Universality of random matrices, Dyson Brownian Motion and Quantum Unique Ergodicity

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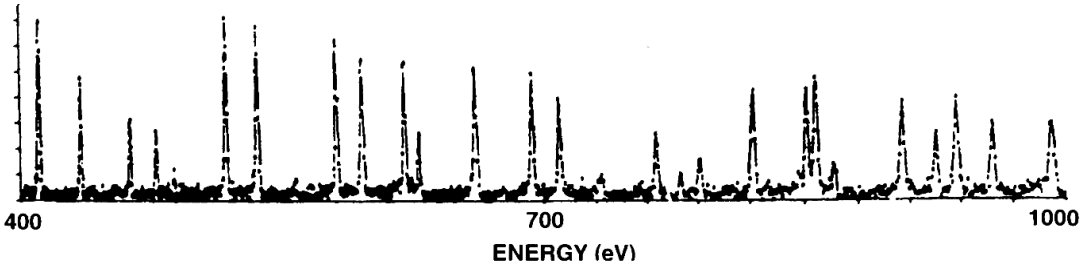
Outlines

Experimental data for excitation spectra of heavy nuclei:

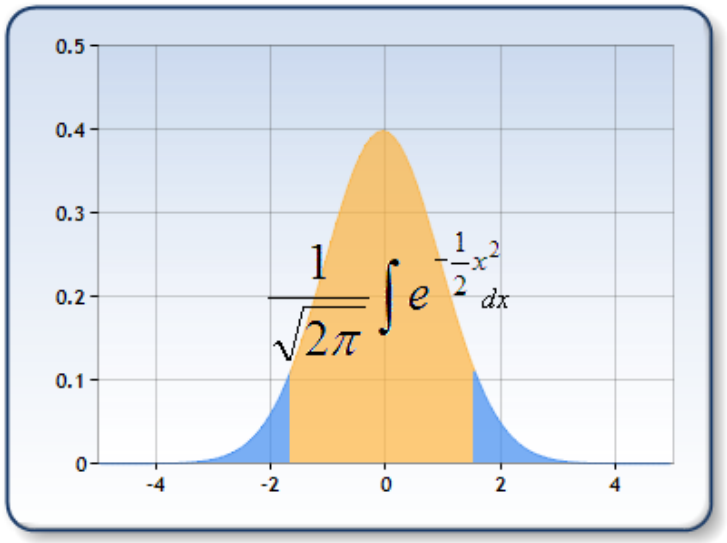


Wigner (1955): This pattern can be modeled by the spacing distribution of eigenvalues of random matrices.

> Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels (of energies of heavy nuclei). Theoretically, the situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients.
> — *E. Wigner*

Gaussian Orthogonal Ensemble (GOE):

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1N} \\ h_{21} & h_{22} & \dots & h_{2N} \\ \vdots & \vdots & & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NN} \end{pmatrix}$$

