Dynamical rigidity of Ginibre interacting Brownian motions

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Interacting Brownian motions in infinite-dimensions $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ are stochastic dynamics in $(\mathbb{R}^d)^{\mathbb{N}}$ given by ISDE

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \in \mathbb{N},\, j \neq i} \nabla \Psi(X_t^i - X_t^j) dt \quad (i \in \mathbb{N})$$

Here Ψ is an interaction potential and β is inverse temperature.
This ISDE has been studied by Lang, Fritz, and others.
So far Ψ is taken to be Ruelle's class potentials.
We take Ψ is the 2D Coulomb potential (logarithmic potential):

$$\Psi(x) = -\log |x|.$$

Set up: Ginibre interacting Brownian motions in infinite-dimensions.

Let Ψ be the 2D Coulomb, $d = 2$, and $\beta = 2$. Then

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j|<r}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \qquad (1)$$

$$X_0^i = x_i.$$

We call $\mathbf{X} = (X_t^i)_{i\in\mathbb{N}}$ the Ginibre interacting Brownian motions (IBMs) in infinite-dimensions.

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Set up: Ginibre interacting Brownian motions in infinite-dimensions.

- This is very complicated SDEs:

$$dX_t^1 = dB_t^1 + \lim_{r\to\infty} \sum_{j\neq 1,\, |X_t^1 - X_t^j|<r}^{\infty} \frac{X_t^1 - X_t^j}{|X_t^1 - X_t^j|^2} dt$$

$$dX_t^2 = dB_t^2 + \lim_{r\to\infty} \sum_{j\neq 2,\, |X_t^2 - X_t^j|<r}^{\infty} \frac{X_t^2 - X_t^j}{|X_t^2 - X_t^j|^2} dt$$

$$dX_t^3 = dB_t^3 + \lim_{r\to\infty} \sum_{j\neq 3,\, |X_t^3 - X_t^j|<r}^{\infty} \frac{X_t^3 - X_t^j}{|X_t^3 - X_t^j|^2} dt$$

$$dX_t^4 = dB_t^4 + \lim_{r\to\infty} \sum_{j\neq 4,\, |X_t^4 - X_t^j|<r}^{\infty} \frac{X_t^4 - X_t^j}{|X_t^4 - X_t^j|^2} dt$$

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Set up: Ginibre interacting Brownian motions in infinite-dimensions.

Ginibre interacting Brownian motions:

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j|<r}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \tag{2}$$

$$X_0^i = x_i.$$

Let μ_{gin} be the Ginibre random point field (RPF).

Let $\mathsf{x} = \sum_i \delta_{x_i} \mapsto (x_i)$ be a label.

Thm [O.(PTRF12)] For μ_{gin}-a.s. $\mathsf{x} = \sum_i \delta_{x_i}$, the ISDE (2) has a weak solution $(\mathbf{X}, \mathbf{B})$ whose unlabeled diffusion

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}, \quad \mathsf{X}_0 = \mathsf{x} = \sum_i \delta_{x_i}. \tag{3}$$

is reversible w.r.t. to the Ginibre RPF μ_{gin}.

Set up: Ginibre interacting Brownian motions in infinite-dimensions.

Ginibre interacting Brownian motions:

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j|<r}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \qquad (2)$$

Thm [O.-Tanemura, 14] (Existence & Uniqueness of strong sols)
(1) For μ_{gin}-a.s. x (2) has a strong solution whose unlabeled diffusion

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}, \quad \mathsf{X}_0 = \mathsf{x} = \sum_i \delta_{x_i}. \qquad (4)$$

is reversible w.r.t. to the Ginibre RPF μ_{gin}.
(2) A pathwise uniqueness holds under the absolutely cont cond (AC):

$$\mu_{\mathrm{gin}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathrm{gin}} \quad \text{for all } t \in [0,\infty). \qquad \text{(AC)}$$

A family of weak solutions whose unlabeled dynamics satisfies (AC) is unique.

- Thm is an application of a general theory of O.-Tanemura.

Ginibre IBMs

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i, |X_t^i - X_t^j| < r}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \qquad (2)$$

Not Ginibre but similar potentials

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i, |X_t^i - X_t^j| < r}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^4} dt. \qquad (5)$$

- What is the difference of these models?

Set up :
Ginibre random point field

Set up

Let $S = \mathbb{R}^2$. We regard S as $\mathbb{C}$.

S: configuration space over S

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\ s_i \in S,\ \mathsf{s}(|s| < r) < \infty\ (\forall r \in \mathbb{N})\}$$

- S is a Polish space with the vague topology.
- A prob meas. μ on S is called random point field (point process) on S.

