# Large Subalgebras of Crossed Product C*-Algebras

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Differences from the talk as presented:

- Some misprints corrected.
- Reformatted for convenient printing.
- Pictures omitted.
- At the actual talk, pages 20 and 22 weren't shown at all, and page 21 was shown only briefly.

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## Outline

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1. Motivation for large subalgebras.
2. $C^*(\mathbb{Z}, X, h)_Y$ and some applications.
3. Other applications of large subalgebras.
4. The definition of a large subalgebra.
5. Statements of some theorems.
6. Something about what makes some of the proofs work.

One other subject I could have talked about: Operator algebras on $L^p$ spaces. To my surprise, there appears to be a substantial theory of such algebras, including analogs of AF C*-algebras and Cuntz algebras, full and reduced group $L^p$ operator algebras, irrational rotation algebras, and crossed products. In particular, interesting things can be proved about these examples, many of them parallel to what happens for C*-algebras, but some not. (There is a link to at least one talk about this on my website, `http://pages.uoregon.edu/ncp/`.)



## Motivation

Large subalgebras are an abstraction of the Putnam algebra $C^*(\mathbb{Z}, X, h)_{\{y\}}$, originally introduced by Putnam in 1989 when $X$ is the Cantor set and $h\colon X \to X$ is a minimal homeomorphism. Putnam used this subalgebra to determine the order on $K_0(C^*(\mathbb{Z}, X, h))$.

### Notation

For a locally compact Hausdorff space $X$ and an open subset $U \subset X$,

$$C_0(U) = \big\{ f \in C_0(X) \colon f(x) = 0 \text{ for all } x \in X \setminus U \big\} \subset C_0(X).$$

It is of course the usual $C_0(U)$ when $U$ is considered by itself.

### Definition

Let $X$ be a locally compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $Y \subset X$ be a nonempty closed subset. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary generator. Define

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C_0(X),\ C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

We call it the *$Y$-orbit breaking subalgebra* of $C^*(\mathbb{Z}, X, h)$.

## Motivation (continued)

Crossed products by minimal homeomorphisms, or more generally by essentially free minimal actions of countable groups, are among the most intensively studied examples of C*-algebras.

The famous irrational rotation algebra is the crossed product by an irrational rotation homeomorphism on the circle. The Giordano-Putnam-Skau theorem for crossed products by minimal homeomorphisms of the Cantor set states that two of them are isomorphic if and only if the homeomorphisms are what is called strongly orbit equivalent. Its proof uses the applicability of the Elliott classification program to these crossed products.

Ultimately, one hopes that if $\Gamma$ is amenable, $X$ is compact metric with $\dim(X) < \infty$, and the action of $\Gamma$ on $X$ is essentially free and minimal, then $C^*(\Gamma, X)$ is covered by the Elliott classification program. (This would imply that all countable amenable groups have quasidiagonal C*-algebras.)

Generalizations of the Putnam subalgebra have been one of the main tools used so far in the study of these crossed products.



## $C^*(\mathbb{Z}, X, h)_Y$

Let $h\colon X \to X$ be a homeomorphism, let $Y \subset X$ be a nonempty closed subset, and set

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\ C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

Putnam used $uC_0(X \setminus Y)$ instead of $C_0(X \setminus Y)u$. The choice here gives better compatibility with Rokhlin towers.

Putnam used $C^*(\mathbb{Z}, X, h)_{\{y\}}$, taking $Y$ to be a one point set. He also used the following fact. Suppose $Y_0 \supset Y_1 \supset Y_2 \supset \cdots$ and $Y = \bigcap_{n=0}^{\infty} Y_n$. Then

$$C^*(\mathbb{Z}, X, h)_Y = \overline{\bigcup_{n=0}^{\infty} C^*(\mathbb{Z}, X, h)_{Y_n}} = \varinjlim C^*(\mathbb{Z}, X, h)_{Y_n}.$$

This is used with $\mathrm{int}(Y_n) \neq \varnothing$, which implies that $C^*(\mathbb{Z}, X, h)_{Y_n}$ is subhomogeneous. (In Putnam's case, $C^*(\mathbb{Z}, X, h)_{Y_n}$ is actually homogeneous.)