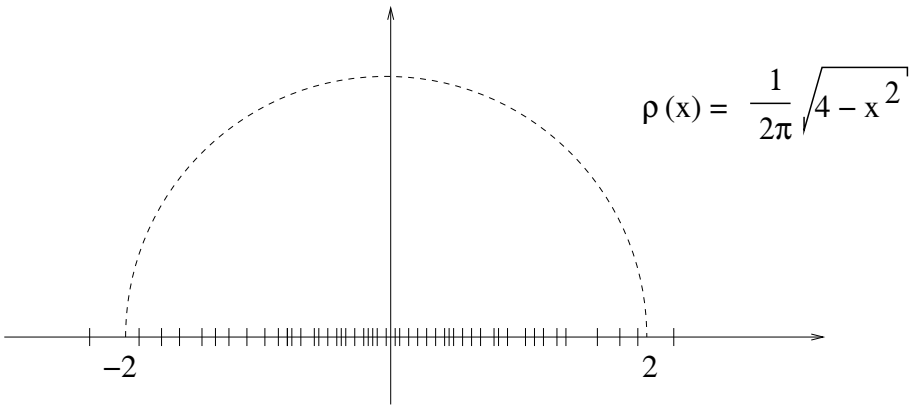


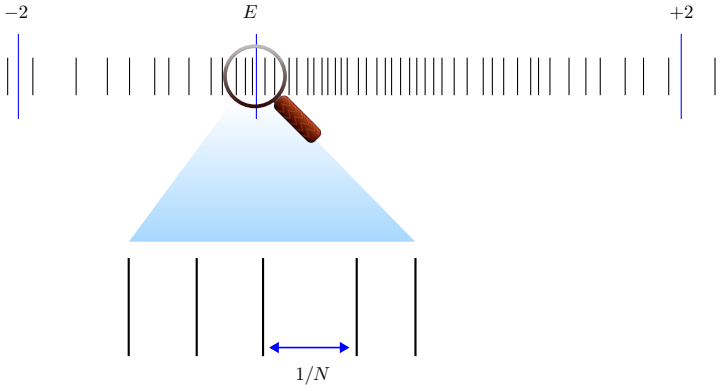
$h_{jk} = h_{kj}$ (for $j < k$) are real independent normal random variables

$$\mathbb{E}h_{jk} = 0, \qquad \mathbb{E}|h_{jk}|^2 = \frac{1}{N}$$

The eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ are of order one.
Also, Gaussian unitary or symplectic (GUE, GSE) ensembles.
Wigner ensembles: h_{ij} are just independent (not necessarily normal) distributions.

Wigner semicircle law: Density of eigenvalues





$$\mathbb{P}\Big(\lambda_1 \sim E + \frac{x_1}{N}, \lambda_2 \sim E + \frac{x_2}{N}\Big) \sim 1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2$$

Gaudin-Mehta and Dyson

Denote the prob. distribution of eigenvalues by $p_N(\lambda_1, ..., \lambda_N)d\lambda_1 \dots d\lambda_N$
Correlation function for two eigenvalues:

$$p_N^{(2)}(x_1, x_2) = \int_{\mathbb{R}^{N-2}} p_N(x_1, x_2, \lambda_3..., \lambda_N)d\lambda_3...d\lambda_N$$

Gaudin, Mehta, Dyson ['60]: local statistics of level correlation

$$\widetilde{p}_N^{(2)}(E; a_1, a_2) := \rho(E)^{-2} p_N^{(2)}\Big(E + \frac{a_1}{N\rho(E)}, E + \frac{a_2}{N\rho(E)}\Big)$$

$$\rightharpoonup \det\left\{S(a_i - a_j)\right\}_{i,j=1}^{2}, \quad S(a) = \frac{\sin \pi a}{\pi a} \text{ (for GUE)}, \ |E| < 2$$

Spacing distribution can be computed from correlation functions.
Different behavior at the edge by Tracy-Widom.

Is this exact computation for Gaussian models valid for general systems?
Fundamental belief of universality: Random matrix statistics are a new class of universal laws for highly correlated systems.

A simple manifestation (Wigner-Dyson-Mehta conjecture) : If h_{ij} are independent, then the local eigenvalue statistics are the same as those of the Gaussian ensembles.

Only symmetry types of the ensembles matter.

Central limit theorem:
$\frac{1}{\sqrt{N}}(X_1 + X_2 + \ldots + X_N) \sim \mathcal{N}(0, \sigma^2)$, normal (Gaussian) distribution

Quantum Chaos and Anderson Model

Anderson (1958): V_ω random potential on $\mathbb{R}^d$ or $\mathbb{Z}^d$.
$H = -\Delta + \lambda V_\omega$. Local eigenvalue statistics are either GOE or Poisson.