- S is a set of unlabeled particles.
- $S^{\mathbb{N}}$ is the space of labeled particles.

Definition of Ginibre RPF

- A symmetric function ρ^n is called the n-correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^{n} m(dx_i) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S)$, $k_i \in \mathbb{N}$ s.t. $k_1 + \ldots + k_m = n$.

- μ is called the determinantal RPF generated by (K, m) if its n-correlation fun. ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1\le i,j\le n}$$

- Ginibre RPF $S = \mathbb{C}$. μ_{gin} is generated by $(K_{\mathsf{gin}}, \mathrm{g})$

$$K_{\mathsf{gin}}(x,y) = e^{x\bar{y}} \qquad \mathrm{g}(dx) = \pi^{-1} e^{-|x|^2} dx$$

- A simulation is as follows:

Basic properties of Ginibre RPF

- μ_{gin} is rotation and translation invariant.
- μ_{gin} has a N-particle approximation μ_{gin}^N, i.e.

$$\lim_{N\to\infty} \mu_{\text{gin}}^N = \mu_{\text{gin}} \qquad \text{weakly}$$

and μ_{gin}^N is the RPF supported on the N-particles defined by the labeled density m^N

$$m^N(\mathbf{x}_N) = \frac{1}{\mathcal{Z}} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \mathrm{g}(x_k) d\mathbf{x}_N$$

- μ_{gin}^N is the distribution of the eigen values of non-Hermitian Gaussian random matricies called Ginibre ensemble.

Two intuitive representations of Ginibre RPF

- From the N-particle density

$$m^N(\mathbf{x}_N) = \frac{1}{\mathcal{Z}} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \mathrm{g}(x_k) \prod_{l=1}^{N} dx_l,$$

we have the first intuitive representation of Ginibre RPF:

$$\mu_{\mathsf{gin}} \sim \frac{1}{\mathcal{Z}} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} \mathrm{g}(x_k) \prod_{l=1}^{\infty} dx_l. \tag{6}$$

- Taking the translation invariance of μ_{gin} into account, we have the second intuitive representation:

$$\mu_{\mathsf{gin}} \sim \frac{1}{\mathcal{Z}} \prod_{i<j}^{\infty} |x_i - x_j|^2 \prod_{k=1}^{\infty} dx_k. \tag{7}$$

- I will justify these two informal expressions. This fact is one of geometric rigidities.

Gibbs measures & Quasi-Gibbs measures

Gibbs measure

- Ψ: Ruelle's class interaction potential,

$S_r = \{|x| \le r\}$, $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\xi))$$

- μ is called (Φ, Ψ)-Gibbs m. if it satisfies DLR eq:

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}} e^{-\mathcal{H}_r(\mathsf{s}) - \mathcal{W}_{r,\xi}(\mathsf{s})} \prod_{k=1}^m e^{-\Phi(s_k)} ds_k$$

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i - s_j), \ \mathcal{W}_{r,\xi} = \sum_{s_i \in S_r, \xi_j \in S_r^c} \Psi(s_i - \xi_j)$$

- Let $\Psi(x) = -2\log|x|$. Then

$$\mathcal{W}_{r,\xi} = \sum_{s_i \in S_r, \xi_j \in S_r^c} -2\log|s_i - \xi_j| = \infty$$

Hence, $\mathcal{W}_{r,\xi}$ diverges and DLR does not make sense as it is.

(Φ,Ψ)-Quasi Gibbs measures

Let $\Lambda_r^m = \Lambda_r(\cdot|\mathsf{s}(S_r)=m)$, where Λ_r is the Poisson RPF with $1_{S_r}ds$.

(Φ,Ψ)-Gibbs m.

$$d\mu_{r,\xi}^m = \frac{1}{z_{r,\xi}^m} e^{-\mathcal{H}_r - \mathcal{W}_{r,\xi}} d\Lambda_r^m \qquad \text{(DLR eq)}$$

(Φ,Ψ)-quasi Gibbs m. $\exists\ c_{r,\xi}^m$

$$c_{r,\xi}^{m\ -1} e^{-\mathcal{H}_r} d\Lambda_r^m \le \mu_{r,\xi}^m \le c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m$$

- If μ is Airy RPF, $\mathcal{W}_{r,\xi}$ and $z_{r,\xi}^m$ diverge. But $e^{-\mathcal{W}_{r,\xi}}/z_{r,\xi}^m$ conv.

$$c_{r,\xi}^{m\ -1} \le e^{-\mathcal{W}_{r,\xi}}/z_{r,\xi}^m \le c_{r,\xi}^m$$

- Quasi-Gibbs is very mild restriction.
- If μ is (Φ,Ψ)-quasi-Gibbs m, then μ is also $(\Phi+f,\Psi)$-quasi Gibbs m for any loc bdd m'able f.

