Global heat kernel estimates for symmetric Markov processes dominated by stable-like processes in exterior $C^{1,\eta}$ open sets

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Outline

1. Introduction
2. Heat Kernel Estimates for Z
3. Dirichlet Heat Kernel Estimates for Z^D
4. Main Result

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2 Heat Kernel Estimates for Z

3 Dirichlet Heat Kernel Estimates for Z^D

4 Main Result

Background

For a large class of elliptic differential operators $\mathcal{L}$, there is a diffusion process X associated with it so that $\mathcal{L}$ is the infinitesimal generator of X, i.e.,

$$\mathcal{L}f(x) = \lim_{t \downarrow 0} t^{-1}(\mathbb{E}_x[f(X_t)] - f(x)).$$

Analytic point of view

$p(t,x,y)$ is called the heat kernel for $\mathcal{L}$ if

$$u(t,x) := \int_{\mathbb{R}^d} p(t,x,y) f(y) dy$$

is the solution to

$$\mathcal{L}u = \partial_t u, \quad u(0,x) = f(x).$$

Probabilistic point of view

$p(t,x,y)$ is the transition density function of X. i.e.,

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Discontinuous Markov Process

Such relation is also true for Markov processes with discontinuous sample paths and the infinitesimal generator is no longer a differential operator but rather a non-local (or, integro-differential) operator .

For example, the infinitesimal generator of symmetric α-stable process is the fractional Laplacian $-(-\Delta)^{\alpha}$. It is well known that on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$

$$p(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}.$$

Recently two-sided heat kernel estimates for Markov processes (in $\mathbb{R}^d$) have been studied a lot. Kolokoltsov (00), Bass & Levin (02), Chen & Kumagai (03, 08, 10), Bogdan & Jakubowski (07), Kim & Song (13), Song & Vondraček (07), Chen, Kim & Kumagai (08), Sztonyk (10, 11)

In particular, Chen, Kim & Kumagai (11) cover a large class of pure jump symmetric Markov process including $J(x,y) \asymp e^{-c|x-y|^{\beta}}|x-y|^{-d-\alpha}$ for $\beta\in(0,\infty)$.

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Dirichlet Heat Kernel Estimates of Discontinous Markov Process

Heat kernel estimates in subsets of $\mathbb{R}^d$ are much more difficult to obtain.

Let Y be a discontinuous Markov process in $\mathbb{R}^d$ whose infinitesimal generator is $\mathcal{L}$ and we assume $\mathbb{P}_x(Y_t \in dy)$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^d$.

For any open subset $D \subset \mathbb{R}^d$, let Y^D be a subprocess of Y killed upon leaving D and $p_D(t,x,y)$ be a transition density of Y^D. $p_D(t,x,y)$ describes the distribution of Y^D: i.e., $\mathbb{P}_x(Y_t^D \in A) = \int_A p_D(t,x,y)dy$.

An infinitesimal generator $\mathcal{L}|_D$ of Y^D is the infinitesimal generator $\mathcal{L}$ with zero exterior condition. $p_D(t,x,y)$ is also called the Dirichlet heat kernel for $\mathcal{L}|_D$ since $u(t,x) := \int_{\mathbb{R}^d} p_D(t,x,y)f(y)dy$ is the solution to exterior Dirichlet problem: $\mathcal{L}u(t,\cdot) = \partial_t u(t,\cdot), u(0,\cdot) = f(\cdot)$ on D and $u = 0$ on D^c.

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Thus, obtaining sharp two-sided estimates of $p_D(t, x, y)$ is a fundamental problem in both analysis and probability theory.

DHKE for disontinou Lévy process in open set

Chen, Kim, and Song (10)
DHKE for α-stable process in $C^{1,1}$ open sets.

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Extension

- Extension to a large class of discontinuous Lévy Process

  Chen, Kim and Song (14): A Varopoulos type factorization estimate for a large class of purely discontinuous SBM in non-smooth open sets and DHKE of SBM in $C^{1,1}$ open sets.

  Chen, Kim and Song (to appear) :DHKE for SBM with Gaussian Components in $C^{1,1}$ open sets .

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4 Main Result

Setup

- Let ψ be an increasing function on $[0,\infty)$ with $\psi(r)=1$ for $0<r\le 1$ and there exist constants $\gamma_2\ge\gamma_1>0$ so that for every $\beta\in[0,\infty]$

$$c_1\exp(\gamma_1 r^{\beta})\le\psi(r)\le c_2\exp(\gamma_2 r^{\beta}) \qquad \text{for } 1<r<\infty.$$

- Let J be a symmetric measurable function on $\mathbb{R}^d\times\mathbb{R}^d\setminus\{x=y\}$ defined by

$$J(x,y):=\begin{cases}\kappa(x,y)\psi(|x-y|)^{-1}|x-y|^{-d-\alpha} & \text{when } \beta\in[0,\infty)\\ \kappa(x,y)|x-y|^{-d-\alpha}\mathbf{1}_{\{|x-y|\le 1\}} & \text{when } \beta=\infty\end{cases}$$

where $\kappa(x,y)$ is symmetric,

$$\kappa_1^{-1}\le\kappa(x,y)\le\kappa_1$$

and

$$|\kappa(x,y)-\kappa(x,x)|\le\kappa_2|x-y|^{\rho},\quad \forall|x-y|<1$$

for some positive constants $\kappa_1,\kappa_2>0$ and $\rho>\alpha/2$.