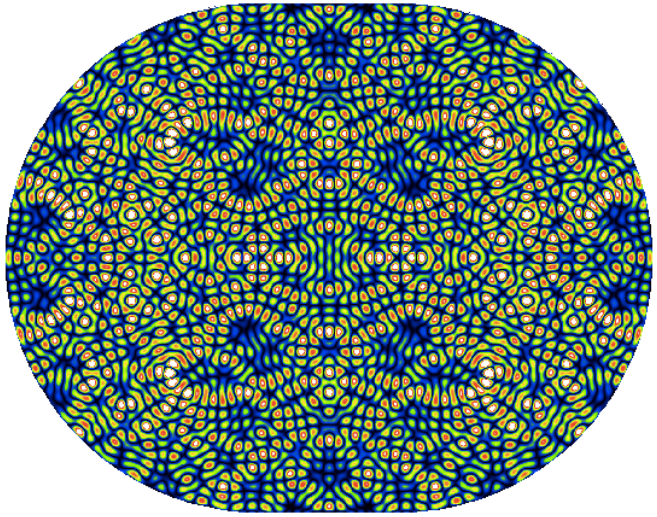
Similarly for $-\Delta$ on a domain with chaotic classical dynamics.
Quantum Chaos conjecture: Bohigas-Giannoni-Schmit (1984)
For integrable dynamics: Berry-Tabor (1977)

Quantum unique ergodicity conjecture (QUE) [Rudnick-Sarnak]:

$(\psi_k)_{k\geqslant 1}$ orthogonal eigenfunctions of M with negative curvature:

$$\int a(x)|\psi_j(x)|^2 \mathrm{dVol}(x) \to \int a(x)\mathrm{dVol}(x)$$

for any large energy limit.



Quantum Ergodicity Theorem: Averaged version of QUE [Snirelman, Colin de Verdiere, Zelditch] Anantharaman. Arithmetic QUE (Luo-Sarnak, Bourgain, Lindenstrauss, Soundararajan)

Question: Does QUE hold for Wigner matrices? Yes for GUE/GOE because the eigenvectors are uniformly distributed on $U(N)/O(N)$.

Quantum (unique) ergodicity/delocalization
$\Longleftrightarrow$ random matrix statistics ?

Two basic mathematical questions:
1. Universality conjecture for Wigner matrices
2. QUE for Wigner matrices.

Key ingredient in solving these problems: Dyson Brownian motion.

Generalized Wigner Ensembles, $H = (h_{ij})_{1 \leqslant i,j \leqslant N}$

$$h_{ji} = h_{ij}, \mathbb{E}h_{ij} = 0, \quad \mathbb{E}|h_{ij}|^2 = s_{ij}, \sum_i s_{ij} = 1, \frac{c}{N} \leq s_{ij} \leqslant \frac{C}{N}$$

Solution to the Universality conjecture

Theorem [Erdos-Schlein-Y-Yin, 2009-2010] Suppose that $4 + \varepsilon$ moments of matrix elements are uniformly bounded. Then local eigenvalue statistics for generalized Wigner ensembles are universal in the bulk (and at the edges).

Matrices with Bernoulli entries with varying variances are included.

Extensions to sparse matrices, β-ensembles, single gap universality, the edge universality [jointly with Bourgade, Erdos, Knowles, Yin].

1. average energy universality: vague convergence of correlation functions (i.e., in the a-variables of $\tilde{p}^{(2)}(E; a_1, a_2)$) and weak convergence in the energy (in the E-variable) . This is the topology used in the previous theorem.

2. fixed energy universality: vague convergence in the a-variables and pointwise convergence in the energy E.
Bourgade-Erdos-Y-Yin [2014]. General results for all symmetry classes.

Hermitian Wigner matrices by Johansson (with Gaussian component)
Erdos-Peche-Ramirez-Schlein-Y (for smooth Wigner matrices, 2009),
Tao-Vu, [2009-10] Wigner matrices with four moments matching those of $N(0,1)$ or other comparison ensembles, ...

3. single gap universality:weak convergence of the eigenvalue gap distribution [Erdos-Y, 2013, using parabolic regularity applied to DBM].