Thm 1 (O. [AOP13]). *Ginibre RPF* μ_{gin} *is a quasi-Gibbs measure with* $\Phi = 0$ *and 2D Coulomb potential:*

$$\Psi(x) = -2\log|x|.$$

Namely, there exists a constant $c = c_{r,\xi}^{m}$ *such that*

$$\frac{1}{c} e^{\sum_{i<j}^{m} 2\log|s_i - s_j|} d\Lambda_r^m \le \mu_{r,\xi}^m \le c e^{\sum_{i<j}^{m} 2\log|s_i - s_j|} d\Lambda_r^m.$$

Equivalently,

$$\frac{1}{c} \prod_{i<j}^{m} |s_i - s_j|^2 d\Lambda_r^m \le \mu_{r,\xi}^m \le c \prod_{i<j}^{m} |s_i - s_j|^2 d\Lambda_r^m$$

- Sine_β, Airy_β, Bessel_β ($\beta = 1, 2, 4$) are quasi-Gibbs measures.

As a corollary of Thm 2 we have

Thm 2 (O. [AOP13]). *The standard bilinear form*

$$\mathcal{E}^{\mu_{\mathsf{gin}}}(\mathsf{f},\mathsf{g}) = \int_{\mathsf{S}} \mathbb{D}[\mathsf{f},\mathsf{g}]\,\mu_{\mathsf{gin}}(d\mathsf{s})$$

associated with the Ginibre RPF μ_{gin} is closable on $L^2(\mathsf{S},\mu_{\mathsf{gin}})$, and its closure is a quasi-regular Dirichlet form. Here $\mathbb{D}$ is defined as

$$\mathbb{D}[\mathsf{f},\mathsf{g}](\mathsf{s}) = \frac{1}{2}\sum_i \frac{\partial \tilde{f}}{\partial s_i}\frac{\partial \tilde{g}}{\partial s_i} \qquad\qquad (\mathsf{s} = \sum_i \delta_{s_i})$$

Corollary 1. *There exists a unlabeled diffusion $\mathsf{X} = \sum_i \delta_{X_t^i}$ associated with the Dirichlet space $(\mathcal{E}^{\mu_{\mathsf{gin}}},\mathcal{D}^{\mu_{\mathsf{gin}}})$ on $L^2(\mathsf{S},\mu_{\mathsf{gin}})$.*

• For given quasi-Gibbs measures with upper semi-continuous potentials with marginal assumptions, the associated bilinear forms are closable and their closures become quasi-regular Dirichlet forms (O. [CMP96, AOP13]). So Corollary follows from Dirichlt form theory.

Log derivatives:

To give a precise correspondence between Coulomb potentials and quasi-Gibbs measures, we introduce the notion of Log derivative.
As application, we solve the ISDEs.

Log derivative of μ

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | \mathsf{s}(\{x\}) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times \mathsf{S}$:

$$\mu^1(A \times B) = \int_A \rho^1(x) \mu_x(B) dx$$

- $\mathsf{d}_\mu \in L^1(\mathbb{R}^d \times \mathsf{S}, \mu^1)$ is called the log derivative of μ if

$$\int_{\mathbb{R}^d \times \mathsf{S}} \nabla_x f d\mu^1 = -\int_{\mathbb{R}^d \times \mathsf{S}} f \mathsf{d}_\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ$$

Here ∇_x is the nabla on $\mathbb{R}^d$, $\mathcal{D}_\circ = \{$bdd, loc smooth funs f on $\mathsf{S}\}$.

- Very informally

$$\mathsf{d}_\mu = \nabla_x \log \mu^1$$

Log derivative of Ginibre RPF

Thm 3 (O. [PTRF12]).

The log derivative $\mathsf{d}^{\mu_{\mathsf{gin}}}$ *of Ginibre RPF is given by*

$$\mathsf{d}^{\mu_{\mathsf{gin}}}(x, \mathsf{s}) = 2 \lim_{r\to\infty} \sum_{|x-s_i|<r} \frac{x - s_i}{|x - s_i|^2}. \tag{8}$$

Moreover, for all $a \in \mathbb{R}^2$,

$$\mathsf{d}^{\mu_{\mathsf{gin}}}(x, \mathsf{s}) = -2(x - a) + 2 \lim_{r\to\infty} \sum_{|s_i-a|<r} \frac{x - s_i}{|x - s_i|^2}. \tag{9}$$

These give the same log derivative.

