$$

on $M(X)$ ($T \in \mathcal{K}(X)$, $a \in A$).

# Tensor product

The tensor product of $C^*$-algebras is always minimal.

For a $C^*$-correspondence $(X, A)$ and a $C^*$-algebra $B$,

- The tensor product $X \otimes B$ of the Hilbert modules $X$ and $B$ is exterior:

$$(\xi \otimes b) \cdot (a \otimes b') = \xi \cdot a \otimes bb'$$
$$\langle \xi \otimes b, \xi' \otimes b' \rangle_{A \otimes B} = \langle \xi, \xi' \rangle_A \otimes b^* b',$$

- $\varphi_{A \otimes B} := \varphi_A \otimes \mathrm{id}_B : A \otimes B \to \mathcal{L}_A(X) \otimes \mathcal{L}_B(B) \subseteq \mathcal{L}_{A \otimes B}(X \otimes B)$.

Then $(X \otimes B, A \otimes B)$ becomes a $C^*$-correspondence.

# Hopf $C^*$-algebra; reduced Hopf $C^*$-algebra

By a *Hopf $C^*$-algebra* we mean a pair $(S, \Delta)$ of a $C^*$-algebra $S$ and a nondegenerate homomorphism $\Delta : S \to M(S \otimes S)$ satisfying

(i) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$;

(ii) $\overline{\Delta(S)(1 \otimes S)} = S \otimes S = \overline{\Delta(S)(S \otimes 1)}$.

A unitary $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ which is

(i) *multiplicative*: $V_{12} V_{13} V_{23} = V_{23} V_{12}$,

(ii) *regular*: $\{\overline{\mathrm{id} \otimes \omega}(\sum V) : \omega \in \mathcal{L}(\mathcal{H})_*\} = \mathcal{K}(\mathcal{H})$

determines a pair of Hopf $C^*$-algebras

$$S_V := \overline{\{\omega \otimes \mathrm{id}(V) : \omega \in \mathcal{L}(\mathcal{H})_*\}}, \quad \Delta_V(s) = V(s \otimes 1)V^*,$$
$$\widehat{S}_V := \overline{\{\mathrm{id} \otimes \omega(V) : \omega \in \mathcal{L}(\mathcal{H})_*\}}, \quad \widehat{\Delta}_V(x) = V^*(1 \otimes x)V,$$

called the *reduced* and *dual reduced* Hopf $C^*$-algebras, respectively. They are nondegenerate subalgebras of $\mathcal{L}(\mathcal{H})$ (Baaj and Skankalis, 1993).

# Commutative and cocommutative Hopf $C^*$-algebras

The followings are motivated examples of Hopf $C^*$-algebras:

- $(C_0(G), \Delta_G)$ with $\Delta_G(f)(r,s) = f(rs)$;
- $(C^*(G), \delta_G)$ with $\delta_G(r) = r \otimes r$, and similarly for $(C_r^*(G), \delta_G)$.

One can verify that

$$(C_0(G), \Delta_G) \cong (S_{\widehat{W}_G}, \Delta_{\widehat{W}_G}), \quad (C_r^*(G), \delta_G) = (\widehat{S}_{\widehat{W}_G}, \widehat{\Delta}_{\widehat{W}_G}),$$

for $\widehat{W}_G \in \mathcal{L}(G \times G)$ given by $\widehat{W}_G(h)(r,s) = h(sr, s)$.

# Coaction of $S$ on $(X, A)$

## Definition (K.)

A *coaction* of a Hopf $C^*$-algebra $(S, \Delta)$ on $(X, A)$ is a pair

$$(\sigma, \delta) : (X, A) \to (M(X \otimes S), M(A \otimes S))$$

of a linear map $\sigma$ and a homomorphism $\delta$ satisfying

$$\sigma(\varphi_A(a)\xi) = \varphi_{M(A\otimes S)}(\delta(a))\,\sigma(\xi)$$
$$\delta(\langle \xi, \eta \rangle_A) = \langle \sigma(\xi), \sigma(\eta) \rangle_{M(A\otimes S)}$$

such that

(i) $\overline{\varphi_{M(A\otimes S)}(1_{M(A)} \otimes S)\,\sigma(X)} = X \otimes S$; $\overline{(1_{M(A)} \otimes S)\,\delta(A)} = A \otimes S$;

(ii) $(\sigma \otimes \mathrm{id}_S) \circ \sigma = (\mathrm{id}_X \otimes \Delta) \circ \sigma$; $(\delta \otimes \mathrm{id}_S) \circ \delta = (\mathrm{id}_X \otimes \Delta) \circ \delta$.