## Identifying $C^*(\mathbb{Z}, X, h)_Y$

$h\colon X \to X$ is a homeomorphism and $u \in C^*(\mathbb{Z}, X, h)$ is the standard unitary generator. If $Y \subset X$ is a nonempty closed subset, then

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\ C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

**Proposition**

Let $X$ be a compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ be the usual conditional expectation. Let $Y \subset X$ be nonempty and closed. For $n \in \mathbb{Z}$, set

$$Z_n = \begin{cases} \bigcup_{j=0}^{n-1} h^j(Y) & n > 0 \\ \varnothing & n = 0 \\ \bigcup_{j=1}^{-n} h^{-j}(Y) & n < 0. \end{cases}$$

Then

$$C^*(\mathbb{Z}, X, h)_Y = \big\{a \in C^*(\mathbb{Z}, X, h)\colon E(au^{-n}) \in C_0(X \setminus Z_n) \text{ for all } n \in \mathbb{Z}\big\}.$$

If $\mathrm{int}(Y) \neq \varnothing$, then $Z_n = Y$ for $|n|$ large. This can be seen to imply that $C^*(\mathbb{Z}, X, h)_Y$ is a recursive subhomogeneous algebra. (Definition omitted.)



## Large subalgebras of $C^*(\mathbb{Z}, X, h)$

We give the definition and properties of large subalgebras below. The important point is that certain properties of a large subalgebra $B \subset A$ imply the same property for $A$. Example: strict comparison in the Cuntz semigroup. In some cases, one wants a stronger condition. For example, if $B \subset A$ is centrally large, and $\mathrm{tsr}(B) = 1$, then $\mathrm{tsr}(A) = 1$.

**Theorem**

Let $X$ be a compact metric space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact set such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h.)$

Assume $\dim(X) < \infty$, and let $Y$ be as above. Choose $Y_0 \supset Y_1 \supset \cdots$ with $Y = \bigcap_{n=0}^{\infty} Y_n$ and $\mathrm{int}(Y_n) \neq \varnothing$. Then $C^*(\mathbb{Z}, X, h)_Y = \varinjlim C^*(\mathbb{Z}, X, h)_{Y_n}$ is a simple direct limit, with no dimension growth, of recursive subhomogeneous algebras. Such direct limits have many good properties (including strict comparison and stable rank one).

Weakness: We don't know how to get anything about decomposition rank.

## Further applications of $C^*(\mathbb{Z}, X, h)_Y$

Toms and Winter considered a minimal homeomorphism $h\colon X \to X$ of an infinite finite dimensional compact metric space $X$ such that the projections in $C^*(\mathbb{Z}, X, h)$ distinguish the tracial states on $C^*(\mathbb{Z}, X, h)$ (equivalently, distinguish the invariant probability measures on $X$). Using $C^*(\mathbb{Z}, X, h)_{\{y_1\}}$, $C^*(\mathbb{Z}, X, h)_{\{y_2\}}$, and $C^*(\mathbb{Z}, X, h)_{\{y_1, y_2\}}$, with $y_1$ and $y_2$ in different orbits of $h$ (and other techniques), they proved that $C^*(\mathbb{Z}, X, h)$ is covered by the Elliott classification program.

Elliott and Niu used $C^*(\mathbb{Z}, X, h)_{\{y\}}$ and its expression as $\varinjlim C^*(\mathbb{Z}, X, h)_{Y_n}$ as on the previous slide to prove that if $h\colon X \to X$ is a minimal homeomorphism of an infinite compact metric space $X$ and $h$ has mean dimension zero, then $C^*(\mathbb{Z}, X, h)$ has radius of comparison equal to zero (that is, strict comparison in the Cuntz semigroup). This is a special case of the conjecture

$$\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \tfrac{1}{2}\mathrm{mdim}(h).$$

Here $\mathrm{mdim}(h)$ is a dynamical version of covering dimension. (It is zero whenever $\dim(X) < \infty$ or $h$ is uniquely ergodic.)