- This result is a geometric rigidity. In fact, μ_{gin} is supported on the set that these representations define the same functions. These functions are not the same on the whole space.
- This result justifies the both intuitive representations of μ_{gin}.
- Periodic rpfs satisfy the identity of the rhs (but no log derivative).

Consider uncountably many ISDEs $\mathbf{X} = (X_t^i)_{i\in\mathbb{N}}$:

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{j\neq i,\,|X_t^i - X_t^j|<r} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt, \tag{2}$$

$$dX_t^i = dB_t^i - (X_t^i - a)dt + \lim_{r\to\infty} \sum_{j\neq i,\,|X_t^j - a|<r} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{10}$$

Thm 4 (O.PTRF12, O.-Tanemura 14)**.**
(1) *For μ_{gin}-a.s.* x *both of* (2), (10) *have a unique, strong solution whose unlabeled diffusion is reversible w.r.t. the Ginibre rpf μ_{gin}.*
(2) *All SDEs have the same strong solutions.*

- This result is a dynamical rigidity. The support of μ_{gin}concentrates on the very thin subset in $\mathbb{C}^{\mathbb{N}}$. There the log derivatives takes the same value. Log derivatives are tangent vectors in infinite dimensions.

We see that Ginibre IBMs satisfies relation for all $a \in \mathbb{R}^2$ for all t:

$$\lim_{r\to\infty} \sum_{j\neq i,\, |X_t^i - X_t^j| < r} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt$$

$$= -(X_t^i - a)dt + \lim_{r\to\infty} \sum_{j\neq i,\, |X_t^j - a| < r} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt$$

 Recall that the denisity of finite Ginibre is:

$$m^N(\mathbf{x}_N) = \frac{1}{\mathcal{Z}} \prod_{i<j}^{N} |x_i - x_j|^2 \prod_{k=1}^{N} \mathrm{g}(x_k) d\mathbf{x}_N$$

Then the assosiated SDE is

$$dX_t^{N,i} = dB_t^i - X_t^{N,i} dt + \sum_{j \neq i}^{N} \frac{X_t^{N,i} - X_t^{N,j}}{|X_t^{N,i} - X_t^{N,j}|^2} dt.$$

Let $\ell(\mathsf{x}) = (x^1, \ldots, x^n, ..)$ be a label. We take $\ell(\mathsf{x})_N = (x_1, \ldots, x_N)$ as the ini cond.

Thm 5. *The first* m *component* $\mathbf{X}^{N,\mathsf{m}} = (X^{N,1}, \ldots, X^{N,\mathsf{m}})$ *converge to that of Ginibre IBMs:*

$$\lim_{N\to\infty} \mathbf{X}^{N,\mathsf{m}} = \mathbf{X}^{\mathsf{m}} \qquad \textit{weakly in } C([0,\infty); \mathbb{C})$$

Proof. Application of uniqueness of ISDE and control of tail part of log derivative. □

Idea to solve ISDE

Idea to solve ISDE: From S to $S^{\mathbb{N}}$.

(Step 1) To construct unlabeled diffusion X by Dirichlet forms. To prove μ is quasi-Gibbs.

There exist map from probabilities on S to bilenear forms :

$$\mu \mapsto \mathcal{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g] d\mu \quad \text{on } L^2(\mathsf{S},\mu)$$

We can do this with quasi-Gibbs (local density) + marginal ass.
At this step, diffusions are unlabeled.
Then we have unlabeled diffusion

$$\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}.$$

From $C([0,\infty);\mathsf{S})$ to $C([0,\infty);S^{\mathbb{N}})$:

(Step 2) The difficulty to construct $S^{\mathbb{N}}$-valued diffusion, there is no good measure on $S^{\mathbb{N}}$. Even if Brownian motions, the measure should be $dx^{\mathbb{N}}$. Hence consider m-Campbell measure $\mu^{[\mathsf{m}]}$ of μ.
Introduce the coutable family of Dirichlet forms:

$$(\mathcal{E}^{\mu^{[\mathsf{m}]}}, L^2(S^{\mathsf{m}}\times\mathsf{S},\mu^{[\mathsf{m}]})), \quad \mathbf{X}^{[\mathsf{m}]} := (X^{\mathsf{m}1},\ldots,X^{\mathsf{m},\mathsf{m}},\sum_{i=\mathsf{m}+1}^{\infty}\delta_{X^{\mathsf{m},i}})$$

There is natural coupling associated diffusions. $\Rightarrow$

$X^{\mathsf{m},i}$ are independent of m. $\Rightarrow$

From this consistency we can construct the labeled diffusion on $S^{\mathbb{N}}$.