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- For any $u \in L^2(\mathbb{R}^d, dx)$ and $a > 0$, define

$$\mathcal{E}(u,u) := \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 J(x,y) dxdy$$

$$\mathcal{D}(\mathcal{E}) := \{u \in C_c(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\} \qquad \text{and}$$

$$\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \int_{\mathbb{R}^d} u(x)^2 dx.$$

- By Chen & Kumagai (08) [Proposition 2.2], $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$, where $\mathcal{F} := \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$ (i.e., $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $\mathcal{F}$ with the norm $\mathcal{E}_1$ and $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $C_c(\mathbb{R}^d)$ with the uniform norm).

- So there exists a Hunt process Z associated with it on $\mathbb{R}^d$. (See Fukushima, Oshima & Takeda (94)) Also, Z is a subclass of the processes considered in Chen, Kim & Kumagai (11).

  Thus it has a continuous transition density $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

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- By Chen & Kumagai (08) [Proposition 2.2], $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$, where $\mathcal{F} := \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$ (i.e., $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $\mathcal{F}$ with the norm $\mathcal{E}_1$ and $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $C_c(\mathbb{R}^d)$ with the uniform norm).

- So there exists a Hunt process Z associated with it on $\mathbb{R}^d$. (See Fukushima, Oshima & Takeda (94)) Also, Z is a subclass of the processes considered in Chen, Kim & Kumagai (11).

  Thus it has a continuous transition density $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

Heat Kernel Estimates of Z

Let $a \wedge b := \min\{a, b\}$.
For constants $a, b, T > 0$, define functions on $(0, T] \times [0, \infty)$ as

$$\Psi^1_{a,b,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge tr^{-d-\alpha} e^{-br^\beta} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge tr^{-d-\alpha} & \text{if } \beta \in (1,\infty] \text{ with } r < 1, \\ t \exp\left(-a\left(r\left(\log \frac{Tr}{t}\right)^{\frac{\beta-1}{\beta}} \wedge r^\beta\right)\right) & \text{if } \beta \in (1,\infty) \text{ with } r \geq 1, \\ (t/(Tr))^{ar} & \text{if } \beta = \infty \text{ with } r \geq 1. \end{cases}$$

and on $[T, \infty) \times (0, \infty)$ as

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There are positive constants $c_1, c_2 \geq 1$ and $\gamma = \gamma(\gamma_1, \gamma_2) \geq 1$ such that

$$c_1^{-1}\Psi^1_{c_2,\gamma,T}(t,|x-y|) \leq p(t,x,y) \leq c_1\,\Psi^1_{c_2^{-1},\gamma^{-1},T}(t,|x-y|) \tag{1}$$

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This is mainly from Chen, Kim and Kumagai(11). But when $\beta \in [0,1]$, the upper bound of (1) is proved by Kateta & Sztonyk(12) and when $\beta \in [0,1]$ or $\beta \in (1,\infty)$ with $|x-y| \geq 1$, the lower bounds in (1) are proven by K and Kim(14).

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Outline

1 Introduction

2 Heat Kernel Estimates for Z

3 Dirichlet Heat Kernel Estimates for Z^D

4 Main Result

Assumptions on Open Sets

- An open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a $C^{1,\eta}$ open set for $\eta \in (0,1]$: if there exists a localization radius $r_0 > 0$ and a constant $\Lambda_0 > 0$ such that for any $z \in \partial D$, there exists a $C^{1,\eta}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$\phi(0) = 0, \nabla\phi(0) = (0, \cdots, 0), \ \|\nabla\phi\|_\infty \leq \Lambda_0,$$

$$|\nabla\phi(x) - \nabla\phi(y)| \leq \Lambda_0 |x - y|^\eta$$

and there exists an orthonormal coordinate system CS_z of $z = (z_1, \cdots, z_{d-1}, z_d) := (\widetilde{z}, y_d)$ such that

$$B(z, r_0) \cap D = \{y = (\widetilde{y}, y_d) \in B(z, r_0) \text{ in } CS_z : y_d > \phi(\widetilde{y})\}.$$

The pair (r_0, Λ_0) is called the characteristic of the $C^{1,\eta}$ open set D.

- The path distance in each component of D is comparable to the Euclidean distance with characteristic λ_1: if for any x, y in the same component of D, there exists a curve l in D connecting x to y such that $|l| \leq \lambda_1 |x - y|$.

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Dirichlet Heat Kernel Estimates of Z^D in $C^{1,\eta}$ Open Set

Let $\alpha \in (0,2)$ and $\eta \in (\alpha/2, 1]$. Suppose that D is a $C^{1,\eta}$ open set in $\mathbb{R}^d$.