## Actions on $C(X, D)$

Let $X$ be a compact metric space, let $h\colon X \to X$ be a minimal homeomorphism, let $D$ be a simple unital C*-algebra, and let $\beta \in \mathrm{Aut}(C(X, D))$ be an automorphism which "lives over $h$". On $C(X) \otimes D$ this means that $\beta(f \otimes 1) = (f \circ h^{-1}) \otimes 1$ for all $f \in C(X)$. Then for $Y \subset X$ closed and nonempty, one can form

$$\begin{aligned} C^*(\mathbb{Z},\, C(X, D),\, h)_Y &= C^*\big(C(X, D),\, C_0(X \setminus Y,\, D)u\big) \\ &\subset C^*(\mathbb{Z},\, C(X, D),\, h). \end{aligned}$$

Under suitable conditions on $D$ (such as nuclear and tracial rank zero), and for small enough $Y$, this will be a centrally large subalgebra. It is important in work of Julian Buck on the structure of $C^*(\mathbb{Z},\, C(X, D),\, h)$. If $X$ has finite covering dimension (and for some choices of $h$ even when $\dim(X) = \infty$), and $D$ is as above, Buck proves that $C^*(\mathbb{Z},\, C(X, D),\, h)$ has strict comparison of positive elements.



## Actions of $\mathbb{Z}^d$

**Theorem**

Let $d \in \mathbb{Z}_{>0}$, and let $X$ be a finite dimensional compact metric space, with a free minimal action of $\mathbb{Z}^d$. Then there exists a subgroupoid $G$ of the transformation group groupoid $\mathbb{Z}^d \times X$ such that:

1. $C^*_{\mathrm{r}}(G)$ is a centrally large subalgebra of $C^*(\mathbb{Z}^d, X)$.
2. $C^*_{\mathrm{r}}(G)$ is a simple direct limit, with no dimension growth, of recursive subhomogeneous algebras.

This result gave the first proof that $C^*(\mathbb{Z}^d, X)$ has strict comparison of positive elements. (Szabo now has stronger results using finite Rokhlin dimension.)

Recall: $C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h)$. This comes from a subgroupoid $G \subset \mathbb{Z} \times X$. (Omitted; not hard.)

For $\mathbb{Z}^d$, there seems to be no nice description of any choice of $G$, or any nice description (not necessarily involving groupoids) of a subalgebra of $C^*(\mathbb{Z}^d, X)$ satisfying (1) and (2). So an abstract definition is needed.



## Extended irrational rotation algebras

Elliott and Niu have studied C*-algebras obtained from irrational rotation algebras by "cutting" each of the standard unitary generators at one or more points in its spectrum, say by adding logarithms of them or adding some spectral projections. They prove that such algebras are AF. Large subalgebras of such algebras play a significant role in their arguments: the original irrational rotation algebra is large in the new algebra.

These examples differ from the others here in two ways. The containing algebra isn't given as a crossed product, and it isn't known whether the subalgebras are centrally large.

## Work in progress: Actions of amenable groups

We think we know how to do most of the proof of the following.

**Conjecture**

Let $X$ be the Cantor set, and let $\Gamma$ be a countable amenable group. Suppose we have an action of $\Gamma$ on $X$ which is free, minimal, and uniquely ergodic. Then there is a subgroupoid $G$ of the transformation group groupoid $\Gamma \times X$ such that:

1. $C^*_{\mathrm{r}}(G)$ is a centrally large subalgebra of $C^*(\Gamma, X)$.
2. $C^*_{\mathrm{r}}(G)$ is a simple AF algebra.

