Obliquely Reflected Brownian motions (ORBMs) in Non-Smooth Domains

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Why Study Obliquely Reflected Diffusions?

Applications

- limits of interacting particle systems such as TASEP;
- diffusion approximations of stochastic networks;
- rank-dependent diffusion models (in finance);
- biological models of gene networks;

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Fundamental Mathematical Object

- a (non-symmetric) Markov process in a domain
- many basic questions are still not fully understood

Obliquely reflected Brownian motions (ORBMs)
in non-smooth domains:
rough planar domains

Motivation

- Diffusions in fractal domains have unusual and interesting properties: Goldstein ('87), Kusuoka ('87), Barlow-Perkins ('88)
- analysis on fractals (disordered media – physics and biology)

Challenge

Existing techniques for

- reflected diffusions in piecewise smooth domains
- normally reflected diffusions in fractal domains

are not applicable

Acknowledgments

Draws on various joint works with

W. Kang
UMBC
and
K. Burdzy, Z.-Q.-Chen and D. Marshall
University of Washington, Seattle

Outline of the Talk

1. Introduction to ORBMs and some existing tools.
2. ORBMs in piecewise smooth domains: an equivalence result
3. ORBMs in simply connected planar domains

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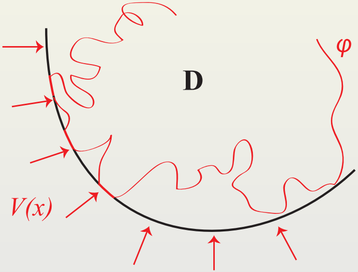
A Heuristic Description

Given a domain D with a vector field $v(\cdot)$ on the boundary ∂D, ORBM behaves infinitesimally like Brownian motion in the interior, is constrained to stay within the closure $\bar{D}$ of the domain and spends zero Lebesgue time on the boundary



Tools to study ORBMs in smooth domains

A. (Extended) Skorokhod problem (and stochastic differential equations with reflection – SDER)

B. Submartingale problem

The 1-dimensional Skorokhod Map $D = [0, \infty)$, $v(0) = e_1$

Definition (Skorokhod Problem (Skorokhod '61))

For every continuous $\mathbb{R}$-valued path ψ, find a continuous path ϕ s.t. $\forall t \geq 0$,

1. $\phi(t) \geq 0$ i.e., $\phi(t)$ lies in $[0, \infty)$
2. $\eta = \phi - \psi$ is non-decreasing
3. "η increases only when ϕ is on the boundary"

$$\int_0^\infty \phi(s) d\eta(s) = 0.$$

The Skorokhod Map Γ_0 on $[0, \infty)$

$$\phi = \psi + \eta \geq 0, \qquad \eta \text{ non-decreasing}, \qquad \int_0^\infty \phi(s) d\eta(s) = 0.$$

An explicit formula (Skorokhod '61)

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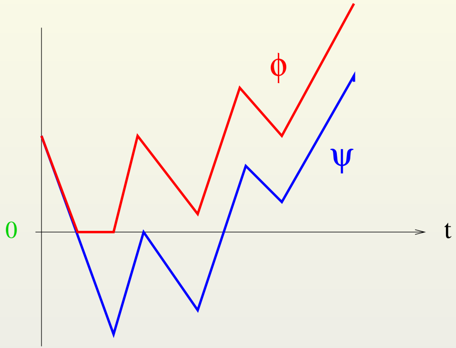
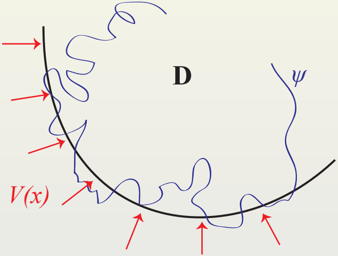