Theorem [Bourgade-Y, 2013] [Probabilistic local QUE]
$u_i, i = 1, \ldots, N$: normalized e-vectors of generalized Wigner matrices.
$\mathscr{N}_1, \mathscr{N}_2$: independent normal distribution $N(0,1)$.Then for any j, k in the bulk and $\mathbf{q}$ a unit vector in $\mathbb{R}^N$, we have

$$\sqrt{N}\Big(|\langle \mathbf{q}, u_j \rangle|,\ |\langle \mathbf{q}, u_k \rangle| \Big) \to \Big(|\mathscr{N}_1|,\ |\mathscr{N}_2| \Big)$$

Probabilistic local QUE holds: for any $k \in [\![1, \cdots, N]\!]$ and for any set $A \subset [\![1, \cdots, N]\!]$ with $|A| \geq N^{C\varepsilon}$, we have with high probability that

$$\Big| \frac{1}{|A|} \sum_{a \in A} \Big(N|u_k(a)|^2 - 1 \Big) \Big| \ll 1$$

Our results hold for Bernoulli random matrices and the extension to the Erdos-Renyi graphs also hold. This gives an example of prob-QUE for the Laplacian on a random graph.

Previous results: Delocalization [Erdos-Schlein-Y],

Knowles-Yin: Some version of local QUE holds for eigenvectors of Wigner matrices near the spectral edge. In the bulk, same property holds if the first four moments of the matrix elements matching those of the standard Gaussian.

In the bulk this was also proved by Tao-Vu under similar assumption (i..e, four or five moment matching).

Why study e vector distributions of Wigner matrices?

1. QUE for a random graph is an important model for Laplacian in a domain or in a random potential.

2. QUE is an important tool in the study of eigenvalue statistics of non-mean field matrix models such as band matrices.

3. E-vectors are important in statistical analysis of large data sets.

Matrix Dyson Brownian Motion (Matrix-DBM)

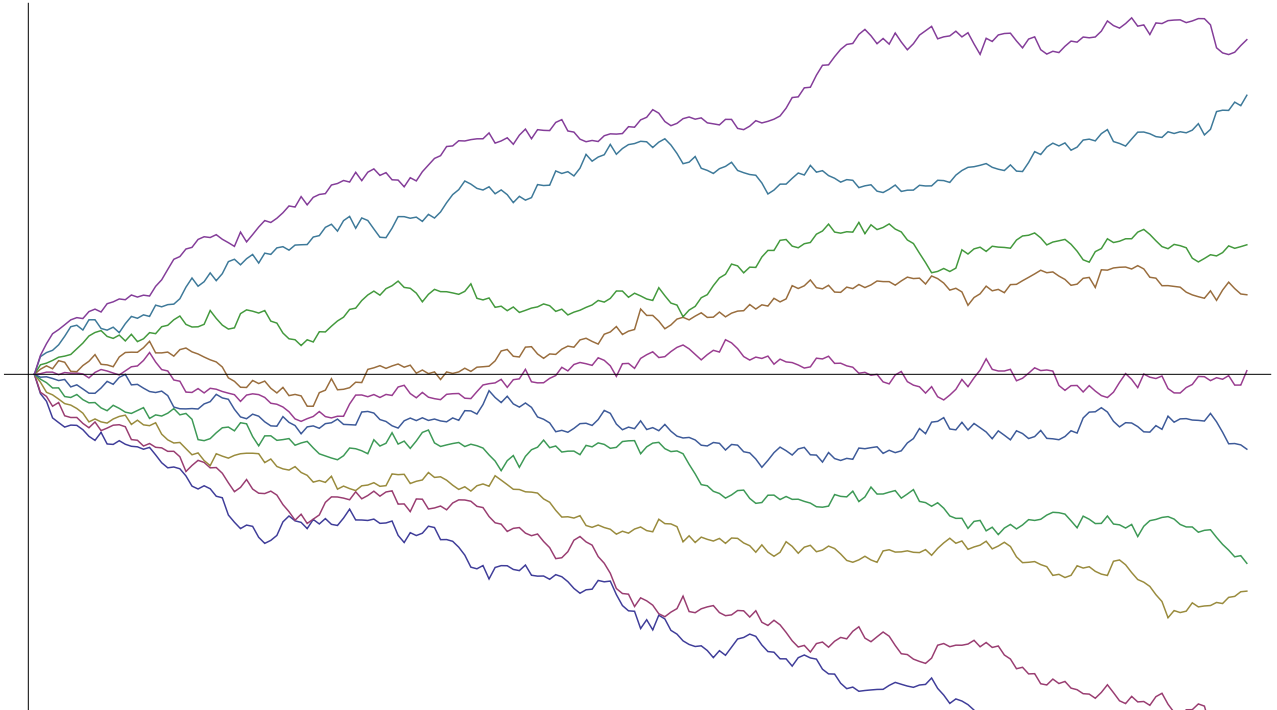
Evolve the matrix with a matrix Ornstein-Uhlenbeck process :