It would follow (modulo some details to check) that $C^*(\Gamma, X)$ is stable under tensoring with the Jiang-Su algebra—an important regularity condition related to the Elliott classification program.



## Cuntz subequivalence

In preparation for the definition of a large subalgebra, we recall Cuntz subequivalence. Let $A$ be a C*-algebra.

1. For $a, b \in (K \otimes A)_+$, we say that $a$ is *Cuntz subequivalent to $b$ over $A$*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n\to\infty} v_n b v_n^* = a$. (We write $\precsim_A$ since we will need different choices of $A$.)
2. We say that $a$ and $b$ are *Cuntz equivalent in $A$*, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$. This relation is an equivalence relation.
3. The *Cuntz semigroup* $\mathrm{Cu}(A)$ is the semigroup of equivalence classes with direct sum as addition.

The Cuntz semigroup does for "supports of positive elements" something like what the semigroup of Murray-von Neumann equivalence classes of projections does for projections. Example: for $A = C(X)$, we have $f \precsim g$ if and only if $f(x) = 0$ whenever $g(x) = 0$. But $\mathrm{Cu}(A)$ is more complicated. For example, if projections $p, q$ satisfy $\|p - q\| < 1$, then $p \sim q$. If $a, b$ are positive, and $\|a - b\| < \varepsilon$, the best we can do is as follows. Let $(a - \varepsilon)_+$ be the positive part of $a - \varepsilon$. Then $(a - \varepsilon)_+ \precsim b$.



## The definition of a large subalgebra

We assume $A$ is stably finite throughout. (Without this, more care is needed.) Also, we do not know the right definition when $A$ is not simple.

**Definition**

Let $A$ be an infinite dimensional stably finite simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in $A$ if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.

**Definition**

$B$ as above is *centrally large* if, in addition, we can require:

5. For $j = 1, 2, \ldots, m$ we have $\|ga_j - a_j g\| < \varepsilon$.



## About the definition of a large subalgebra

Let $A$ be an infinite dimensional stably finite simple unital C*-algebra. Then $B \subset A$ is large if whenever $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.

It says: Given $a_1, a_2, \ldots, a_m \in A$, we can perturb them slightly so that a very large piece is actually in $B$, namely, the cutdown by $1 - g$, with $g$ small in the Cuntz sense (relative to both $A$ and $B$).

Fact (needs some functional calculus): One can replace (2) and (3) by the condition

$$\mathrm{dist}\big((1 - g)a_j,\, B\big) < \varepsilon.$$

It is very important that we do not ask that $g$ be a projection. This avoids the need for Berg's technique.

## About the definition of a large subalgebra (continued)

Let $A$ be an infinite dimensional stably finite simple unital C*-algebra. Then $B \subset A$ is large if whenever $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1-g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.

In previous talks, in (3) I have also required $c_j(1-g) \in B$. This is not needed for any of the proofs, and causes trouble in the application to the extended irrational rotation algebras.

Mostly, one uses the following. If $a_j \geq 0$, then require $c_j \geq 0$ and $(1-g)c_j(1-g) \in B$. (Apply the definition to $a_j^{1/2}$. One apparently can't get both $c_j \geq 0$ and $(1-g)c_j \in B$.)



## Some results for large subalgebras

**Theorem**

Let $A$ be an infinite dimensional stably finite simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then:

1. $B$ is simple, infinite dimensional, and stably finite.
2. The restriction maps $T(A) \to T(B)$ and $QT(A) \to QT(B)$ (on tracial states and quasitraces) are bijective.
3. The inclusion of $B$ in $A$ induces an isomorphism on the semigroups that remain after deleting from $\mathrm{Cu}(B)$ and $\mathrm{Cu}(A)$ all the classes of nonzero projections.
4. $B$ and $A$ have the same radius of comparison.

It is known that if $A$ be an infinite dimensional stably finite simple unital C*-algebra, and one deletes from $\mathrm{Cu}(A)$ all the classes of nonzero projections, then what remains is in fact a subsemigroup.