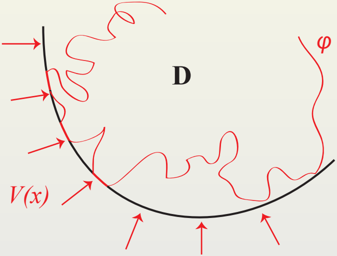
$$\phi(t) = \Gamma_0(\psi)(t) = \psi(t) + \sup_{s \in [0,t]} [-\psi(s)]^+$$

$Z = \Gamma_0(Z_0 + B)$ is *RBM* in 1-d

The Multidimensional Skorokhod Problem

Obtain reflected Brownian motion as a constrained version of Brownian motion





Formulation of the Skorokhod Map in $\mathbb{R}^d$

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$$\int_0^\infty \phi(s) d\eta(s) = 0.$$

1. Property 2 is equivalent to $\eta(t) - \eta(s) \geq 0$ for all $0 \leq s \leq t$;
2. Setting $v(x) = 0$ if $x > 0$, properties 2 and 3 are equivalent to

$$\eta(t) - \eta(s) \in \overline{\text{co}} \left(\cup_{u \in (s,t]} v(\phi(u)) \right),$$

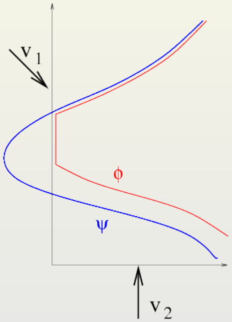
where, for $A \subset \mathbb{R}^d$, $\overline{\text{co}}[A]$ is the closure of the convex cone generated by the vectors in A.

The Multidimensional Skorokhod Problem

Obtain reflected Brownian motion as a constrained version of Brownian motion

Natural to consider piecewise smooth domains where v is multi-valued

$$v(0) = \{\alpha_1 v_1 + \alpha_2 v_2 : \alpha_1, \alpha_2 \geq 0\}$$



The Extended Skorokhod Map on $(D, v(\cdot))$

Extend v to $\bar{D}$ by setting $v(x) = 0$ for $x \in D$

Definition (Extended Skorokhod Problem ('R '06))

For every continuous $\mathbb{R}^d$-valued path ψ, find a continuous ϕ s.t. $\forall t \geq 0$,

1. $\phi(t) \in \overline{D}$;
2. $\eta = \phi - \psi$ satisfies for every $0 \leq s \leq t$,

$$\eta(t) - \eta(s) \in \overline{\text{co}} \left(\cup_{u \in (s,t]} v(\phi(u)) \right),$$

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Previous Formulations and Results

Tanaka ('79), Harrison-Reiman ('81), Lions-Sznitman ('84), Bernard El-Kharroubi ('91), Dupuis-Ishii ('91), Costantini ('92), Dupuis-Ramanan ('99), ...

The Extended Skorokhod Map on $(D, v(\cdot))$

- Previous formulations of the Skorokhod Map assumed η is of bounded variation;
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- The ESP formulation enabled one to construct solutions to SDER that are not necessarily semimartingales, thus extending the applicability of the SDER approach.
- The ESP can be used to construct both strong and weak solutions to the associated SDER.

B. The submartingale problem

The Submartingale Problem (Stroock-Varadhan '71)

Given $(D, v(\cdot))$, b, σ, find probability measures $\mathbb{Q}_z$, $z \in \bar{D}$, on $\mathcal{C}([0,\infty) : \mathbb{R}^n)$ such that

- For every $z \in \bar{D}$, $\mathbb{Q}_z(w(0) = z) = 1$;
- Under each $\mathbb{Q}_z$,

$$M_t^f \doteq f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\, ds$$

is a submartingale for all $f \in \mathcal{H}_0$, where

$$\mathcal{H}_0 = \{f \in \mathcal{C}_b^2(D) : \langle \nabla f(x), v(x) \rangle \geq 0\}$$

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Well-posedness of the submartingale problem

The submartingale problem is said to be well posed if there exists a solution to the submartingale problem and it is unique.