$$\mathrm{d}H_t = \frac{1}{\sqrt{N}}\mathrm{d}B_t - \frac{1}{2}H_t\mathrm{d}t \quad B_{ij} : \text{symm. indep. BM}$$

The distribution of $H_t \sim e^{-t/2}H_0 + \sqrt{1-e^{-t}}\,V$ where V is a GOE.

$$\mathrm{d}\lambda_k = \frac{\mathrm{d}B_k}{\sqrt{N}} + \left(\frac{1}{N}\sum_{\ell\neq k}\frac{1}{\lambda_k-\lambda_\ell} - \frac{1}{2}\lambda_k\right)\mathrm{d}t$$

Dyson Brownian Motion (1962) E-value equations are autonomous!



Dyson: The classical Coulomb gas is invariant under the DBM:

$$\mu_\beta \sim e^{-\beta N \mathcal{H}(\lambda)}, \quad \mathcal{H} = \sum_i \frac{\lambda_i^2}{4} - \frac{1}{N} \sum_{i<j} \log(\lambda_j - \lambda_i)$$

Prob. density for the classical ensembles with $\beta = 1$ for the GOE.

Dyson's conjecture: Time to "local equilibrium" for DBM is $\gtrsim N^{-1}$.

Erdős-Schlein-Y-Yin, 2009-10: Dyson conjecture holds for $\beta > 0$.

The Coulomb interactions drive the system locally to equilibrium very fast!

$\Longrightarrow$ Wigner-Dyson-Mehta conjecture holds for H_t with $t \gtrsim N^{-1}$.

Question: How to connect to $t = 0$?

