Lévy-Khintchine random matrices

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Motivation

- Wigner matrices ('55, '58).
- Heavy tailed matrices have i.i.d. entries (up to symmetry) with infinite variance. Cizeau, Bouchaud, Soshnikov, Belinschi, Dembo, Benaych-Georges, Male; Ben Arous, Guionnet (08); Bordenave, Caputo, Chafai ('11).
- Adjacency matrices of Erdös-Rényi graphs with $p = 1/n$. Rogers, Bray, Khorunzhy, Shcherbina, Vengerovsky; Zakharevich ('06), Bordenave and Lelarge ('10).
- We study general symmetric matrices with symmetric i.i.d. entries; Sum of a row converges weakly as $n \to \infty$. Limits are infinitely divisible (σ^2, d, ν).

Wigner's theorem (1955, 58)

- i.i.d. entries up to symmetry $a_{ij} = a_{ji}$.
- The (normalized) empirical measure for random eigenvalues $e_j(\omega) \in \mathbb{R}$:

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{e_j} = \mathrm{ESD}_n.$$

- This random measure-valued sequence converges a.s. if $\mathbf{E}\left(\frac{1}{\sqrt{n}} a_{ij}\right)^2 = \frac{1}{n}$, the limit is non-random:

$$\frac{1}{2\pi}\sqrt{4 - x^2}, \quad x \in [-2, 2].$$

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The semicircle law

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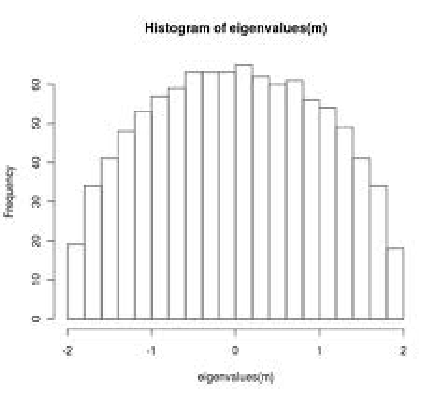

picture from cartesianfaith.com

The normalization

- To normalize note that

$$\mathbf{E}(\mathrm{ESD}_n^2) = \mathbf{E}\frac{1}{n}\,\mathrm{Tr}(A_n^2) = \mathbf{E}\frac{1}{n}\sum_{i,j} a_{ij}a_{ji} = n\mathbf{E}a_{ij}^2.$$

- So we need

$$\mathbf{E}a_{ij}^2 \sim \frac{1}{n}.$$

- Instead of normalizing, change the distribution as n varies:

$$a_{ij} = a_{ji} \;\sim\; \text{Bernoulli}\,(\lambda/n) \quad \text{so that} \quad \mathbf{E}a_{ij}^2 = \lambda/n.$$

Simplification: By concentration results for self-adjoint matrices, it suffices to study the expected ESD_n.

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Main results, J. (2014)

- Suppose each A_n has i.i.d. entries up to self-adjointness satisfying:

$$\lim_{n\to\infty} \sum_{j=1}^{n} A_n(i,j) \stackrel{d}{=} ID(\sigma^2, d, \nu).$$

ESD$_n$ a.s. weakly converge to a symm. prob. meas. μ_∞.

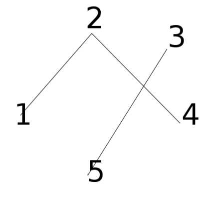
- μ_∞ is the expected spectral measure associated to δ_{root} of a self-adjoint operator on $L^2(G)$.

  (for Wigner matrices, $G = \mathbb{N}$, the free Fock space)

Erdős-Rényi random graphs (rooted at 1)

- We need $\mathbf{E}a_{ij}^2 \sim \frac{\lambda}{n}$.
- The matrices are adjacency matrices of

Erdős-Rényi graphs $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$



Idea of proof

(1) As rooted graphs, Erdős-Rényi(λ/n) *locally converge* to a branching process with a Poiss(λ) offspring distribution.

(2)

Bordenave-Lelarge (2010)

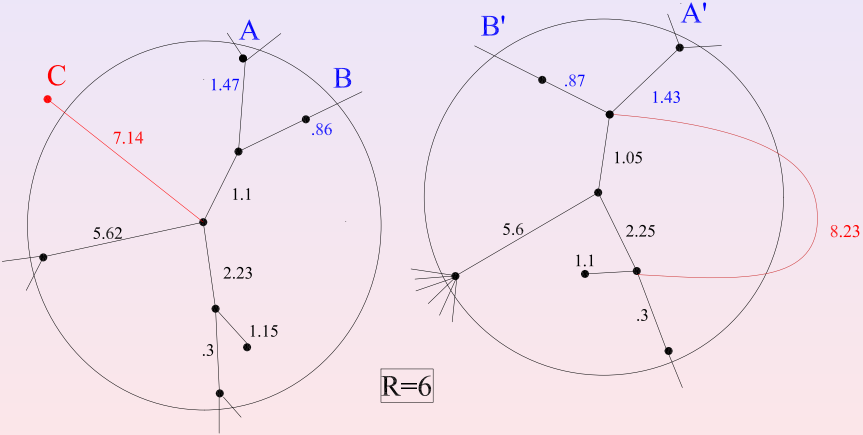
If $G_n[1] \Rightarrow G_\infty[1]$, then one has strong resolvent convergence: for all $z \in \mathbb{C}_+$,