Results on the Submartingale Problem

1. Stroock and Varadhan (1971) established well-posedness of the submartingale problem for bounded $\mathcal{C}^2$ domains with Lipschitz continuous reflection field v satisfying $|\nabla v| \geq 1$.
2. Extended to specific non-smooth domains in Varadhan-Williams (1985); Williams (1987); Deblassie (1987, 1996); Deblassie-Toby (1993), ...

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2. Extended to specific non-smooth domains in Varadhan-Williams (1985); Williams (1987); Deblassie (1987, 1996); Deblassie-Toby (1993), ...
3. A general multi-dimensional formulation was not available ... cited as an open problem (Williams 1995, DeBlassie 1997)

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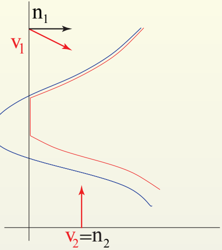
2. ORBMs in piecewise smooth domains: an equivalence result
3. ORBMs in simply connected planar domains

2. ORBMs in Piecewise-smooth Domains



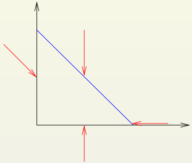
- The direction vector field $v(\cdot)$ can be multi-valued.
- Consider domains of the form $D = \cap_i D_i$, with each D_i a $\mathcal{C}^2$ domain having a continuous vector field v_i, and v on the intersections of multiple smooth boundaries is defined as

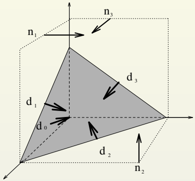
$$v(x) = \left\{ \sum_{i \in I(x)} \alpha_i v_i(x), \alpha_i \geq 0 \right\},$$

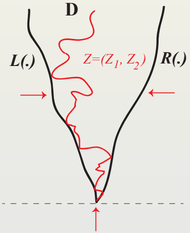
Challenging Aspects of Piecewise smooth Domains

A set $\mathcal{V}$ of irregular points where v contains a half-plane

$$\mathcal{V} = \partial D \setminus \{x \in \partial D : \exists n \in n(x) \text{ such that } \langle n, v \rangle > 0, \forall v \in v(x)\}$$







Submartingale Problem for Piecewise smooth Domains

Submartingale Formulation (Kang-'R, '12)

Given $(D, v(\cdot))$, b, σ, find probability measures $\mathbb{Q}_z$, $z \in \bar{D}$, on $\mathcal{C}([0,\infty) : \mathbb{R}^n)$ such that

- $\mathbb{Q}_z(\omega(0) = 0) = 1$
- $$M_t^f \doteq f(X_t) - \int_0^t \mathcal{L}f(X_s)\,ds, \quad t \geq 0,$$

  is a $\mathbb{Q}_z$-submartingale for all $f \in \mathcal{H}$:

  $$\mathcal{H} \doteq \left\{ \begin{array}{ll} f \in \mathbb{C}_c^2(\bar{D}) \oplus \mathbb{R} : & f \text{ is constant in a neighborhood of } \mathcal{V}, \\ & \langle v, \nabla f(y) \rangle \geq 0 \text{ for } v \in v(y) \text{ and } y \in \partial D \end{array} \right\}$$

- For every $z \in \bar{D}$, $\mathbb{Q}_z$-almost surely,

  $$\mathcal{L}eb\{s \in [0,\infty) : \omega(s) \in \mathcal{V}\} = 0.$$

An Equivalence Theorem

- Stroock and Varadhan (1969) introduced the martingale problem for diffusions and showed, under general conditions on b and σ that it was equivalent to the SDE formulation
- Reflected diffusions can also be defined as solutions to SDERs using the extended Skorokhod map Γ
- Is there a similar equivalence between SDERs and the submartingale formulation here?

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Theorem (Kang-'R '13)

When the set $\mathcal{V}$ is finite, D is piecewise $\mathcal{C}^2$ and v is $\mathcal{C}^1$, b bounded and measurable and σ continuous, then well-posedness of submartingale formulation is equivalent to existence and uniqueness in law of weak solutions to SDERs

Sketch of Proof: Martingale Prob.