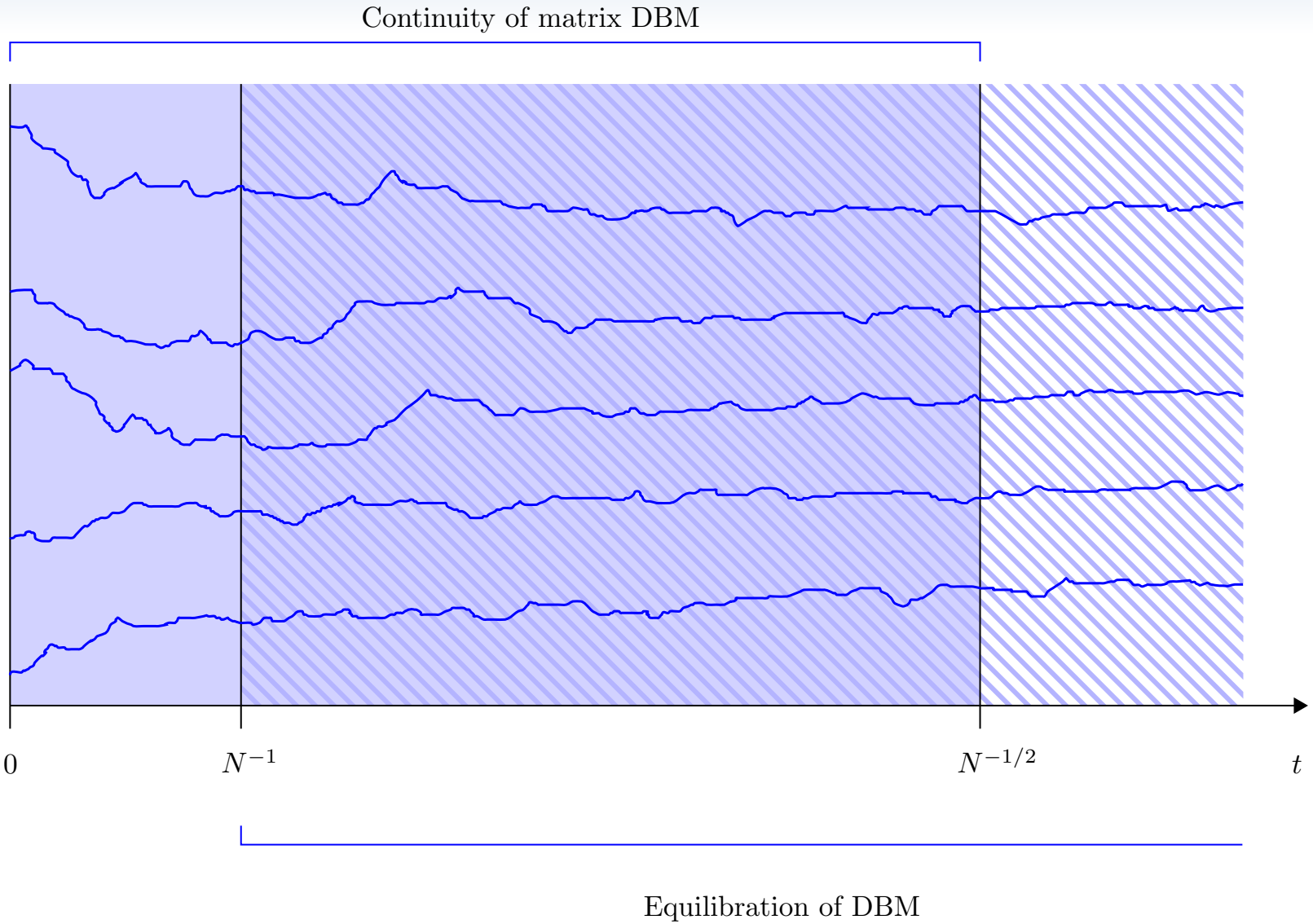
Theorem* [Continuity of matrix-DBM]
F: function of matrix elements with uniformly bounded 3rd derivatives. If $t \ll N^{-1/2}$ then

$$\mathbb{E}F\Big(H_t\Big) - \mathbb{E}F\Big(H_0\Big) \to 0$$

Proof: By Ito's formula and local semicircle laws.

Corollary: Continuity of eigenvalues and eigenfunctions for $t \ll N^{-1/2}$.

* [Bourgade-Y 2013 (see also Erdos-Knowles-Y-Yin)]



No contradiction since we use the matrix structure.

Dyson e-vector flow

$$\mathrm{d}H_t = \frac{1}{\sqrt{N}}\mathrm{d}B_t - \frac{1}{2}H_t\mathrm{d}t \quad B_{ij} : \text{symm. indep. BM}$$

E-vector equations depend on the e-values.

$$\mathrm{d}u_k = \frac{1}{\sqrt{N}}\sum_{\ell\neq k}\frac{u_\ell\,\mathrm{d}B_{k\ell}}{\lambda_k-\lambda_\ell} - \frac{1}{2N}\sum_{\ell\neq k}\frac{u_k\,\mathrm{d}t}{(\lambda_k-\lambda_\ell)^2}$$