If $S = C^*(G)$ and $\delta$ is injective, $(\sigma, \delta)$ is by definition a *nondegenerate coaction* of $G$ on $(X, A)$ (Echterhoff et al., 2005).

# The $C^*$-correspondence $(M(X \otimes C_0(G)), M(A \otimes C_0(G)))$

Recall that $M(A \otimes C_0(G)) = C_b(G, M(A)_s)$ (Akemann et al., 1973).

**Theorem (K.)**

*Let $(X, A)$ be a (nondegenerate) $C^*$-correspondence and $G$ be a locally compact Hausdoff space. Then*

$$\big(M(X \otimes C_0(G)), M(A \otimes C_0(G))\big) = \big(C_b(G, M(X)_s), C_b(G, M(A)_s)\big).$$

# Coaction of $C_0(G)$ on $(X, A)$

Let $Aut(X, A)$ be the group of isomorphisms on $(X, A)$. An *action* of a LCG $G$ on $(X, A)$ is a homomorphism

$$(\gamma, \alpha) : G \to Aut(X, A), \quad r \mapsto (\gamma_r, \alpha_r)$$

such that for each $\xi \in X$ and $a \in A$, both maps

$$G \ni r \mapsto \gamma_r(\xi) \in X, \quad G \ni r \mapsto \alpha_r(a) \in A$$

are continuous (Echterhoff et al., 2005).

**Theorem (K.)**

*There exists a one-to-one correspondence*

$$\{\textit{actions of } G \textit{ on } (X, A)\} \ni (\gamma, \alpha)$$
$$\leftrightarrow (\sigma^\gamma, \delta^\alpha) \in \{\textit{coactions } (X, A) \to \big(M(X \otimes C_0(G)), M(A \otimes C_0(G))\big)\}$$

*such that* $\sigma^\gamma(\xi)(r) = \gamma_r(\xi)$ *and similarly for* $\delta^\alpha$

# Weak $\delta$-invariancy

For $(X, A)$, recall that $J_X$ is the following ideal of $A$:

$$J_X := \{a \in A : \varphi_A(a) \in \mathcal{K}(X),\ a \ker \varphi_A = 0\}.$$

**Definition (K., cf. Kaliszewski et al., for group coaction)**

For a coaction $(\sigma, \delta)$ of $S$ on $(X, A)$, $J_X$ is said to be *weakly $\delta$-invariant* if

$$\delta(J_X)(1_{M(A)} \otimes S) \subseteq J_X \otimes S.$$

If $S = C_0(G)$, this is automatic.

# Induced coaction $\zeta$ of $S$ on $\mathcal{O}_X$

Recall that $\mathcal{O}_X = C^*\langle k_X(X), k_A(A)\rangle$.

**Theorem (K., cf. Kaliszewski et al., for group coaction)**

*Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $J_X$ is weakly $\delta$-invariant. Then there exists a unique coaction $\zeta$ of $S$ on $\mathcal{O}_X$ such that the diagram*

$$\begin{tikzcd}
 & \mathcal{O}_X \arrow[d, dashed, "\exists!\ \zeta := ((k_X \otimes \mathrm{id}_S)\circ\sigma)\times((k_A\otimes \mathrm{id}_S)\circ\delta)"] \\
(X, A) \arrow[ur, "(k_X, k_A)"] \arrow[r, "((k_X\otimes \mathrm{id}_S)\circ\sigma,\ (k_A\otimes \mathrm{id}_S)\circ\delta)"'] & M(\mathcal{O}_X \otimes S)
\end{tikzcd}$$