Part (3) is in some sense the main part. At least heuristically, the other results follow from this.



## Some results for centrally large subalgebras

The definition of a centrally large subalgebra, and the following theorem, are part of joint work with Dawn Archey.

Recall that in the definition of a large sublgebra, we had $a_1, \ldots, a_m \in A$, and we wanted $g \in B_+$ to be "small" in both $\mathrm{Cu}(A)$ and in $\mathrm{Cu}(B)$, and $(1-g)a_j$ to be close to $B$. For centrally large, we add the requirement that $g$ approximately commute with $a_j$.

**Theorem**

Let $A$ be an infinite dimensional stably finite simple unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Then:

1. If $B$ has stable rank one, so does $A$.
2. (A bit still to be checked for this.) If $A$ and $B$ are nuclear, and $B$ is $Z$-stable, then so is $A$.

In part (2), $Z$ is the Jiang-Su algebra. Nuclearity enters through a result of Matui-Sato, via Hirshberg-Orovitz "tracial $Z$-stability".



## Large subalgebras and the Cuntz semigroup

Recall: The Cuntz semigroup is an ordered semigroup made from positive elements. We had $a \precsim_A b$ if there is a sequence $(v_n)_{n=1}^{\infty}$ such that $v_n b v_n^* \to a$. This relation compares "supports of positive elements". (For example, $2b \precsim b$, using $v_n = 2^{1/2} b^{1/n}$.)

For $B \subset A$ large, we want $\mathrm{Cu}(A)$ and $\mathrm{Cu}(B)$ to be "essentially the same".

Recall what largeness gives for positive elements: for $a_1, a_2, \ldots, a_m \in A_+$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A_+$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1-g)c_j(1-g) \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.

Using simplicity, we can ensure that $g$ has "tracially small support" (in $B$) by choosing $y \in B_+$ so that a direct sum of a large number of copies of $y$ is subequivalent to 1.

$B \subset A$ is large. Thus, for $a_1, a_2, \ldots, a_m \in A_+$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A_+$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1-g)c_j(1-g) \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.

We arranged for $g$ to be small in $\mathrm{Cu}(B)$.

If $a \in A_+$, then one can show that

$$(1-g)a(1-g) \precsim_A a \precsim_A (1-g)a(1-g) \oplus g.$$

If $(1-g)a(1-g) \in B$, then we have bounded the Cuntz class of $a$ above and below by elements of $\mathrm{Cu}(B)$ which are very close together.

Actually, we get $\|a - c\| < \varepsilon$ and $(1-g)c(1-g) \in B$. Recall that $(c-\varepsilon)_+$ is the positive part of $c - \varepsilon$, and that $\|a - c\| < \varepsilon$ implies $(c-\varepsilon)_+ \precsim a$. So we need (naively; the real proof isn't as easy):

$$\big[(1-g)c(1-g) - \varepsilon\big]_+ \precsim (c-\varepsilon)_+ \precsim_A \big[(1-g)c(1-g) - \varepsilon\big]_+ \oplus g.$$

This is in fact true.



## Why $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is large in $C^*(\mathbb{Z}, X, h)$

Recall: $B \subset A$ is large if whenever $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1-g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.

We let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary and take $c_j$ to be of the form $c_j = \sum_{n=-N}^{N} c_{j,n} u^n$ with $c_{j,n} \in C(X)$. In this case, the following choice turns out to work. Choose $g_0 \in C(X)$ such that $g(y) = 1$, $\mathrm{supp}(g_0)$ is very small (the disjoint union of $2N+1$ copies of it is still small), and the sets $h^n(\mathrm{supp}(g_0))$, for $-N \leq n \leq N$, are disjoint. Then take

$$g = \sum_{n=-N}^{N} g_0 \circ h^n \in C(X) \subset C^*(\mathbb{Z}, X, h)_{\{y\}}.$$

One needs to show that "small in $C(X)$" implies "small in $C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $C^*(\mathbb{Z}, X, h)$".