[Bourgade-Y 2013] For any unit vector $\mathbf{q}$ fixed, define the conditional expectation given the eigenvalue trajectories

$$f(t,j) = f(t,j,\boldsymbol{\lambda}(\cdot)) = N\mathbb{E}\Big[|\mathbf{q}\cdot u_j(t)|^2\big|\boldsymbol{\lambda}(\cdot)\Big].$$

Then we have the eigenvector moment flow

$$\partial_t f(t,j) = N\sum_{k\neq j}\frac{f(t,k)-f(t,j)}{(\lambda_j-\lambda_k)^2(t)}$$

Random walk (in the index j) in random environments.
There are analogue equations for higher moments:

$$f(t,\mathbf{j}) = N^k\mathbb{E}\Big[|\mathbf{q}\cdot u_{j_1}(t)|^2\ldots|\mathbf{q}_k\cdot u_{j_k}(t)|^2\big|\boldsymbol{\lambda}(\cdot)\Big].$$

Key difficulty: The jump rate $\frac{1}{(\lambda_j-\lambda_k)^2(t)}$ can be very singular. We are in a situation that

$$\partial_t u(t,x) = \nabla[D(x,t)\nabla u(x,t)]$$

with D very singular (our dynamics is also nonlocal).

Theorem [Bourgade-Y 2013] Let H_t denote the DBM with a symmetric generalized Wigner matrix as the initial condition. Then for $t \geq N^{-1/4}$

$$|f(t,k) - 1| \leq C\,(Nt)^{-\varepsilon}.$$

Proof: By maximum principle and (isotropic) local semicircle law (i.e., semicircle law holds in small scales and in every direction).

Together with extension to higher moments $\Longrightarrow$

Corollary. Probabilistic local QUE holds for H_t; $N|\mathbf{q}\cdot u_j(t)|^2 \to$ Gaussian. By continuity of matrix-DBM $\Longrightarrow$ probabilistic local QUE holds for H_0.

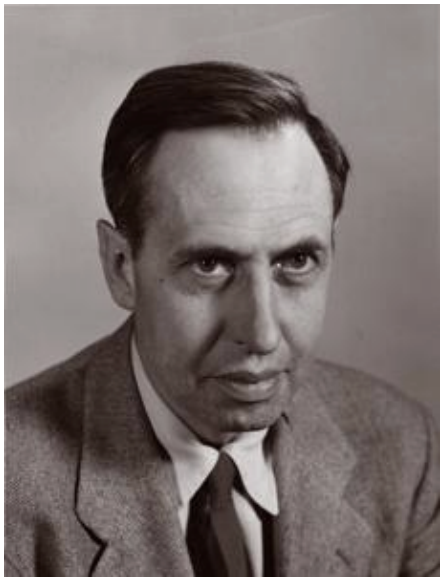
The underlying mechanism of the universality and QUE for random matrices: Dyson Brownian motion (A dynamical idea)

Universality: *Fast relaxation of DBM* + continuity of matrix-DBM

QUE: *Holder regularity of e-vector moment flow* + continuity of matrix-DBM

- Dynamical idea to solve time independent problems: *study the limits of the flow as* $t \to \infty$.
- Dyson Brownian Motion: *study the initial layer to understand* $t = 0$.

If you admit that the Wigner ensemble gives a completely wrong answer for the level density, why do you believe any of the other predictions of random-matrix theory?



George Uhlenbeck

- Local theory for universality was developed.
- Random matrix is a mean-field model. Going beyond mean-field is a key question. One key example: band matrices.

Beyond semicircle necessary law:

Two matrix Models: $A + B$. Example: A Wigner matrices and B diagonal random matrices, i.e., random potentials. One can also view B as signal and A as noise.

Ji Oon Lee, K. Schnelli: The eigenvalue density is no longer semicircle. It is given by a "free convolution" measure. Furthermore, both the bulk universality and edge universality hold.

Idea: 1. Bulk universality is based on estimating the relative entropy of the Dyson Brownian motion.

2 Edge universality can be proved by entropy method and also by comparison method.

3. Edge universality has important application in principal component analysis of large data problems (i.e., sample covariance matrices).