Recall the case of the martingale problem for (unconstrained) diffusions

- To show that a weak solution satisfies the martingale problem is a simple application of Itô's formula
- Main thrust: need to construct a weak solution from a solution to the martingale problem
- Use test functions $f(x) = x_i$, $f(x) = x_i x_j$ to show that

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \langle \nabla f(X_s), b(X_s) \rangle \, ds$$

is a martingale, then use the martingale representation theorem to show that there exists a suitable Brownian motion such that

$$M_t^f = \int_0^t \sigma(s) dB_s,$$

Sketch of Proof: Submartingale Prob. ($\mathcal{V} = \emptyset$)

- Again, to show that a weak solution satisfies the submartingale problem is a simple application of Itô's formula and the thrust is to construct a weak solution from the submartingale prob.
- Choice of test functions limited by derivative conditions; their construction depends on geometry of domain, especially at intersections of faces;
- Test functions yield submartingales, not martingales

$$S_t^f = f(X_t) - f(X_0) - \int_0^t \langle \nabla f(X_s), b(X_s) \rangle \, ds$$

- Applying the Doob-Meyer decomposition, $S^f(t) = M^f(t) + A^f(t)$, where M^f is a local martingale and A^f is a BV process
- Need to identify M^f as a stochastic integral and A^f as the "boundary local time" process
- this requires the study of the behavior of the quadratic variation of M^f on on ∂D, the boundary of the domain

Comments on the Proof

- The case $\mathcal{V} = \emptyset$ entails yet more analysis
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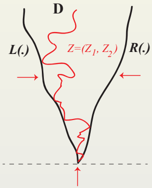
- The case $\mathcal{V} = \emptyset$ entails yet more analysis
- In summary, proof much more subtle than the martingale prob. case
- In fact, the equivalence fails if $\mathcal{V}$ is not a finite set

$$D = \{y \in \mathbb{R}^3 : y_2 \geq 0, L(y) \leq y_2 \leq R(y)\}$$



- Suggests that additional conditions may need to be imposed near $\mathcal{V}$ ...
- Are there a natural set of conditions ?

Other Related Questions

- When does the ORBM have the semimartingale property?
  Taylor-Williams (PTRF ’93); ’R (EJP, ’06), Kang-’R (AoP, ’10),
  Burdzy-Kang-’R (SPA, ’09)
- characterization of stationary distributions
  Harrison-Williams (AoP ’87), Kang-’R, AAP ’14)
- flow properties for ORBM
  S. Andres (AIHP ’09), Mandelbaum-’R, (’06, ’10),
  Lipshutz-’R (forthcoming)
- ...

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How does one define even normally reflected BMs in “rough” domains?

Challenge

No way to make sense of the normal vector field

Dirichlet form approach to construct process X

Setup:

- E Hausdorff topological space, a Borel σ-field $\mathcal{B}(E)$, a σ-finite Borel measure m;

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- A pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ that satisfies suitable conditions (closed, coercive biliear form with a (unit) contraction property)

$$\mathcal{E}(u,u) = \lim_{t \downarrow 0} \frac{1}{2t} \int_E \mathbb{E}_z \left[(u(X_t) - u(X_0))^2 \right] m(dz).$$

$$(\mathcal{E}, \mathcal{D}(\mathcal{E})) \mapsto (T_t) \mapsto (p_t) \mapsto \{X_t\}_{t \geq 0}$$

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Results using the Dirichlet Form Approach

- Beurling and Deny (1959)
- Silverstein and Fukushima (1970s)
- Fukushima ('80s): If a Dirichlet form on a locally compact state space is symmetric and regular, one can construct an associated Markov process (Hunt process) with RCLL paths.