*commutes.*

If $S = C_0(G)$, then we can recover Hao and Ng's Theorem:

$G \overset{(\gamma,\alpha)}{\curvearrowright} (X, A) \rightsquigarrow G \overset{\beta}{\curvearrowright} \mathcal{O}_X$ with

$$\begin{tikzcd}
 & \mathcal{O}_X \arrow[d, dashed, "\exists!\ \beta_r := (k_X\circ\gamma_r)\times(k_A\circ\alpha_r)"] \\
(X, A) \arrow[ur, "(k_X, k_A)"] \arrow[r, "(k_X\circ\gamma_r, k_A\circ\alpha_r)"'] & \mathcal{O}_X
\end{tikzcd}$$

# Reduced crossed product correspondence

From now on, we consider only $S = S_V (\subseteq \mathcal{L}(\mathcal{H}))$. Let $\iota : S \hookrightarrow \mathcal{L}(\mathcal{H})$ be the inclusion map. For a coaction $(\sigma, \delta)$, let $(\sigma_\iota, \delta_\iota)$ be the following composition:

$$\begin{tikzcd}
(X, A) \arrow[rr, "(\sigma_\iota, \delta_\iota)"] \arrow[dr, "(\sigma,\delta)"'] & & (M(X \otimes \mathcal{K}(\mathcal{H})), M(A \otimes \mathcal{K}(\mathcal{H}))). \\
& (M(X \otimes S), M(A \otimes S)) \arrow[ur, "(\mathrm{id}_X \otimes \iota, \mathrm{id}_A \otimes \iota)"'] &
\end{tikzcd}$$

Let $1 \otimes \widehat{S} \subseteq M(A \otimes \mathcal{K}(\mathcal{H}))$ be the canonical embedding of $\widehat{S}$. Define

- $X \rtimes_\sigma \widehat{S} := \overline{\sigma_\iota(X) \cdot (1 \otimes \widehat{S})} \subseteq M(X \otimes \mathcal{K}(\mathcal{H}))$,
- $A \rtimes_\delta \widehat{S} := \overline{\delta_\iota(A)(1 \otimes \widehat{S})} \subseteq M(A \otimes \mathcal{K}(\mathcal{H}))$ (Baaj and Skandalis, 1993).

**Theorem (K.)**

*With respect to the operation of $\big(M(X \otimes \mathcal{K}(\mathcal{H})), M(A \otimes \mathcal{K}(\mathcal{H}))\big)$, $(X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})$ is a (nondegenerate) $C^*$-correspondence.*

# The representation $(\Psi_\sigma, \Pi_\delta)$

- $(\psi, \pi) : (X, A) \to B$ is called a representation if $\psi(\varphi_A(a)\xi) = \pi(a)\psi(\xi)$ and $\pi(\langle \xi, \eta \rangle_A) = \psi(\xi)^* \psi(\eta)$.
- $(\psi, \pi)$ is said to be covariant if $\psi^{(1)} \circ \varphi_A = \pi$ on $J_X$.
- $(k_X, k_A)$ denotes the universal covariant representation of $(X, A)$.
- $(\sigma, \delta)$ on $(X, A)$ indeces $\zeta$ on $\mathcal{O}_X$ under the assumption of weak $\delta$-invariancy of $J_X$.

## Proposition (K.)

*Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $J_X$ is weakly $\delta$-invariant. Then there exists a representation*

$$(\Psi_\sigma, \Pi_\delta) : (X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S}) \to \mathcal{O}_X \rtimes_\zeta \widehat{S}$$

*such that $\Psi_\sigma\big(\sigma_\iota(\xi) \cdot (1_{M(A)} \otimes x)\big) = \zeta_\iota(k_X(\xi))(1_{M(\mathcal{O}_X)} \otimes x)$ and similarly for $\Pi_\delta$.*

# Main result

## Theorem (K.)

*Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $J_X$ is weakly $\delta$-invariant. Then*

(i) *If $(\Psi_\sigma, \Pi_\delta)$ is covariant, then*

$$\Psi_\sigma \times \Pi_\delta : \mathcal{O}_X \rtimes_\zeta \widehat{S} \cong \mathcal{O}_{X \rtimes_\sigma \widehat{S}}.$$

(ii) *The followings are equivalent.*

1. $(\Psi_\sigma, \Pi_\delta)$ *is covariant;*
2. $J_{X \rtimes_\sigma \widehat{S}_V}(A \otimes \mathcal{K}(\mathcal{H})) \subseteq J_X \otimes \mathcal{K}(\mathcal{H})$;
3. $J_{X \rtimes_\sigma \widehat{S}_V}(\ker \varphi_A \otimes \mathcal{K}(\mathcal{H})) = 0$.