(Step 3) Calculate the logarithmic derivative d^{μ}. ISDE becomes

$$dX_t^i = dB_t^i + \frac{1}{2}\mathsf{d}^{\mu}(X_t^i, \sum_{j\neq i} \delta_{X_t^j})dt$$

In the case of Ginibre, Sine_{β}(Dyson), Bessel, and Gibbs measures:

$$\beta\nabla\Phi(x) + \beta \lim_{r\to\infty} \sum_{j,\,|x-s_j|<r} \nabla\Psi(x-s_j)$$

Then we have the ISDE (weak solution):

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla\Phi(X_t^i) - \frac{\beta}{2}\lim_{r\to\infty} \sum_{j\neq i,\,|X_t^i - X_t^j|<r} \nabla\Psi(X_t^i - X_t^j)dt$$

To calculate the logarithmic derivative we use finite particle approximation. In particular, orthogonal polynomials.

The shape of Airy RPF is different.

(Step 4) Introduce the infinitely many finite dimensional SDEs:
Let $(\mathbf{B}, \mathbf{X})$ be a weak solution.
We regard $\mathbf{X}$ as a part of coefficients of SDEs.
For each m consider SDE of $\mathbf{Y}^{\mathsf{m}}$

$$dY_t^{\mathsf{m},i} = dB_t^{\mathsf{m}} - \frac{\beta}{2}\nabla\Phi(Y_t^{\mathsf{m},i}) - \frac{\beta}{2}\sum_{j=1,j\neq i}^{\mathsf{m}} \nabla\Psi(Y_t^{\mathsf{m},i} - Y_t^{\mathsf{m},j})dt - \frac{\beta}{2}\sum_{j=\mathsf{m}+1}^{\infty} \nabla\Psi(Y_t^{\mathsf{m},i} - X_t^j)dt.$$

These SDE have unique strong solution (under suitable assumptions). Hence

$$\mathbf{Y}^{\mathsf{m}} = \mathbf{X}^{\mathsf{m}} := (X^1, \dots, X^{\mathsf{m}})$$

- We solve infinite-many finite-dimensional SDEs with consistency in stead of solving the ISDE.

(Step 5) $\mathbf{Y}^{\mathsf{m}}$ is a functional of $(\mathbf{B}, (X^{\mathsf{m}+1}, \ldots,))$.

$\Rightarrow$ If $\lim_{\mathsf{m}\to\infty} \mathbf{Y}^{\mathsf{m}}$ exists, then $\sigma[\mathbf{B}] \vee \cap_{\mathsf{m}=1}^{\infty} \sigma[X^{\mathsf{m}}, \ldots,]$-measurable.

$\Rightarrow$ Since $\lim_{\mathsf{m}\to\infty} \mathbf{Y}^{\mathsf{m}} = \mathbf{X}$, $\mathbf{X}$ is $\sigma[\mathbf{B}] \vee \cap_{\mathsf{m}=1}^{\infty} \sigma[X^{\mathsf{m}}, \ldots,]$-measurable.

$\Rightarrow$ If $\cap_{\mathsf{m}=1}^{\infty} \sigma[X^{\mathsf{m}}, \ldots,]$ is trivial, then $\mathbf{X}$ is a strong solution.

(Step 6) If absolutely continuity condition satisfied, and if μ is tail trivial with a marginal assumption, then

$$\mathcal{T}_{\mathsf{tail}}(C([0,\infty); S^{\mathbb{N}})) := \cap_{\mathsf{m}=1}^{\infty} \sigma[X^{\mathsf{m}}, \ldots,]$$

is trivial.

- Tail triviality of RPF $\Rightarrow$ tail tiriviality of labeled path space.
- The tail σ-field of the labeled path space w.r.t. the label is a *boundary condition of ISDE*. So if its trivial and unique, then the solution of ISDE is unique.

Tail triviality of μ is not a restriction. Indeed,

Prop 1. *Determinantal RPFs (in continuous spaces) are tail trivial. In particular, Ginibre RPF is tail trivial.*

This result is a generalization of Shirai-Talahashi, and Russel Lyons for discrete spaces.

Prop 2. *Quasi-Gibbs measures μ have decomposition w.r.t. their tail σ-fields $\mathcal{T}(\mathsf{S})$ such that each components are tail trivial: For μ-a.s.* s

$$\mu(\mathsf{A}|\mathcal{T}(\mathsf{S}))(\mathsf{s}) = 1_{\mathsf{A}}(\mathsf{s}) \qquad \textit{for all } \mathsf{A} \in \mathcal{T}(\mathsf{S}).$$

This is an analogy of the result of Georgii on Gibbs measures on discrete spaces.