Normal RBMs in Fractal Domains

- Normal RBMs are symmetric Markov processes (Fukushima '80; Fukushima, Oshima, Takeda '94)
- Dirichlet form techniques have been successively used to construct normally reflected Brownian motions in quite general domains
  (Fukushima '67, Bass-Hsu '90, Williams-Zheng '90, Z.-Q.-Chen '93, Chen-Fukushima '12)

ORBMs in Fractal Domains

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- Ma and Röckner ('92): established a more general result relating (non-symmetric) Dirichlet forms with Markov processes;
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- Ma and Röckner ('92): established a more general result relating (non-symmetric) Dirichlet forms with Markov processes;
- Generalized Dirichlet forms (Stannat '99, Trutnau '03, ...)
- However, not much success so far with ORBMs (except for those that are obtained as perturbations of normal RBMs)

Reboot: ORBMs in Non-smooth Planar Domains

Challenges

- The normal and tangential vector fields are not well defined in the classical sense
- ORBM is not a symmetric Markov process
- A new approach is required ...

ORBMs in Planar Domains: A Useful Parametrization

ORBMs in Smooth Planar domains

- Parametrize ORBMs in smooth domains by “angle of reflection”
- Let B be standard two-dimensional Brownian motion
- Given $D \subset \mathbb{C}$ a smooth bounded open set and $\theta : \partial D \mapsto (-\pi/2, \pi/2)$ Borel measurable function satisfying $\sup_{x \in \partial D} |\theta(x)| < \pi/2$.
- $\mathbf{n}(x)$: unit inward normal vector at $x \in \partial D$
- $\mathbf{t}(x)$: unit tangent vector to ∂D at x
- Vector field v_θ on ∂D associated with θ:

$$\mathbf{v}_\theta(x) = \mathbf{n}(x) + \tan\theta(x)\mathbf{t}(x)$$

- Parametrize vector field v_θ in terms of the angle of reflection θ

ORBMs in Smooth Planar domains

- Recall, given $\theta : \partial D \mapsto (-\pi/2, \pi/2)$,

$$\mathbf{v}_\theta(x) = \mathbf{n}(x) + \tan\theta(x)\mathbf{t}(x)$$

- When θ is $\mathcal{C}^2$, D smooth, Skorokhod Map Γ is well defined; RBM

$$Z = \Gamma(Z_0 + B)$$

or, equivalently, letting L denote the local time of Z on ∂D:

$$Z_t = Z_0 + B_t + \int_0^t \mathbf{v}_\theta(Z_s)\,dL_s,$$

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or, equivalently, letting L denote the local time of Z on ∂D:

$$Z_t = Z_0 + B_t + \int_0^t \mathbf{v}_\theta(Z_s) dL_s,$$

- When $D = D_*$, strong solution exists for θ in

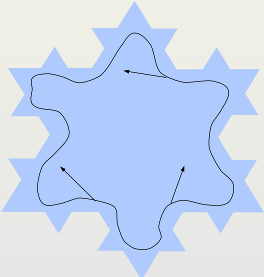
$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2,\ \theta \not\equiv \pi/2,\ \text{and}\ \ \theta \not\equiv -\pi/2\}.$$

A. Domain Approximation

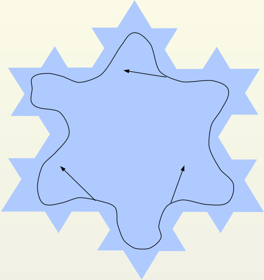
- Given a simply connected Jordan domain D, $y_0 \in D$, approximate it by a sequence of smooth domains D^k in the sense that for all k,

$$y_0 \in D^k \subset D^{k+1} \subset D, \text{ and } \bigcup_k D^k = D$$

- For each k consider a smooth vector field θ^k and let Z^k be ORBM associated with (D^k, θ^k).



ORBMs in Planar Domains: A. Domain Approximation



- When does Z^k converge to some limit process Z, and in what sense ?
- Is the “limit” Z an ORBM in D in any reasonable sense?
- Is there an independent characterization of the limit ORBM ?