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Denote by $\delta_D(x)$ the Euclidean distance between x and D^c.

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K and Kim (14)

For any $T > 0$, on $(0, T] \times D \times D$,

$$p_D(t,x,y) \le c_1 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\Psi^1_{c_2^{-1},\gamma^{-1},T}(t,|x-y|/6)$$

and in addition to D is a connected,

$$p_D(t,x,y) \ge c_1^{-1} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\Psi^1_{c_2,\gamma,T}(t,5|x-y|/4).$$

Suppose in addition that D is bounded and connected. For any $T > 0$, on $[T, \infty) \times D \times D$,

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Motivation

K and Kim (14) give us the Dirichlet heat kernel estimates for the small time.

However the large time estimates are established only for the bounded and connected $C^{1,\eta}$ open sets. The large time Dirichlet heat kernel estimates for unbounded open sets are different depending on the geometry of D.

Assumption on Open Sets

Let $\alpha \in (0,2)$ and $R > 0$. For any $\eta \in (\alpha/2, 1]$, suppose that $D \subset \mathbb{R}^d$ is an exterior $C^{1,\eta}$ open set, i.e., D^c is compact, satisfying $D^c \subset B(0,R)$.

It can be disconnected and there are bounded connected components.

The number of the such bounded connected components is finite, i.e., $D = D_0 \cup D_1 \cup \ldots \cup D_n$ where D_0 be an unbounded connected component and $D_1, \ldots, D_n$ be bounded connected components.

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It can be disconnected and there are bounded connected components.

The number of the such bounded connected components is finite, i.e., $D = D_0 \cup D_1 \cup \ldots \cup D_n$ where D_0 be an unbounded connected component and $D_1, \ldots, D_n$ be bounded connected components.

Outline

1 Introduction

2 Heat Kernel Estimates for Z

3 Dirichlet Heat Kernel Estimates for Z^D

4 Main Result

Heat Kernel Estimates in Exterior $C^{1,\eta}$ Open Set

Recall that

$$\Psi^2_{a,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge tr^{-d-\alpha} & \text{if } \beta = 0, \\ t^{-d/2} \exp\left(-a(r^\beta \wedge r^2/t)\right) & \text{if } \beta \in (0,1], \\ t^{-d/2} \exp\left(-a(r\left(1+\log^+(Tr/t)\right)^{(\beta-1)/\beta} \wedge r^2/t)\right) & \text{if } \beta \in (1,\infty), \\ t^{-d/2} \exp\left(-a(r\left(1+\log^+(Tr/t)\right) \wedge (r^2/t))\right) & \text{if } \beta = \infty. \end{cases}$$

Let $d > 2 \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$ and $T > 0$, and denote $a \vee b := \max\{a, b\}$.

For any $t \geq T$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$,
(1) if $\beta \in [0, \infty]$,

$$p_D(t,x,y) \leq c_1 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{1 \vee \sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{1 \vee \sqrt{t}}\right) \Psi^2_{c_2,T}(t, |x-y|).$$

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$$p_D(t,x,y) \geq c_3 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{1 \vee \sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{1 \vee \sqrt{t}}\right) \Psi^2_{c_4,T}(t,|x-y|).$$

(3) if $\beta \in (1,\infty]$ with $|x-y| \geq 4/5$ and x, y are in a same component of D,

(3.a) (Unbounded connected component: $x, y \in D_0$)

$$p_D(t,x,y) \geq c_5 \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \left(1 \wedge \delta_D(y)^{\alpha/2}\right) \Psi^2_{c_6,T}(t,|x-y|).$$

(3.b) (Bounded connected component: $x, y \in D_j$ for some $j = 1, \ldots n,$)

$$p_D(t,x,y) \geq c_7\, e^{-t\lambda_j}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2}.$$

where $-\lambda_j < 0$ is the largest eigenvalue of the generator $Y^{D_j}, j = 1 \ldots, n.$

(4) if $\beta \in (1,\infty)$ with $|x-y| \geq 4/5$ and x, y are in different components of D,

$$p_D(t,x,y) \geq c_8 \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \left(1 \wedge \delta_D(y)^{\alpha/2}\right) \frac{\exp\left(-c_9(|x-y|^\beta + t)\right)}{|x-y|^{d+\alpha}}.$$

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Let D is an exterior $C^{1,\eta}$ open set, and $d > 2 \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$.

There are positive constants $c_i > 1, i = 1, 2$ such that for every $(t, x, y) \in (0, \infty) \times D \times D$,

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and in addition D is a connected,

$$p_D(t,x,y) \ge c_1^{-1} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{1 \vee \sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{1 \vee \sqrt{t}}\right) \cdot \begin{cases} \Psi^1_{c_2,\gamma,T}(t,5|x-y|/4) & \text{if } t \in (0,T], \\ \Psi^2_{c_2,T}(t,|x-y|) & \text{if } t \in [T,\infty). \end{cases}$$

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Thank you!