*In particular, if either $J_X = A$ or $\varphi_A$ is injective, then $(\Psi_\sigma, \Pi_\delta)$ is covariant.*

(iii) *Suppose that $S$ is defined by an amenable regular multiplicative unitary (in the sense of Baaj and Skandalis). If $A$ is nuclear (or exact) then the same is true for $\mathcal{O}_X \rtimes_\zeta \widehat{S}$.*

# Application to group actions

Considering the multiplicative unitary $\widehat{W}_G$ we can extend Hao and Ng's theorem:

**Corollary (K., cf. HN, 2008 for amenable $G$)**

*Let $(\gamma, \alpha)$ be an action of a LCG $G$ on $(X, A)$ and $\beta$ be the induced action of $G$ on $\mathcal{O}_X$. If either $J_X = A$ or $\varphi_A$ is injective, then*

$$\mathcal{O}_X \rtimes_{\beta,r} G \cong \mathcal{O}_{X \rtimes_{\gamma,r} G}.$$

# Examples

Let $\delta$ be a coaction of $S$ on a $C^*$-algebra $A$ and $\varphi \in Aut(A)$ such that

$$\delta \circ \varphi = (\varphi \otimes \mathrm{id}_S) \circ \delta.$$

Consider $(X, A) := (A(\varphi), A)$. Then $(\delta, \delta)$ defines a coaction of $S$ on $(X, A)$, and induces a coaction $\zeta$ of $S$ on the $C^*$-algebra $\mathcal{O}_X = A \rtimes_\varphi \mathbb{Z}$. Applying the main theorem we have $\mathcal{O}_X \rtimes_\zeta \widehat{S} = \mathcal{O}_{X \rtimes_\delta \widehat{S}}$, or equivalently

$$(A \rtimes_\varphi \mathbb{Z}) \rtimes_\zeta \widehat{S} = (A \rtimes_\delta \widehat{S}) \rtimes_{\varphi \rtimes \mathrm{id}} \mathbb{Z},$$

where $\varphi \rtimes \mathrm{id} \in Aut(A \rtimes_\delta \widehat{S})$ is the automorphism given by

$$(\varphi \rtimes \mathrm{id})(\delta_\iota(a)(1 \otimes x)) = \delta_\iota(\varphi a)(1 \otimes x) \quad (a \in A \ x \in \widehat{S}).$$

# Examples

The quantum permutation group $\mathbb{G} = A_{aut}(X_n)$ is the quantum automorphism group acting on $n$-point set $X_n$ (Wang, 1998). As a $C^*$-algebra, $C(\mathbb{G})$ is a universal $C^*$-algebra generated by $n^2$ projections $u_{ij}$ satisfying $\sum_i u_{ij} = 1 = \sum_j u_{ij}$. It is a compact quantum group.

Consider the $C^*$-correspondence $(\mathbb{C}^n, \mathbb{C})$ $(n \geq 2)$. Define

$$\sigma(e_j) = \sum_i e_i \otimes u_{ij}, \quad \delta(1) = 1.$$

Then $(\sigma, \delta)$ is a coaction of $C(\mathbb{G})$ on $(\mathbb{C}^n, \mathbb{C})$ and induces a coaction $\zeta$ of $C(\mathbb{G})$ on $\mathcal{O}_n$. We see that $\mathcal{O}_n \rtimes_\zeta \mathbb{G} \cong \mathcal{O}_{\mathbb{C}^n \rtimes_\sigma \widehat{\mathbb{G}}}$.