Possible Constructions: B. Conformal Mappings

- Let D_* denote the unit disc in the plane

Possible Constructions: B. Conformal Mappings

- Let D_* denote the unit disc in the plane
- Let θ belong to the fairly general class of vector fields

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2, \ \theta \not\equiv \pi/2, \text{ and } \ \theta \not\equiv -\pi/2\}.$$

- Let Z be a $(D_*, \mathbf{v}_\theta)$ ORBM.
- Let $D \subset \mathbb{C}$ be a simply connected bounded domain
- Let $f : D_* \mapsto D$ be a one-to-one onto analytical function.

Possible Constructions: B. Conformal Mappings

- Let D_* denote the unit disc in the plane
- Let θ belong to the fairly general class of vector fields

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2, \ \theta \not\equiv \pi/2, \text{ and } \ \theta \not\equiv -\pi/2\}.$$

- Let Z be a $(D_*, \mathbf{v}_\theta)$ ORBM.
- Let $D \subset \mathbb{C}$ be a simply connected bounded domain
- Let $f : D_* \mapsto D$ be a one-to-one onto analytical function.
- Define $Y(t) = f(Z_{c^{-1}(t)})$ for $t \in [0, \zeta)$, where

$$c(t) = \int_0^t |f'(Z_s)|^2 ds, \qquad \text{for } t \geq 0,$$
$$\zeta = \inf\{t \geq 0 : c(t) = \infty\},$$

- Then Y is an extension of killed Brownian motion in D, i.e., for every $t \geq 0$ and $\tau_t = \inf\{s \geq t : Y_s \in \partial D\}$, the process $\{Y_s, s \in [t, \tau_t)\}$ is Brownian motion killed upon exiting D.
- Is Y an ORBM in a suitable sense?

Results: I. Alternative Parametrization on D_*

D_* – unit disc in $\mathbb{R}^2$
$\theta(x)$ – angle of reflection at $x \in \partial D$
Z – associated RBM

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2, \ \theta \not\equiv \pi/2, \ \text{and} \ \theta \not\equiv -\pi/2\}.$$

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$$\theta \in \mathcal{T} \leftrightarrow (h, \mu_0) \in \mathcal{H} \times (-\infty, \infty)$$

$$\mathcal{H} = \{h \text{ harmonic in and strictly positive in } D_*, ||h|| = \pi h(0)\}$$

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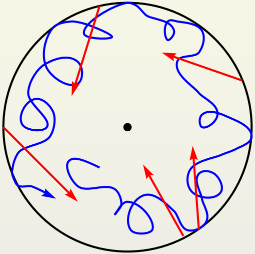
$h(x)dx$ – stationary distribution
μ_0 – "rate of rotation" of Z around zero

Rate of Rotation μ_0

D_* – unit disc in $\mathbb{R}^2$, $\theta(x)$ – angle of reflection at $x \in \partial D$

$$\theta \leftrightarrow (h, \mu_0)$$

$h(x)dx$ – stationary distribution μ_0 – rate of rotation around zero



$$\lim_{t\to\infty} \frac{1}{t} \arg X_t - \mu_0 \Rightarrow \text{Cauchy}.$$

$$\lim_{t\to\infty} \frac{1}{t} \arg^* X_t = \mu_0$$

Results: Alternative Parametrization of ORBMs in D_*

$$\theta \leftrightarrow (h, \mu_0)$$

The correspondence has quite an explicit form

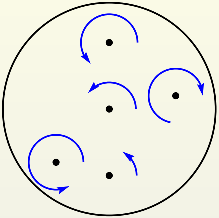
THEOREM (forthcoming; Burdzy, Chen, Marshall, 'R)

$$h(z) = \frac{\mathrm{Re}\,\exp(\widetilde{\theta}(z) - i\theta(z))}{\pi \mathrm{Re}\left(e^{-i\theta(0)}\right)} = \frac{\mathrm{Re}\,\exp(\widetilde{\theta}(z) - i\theta(z))}{\pi \cos\theta(0)}, \qquad z \in D,$$

$$\mu_0 = \tan\theta(0) = \int_D \tan\theta(z) h(z) dz,$$

$$\theta(z) = -\arg\left(h(z) + i\widetilde{h}(z) - i\mu/\pi\right), \qquad z \in D.$$

- We can also parameterize the ORBM in terms of “rotation rates”



- On D_*, this correspondence takes the following form

$$\theta \in \mathcal{T} \leftrightarrow \mu(\cdot) \in \mathcal{R},$$

$$\mathcal{R} = \{\mu : \mu \text{ is harmonic in } D_* \text{ and } \tilde{\mu}(z) > -1, \quad \text{for all } z \in D_*\}.$$

- Again, $\mu(z)$ can be written quite explicitly in terms of θ.
- Probabilistic interpretation of $\mu(z)$:

$$\lim_{t\to\infty} \arg^* (X_t - z)/t = \mu(z).$$

Results: II. ORBMs on more general D

Given simply connected bounded domain D, $\theta \in \mathcal{T}$, conformal mapping $f : D_* \mapsto D$, Z a $(D_*, \mathbf{v}_\theta)$ ORBM, set $Y(t) = f(Z_{c^{-1}(t)})$, where

$$c(t) = \int_0^t |f'(Z_s)|^2 ds, \qquad \text{for } t \geq 0,$$
$$\zeta = \inf\{t \geq 0 : c(t) = \infty\},$$

Theorem (Part 1: forthcoming; Burdzy, Chen, Marshall, 'R)

- *If $\theta \leftrightarrow (h, \mu_0) \leftrightarrow \mu$, and $h \circ f^{-1}$ is in $\mathbb{L}^1(D)$, then $\zeta = \infty$, Y is an extension of a killed RBM on D, Y has a stationary density*

$$\widehat{h} = h \circ f^{-1} / ||h \circ f^{-1}||_1^D,$$

and

$$\lim_{t \to \infty} \arg^* \frac{Y_t - z}{t} = \frac{\mu(f^{-1}(z))}{||h \circ f^{-1}||_1^D}.$$

Results: II. ORBMs on more general D

Theorem (Part 2: forthcoming; Burdzy, Chen, Marshall, 'R)

- *For any $\mu_0 \in \mathbb{R}$ and $\widehat{h}$ a positive harmonic function in D with $||\widehat{h}||_1 = 1$, there exists an ORBM Y in D such that Y has stationary distribution $\widehat{h}$ and*

$$\lim_{t\to\infty} \arg^* \frac{Y_t - z}{t} = \frac{\mu(f^{-1}(z))}{||h \circ f^{-1}||_1^D}.$$

holds with $\mu(\cdot) \leftrightarrow (\mu_0, h)$.

Thus, ORBMs in D can be parametrized either in terms of pairs (θ, f) or triplets $(\widehat{h}, \mu_0, f)$:

$$Y \leftrightarrow (\theta, f) \leftrightarrow (\widehat{h}, \mu_0, f).$$

Roughly, suppose D_k approximate D, with $y_0 \in D_k \subset D$, and Y is an ORBM in D associated with (θ, f).

Theorem (forthcoming; Burdzy, Chen, Marshall, 'R)

Let Y^k be the ORBM in D_k such that $Y_k \leftrightarrow (\theta_k, f_k)$ and $Y_0^k = y_0$. Then $\theta_k \to \theta$ and $f_k \to f$ (by assumption) and Y_k's converge weakly to $Y \leftrightarrow (\theta, f)$.

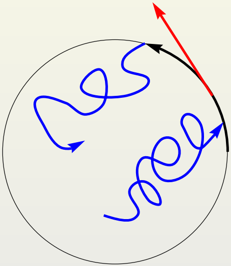
Results: III Consistency with Domain Approximations

Precise Setup:

$D \subset \mathbb{C}$ – open simply connected Jordan domain, $y_0 \in D$
$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries
$f_k : D_* \mapsto D$ conformal maps such that $f_k \to f$ as $k \to \infty$ and
$f_k^{-1}(y_0) = f^{-1}(y_0)$
$\theta : \partial D \mapsto (-\pi/2, -\pi/2)$ continuous
$\theta_* = \theta \circ f$ and Y ORBM
$\theta_* : \partial D_* \mapsto (-\pi/2, -\pi/2)$
Y ORBM in D such that $Y \leftrightarrow (\theta_*, f)$ and $Y_0 = y_0$
For every k, let $g_k : \partial D_k \mapsto \partial D$ be a measurable function such that for every $x \in \partial D_K$, $g_k = y \in \partial D$, and $|x - y| = \text{dist}(x, \partial D)$. Let $\theta_k(x) = \theta(g_k(x))$ for $x \in \partial D_k$.

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2,\ \theta \not\equiv \pi/2,\ \text{and}\ \ \theta \not\equiv -\pi/2\}.$$

- $\theta_k \in \mathcal{T}$ converges to θ in the weak-* toplogy (as elements of the dual space of $\mathbb{L}^1(\partial D_*)$).
- Limit process could have jumps



- So convergence of Z^k to Z is (in general) in a certain M_1 topology

Jumps on the boundary when $\theta(x) = \pi/2$

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : ||\theta||_\infty \leq \pi/2,\ \theta \not\equiv \pi/2,\ \text{and}\ \ \theta \not\equiv -\pi/2\}.$$

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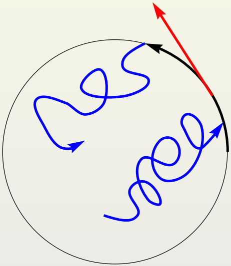

- So convergence of Z^k to Z is (in general) in a certain M_1 topology
- Limit could be "excursion reflected Brownian motion (ERBM)" when limit $\theta \equiv \pi/2$

Summary

- Reflected Diffusions in piecewise-smooth domains arise in a variety of fields ranging from math physics and finance to queueing theory.
- Established an equivalence result between submartingale characterization and weak solution characterization
- Several foundational questions remain even for RBMs in polyhedral domains

Summary

- Reflected Diffusions in piecewise-smooth domains arise in a variety of fields ranging from math physics and finance to queueing theory.
- Established an equivalence result between submartingale characterization and weak solution characterization
- Several foundational questions remain even for RBMs in polyhedral domains
- A new paradigm has been developed for characterization of ORBMs in bounded planar domains (in terms of triplets of harmonic functions, real numbers and conformal maps) including some ORBMs with jumps (excursion-reflected Brownian motions)
- Further interesting questions:
  - Is there a purely analytical interpretation of “rate of rotation” $\mu(\cdot)$?
  - Do these ORBMs arise as scaling limits of random walks?
  - Construction of ORBMs in more general (multiply connected) planar domains as well as multi-dimensional domains

List of Some Relevant Works

- K. Burdzy, Z.-Q. Chen, D. Marshall and K. Ramanan, "Obliquely reflected Brownian motions in non-smooth planar domains," forthcoming, 2014.
- W. Kang and K. Ramanan, "On the Submartingale Problem for reflected diffusions in piecewise smooth domains", Preprint, 2014.
- W. Kang and K. Ramanan, "Characterizations of stationary distributions of reflected diffusions," Ann. Appl. Probab., 2014.
- W.N. Kang and K. Ramanan, "A Dirichlet process characterization of a class of reflected diffusions," Ann. Probab., **38** (2010) 1062–1105.
- K. Burdzy, W.N. Kang and K. Ramanan, "The Skorokhod map in a time-dependent interval," Stoch. Proc. Appl., **119** (2009) 428–452.
- K. Ramanan. "Reflected diffusions defined via the extended Skorokhod map." Elec. Jour. Probab., **11** (2006), 934–992.