$$(zI - A_n)^{-1}_{11} \to (zI - A_\infty)^{-1}_{11}$$

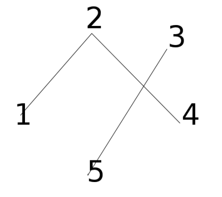
(3)

$$\mathbf{E}(zI - A_n)^{-1}_{11} = \mathbf{E}\frac{\mathrm{Tr}(zI - A_n)^{-1}}{n} = \int \frac{1}{z-x} d\mathbf{E}(\mathrm{ESD}_n)$$

$\epsilon = 1/6$-close graphs



- $a_{ij} \sim$ Bernoulli(λ/n) so the number of offspring is Poisson(λ).



- Fix k, an offspring in generation bigger than 1, the probability that it's also a direct offspring (genereation 1) is:

$$\mathbf{P}(1 \sim k) = 1/n \to 0.$$

Thus, local weak convergence to a Poisson(λ) branching process.

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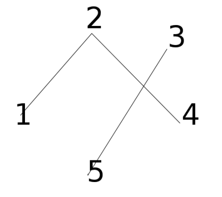

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Easy existence proof in the case of exponential moments

- By Lévy-Itō decomposition, write $A_n = G_n + L_n$
- Local weak convergence implies strong resolvent convergence when $\sigma^2 = 0$ takes care of (L_n).
- Voiculescu's theorem says (G_n) and (L_n) are asymptotically free.
- The LSD of (A_n) is the free convolution of the LSDs of (G_n) and (L_n).

General weighted-edges case: Aldous' Poisson weighted infinite tree

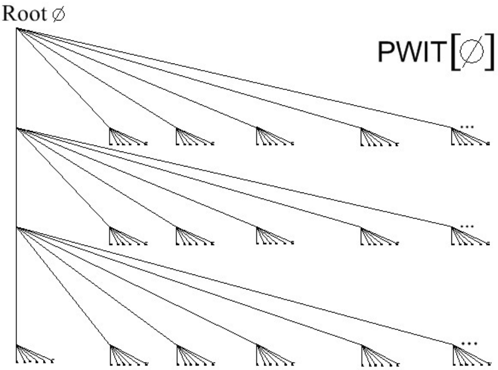

Figure 2: Each represents a copy of the PWIT. Weights on offspring edges from any vertex are determined by a Poisson process.

What about σ^2 and d?

- Lidskii's theorem handles drift.
- For the step in the proof where LWC $\Rightarrow$ Strong resolvent conv. we need

$$\lim_{\varepsilon \searrow 0} \lim_{n \to \infty} \sum_{j=1}^{n} |a_{1j}|^2 1_{\{|a_{1j}|^2 \leq \varepsilon\}} = 0.$$

The Poisson weighted infinite skeleton tree

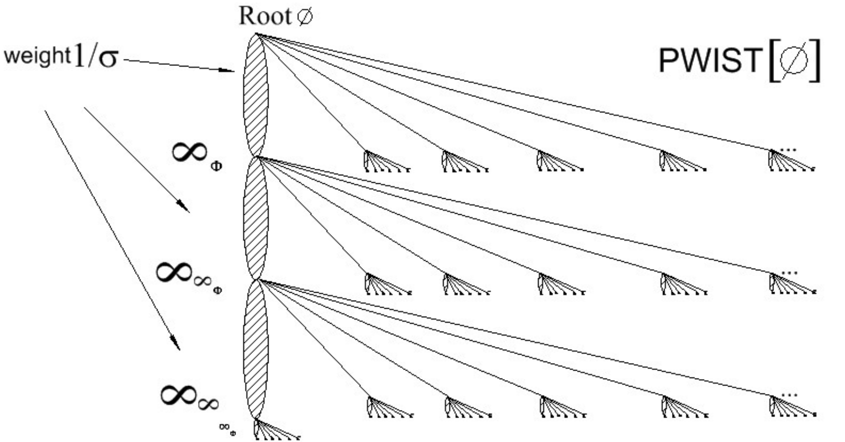

Figure 3: Each [small tree symbol] represents a copy of the PWIST.

Weights on cords to infinities are deterministic. All other weights are random and determined by Poisson processes as before.

Wigner matrices: vacuum state of the free Fock space

We can handle infinite second moments in the Gaussian domain of attraction.

When there is no Levy measure, the PWIST is the half-line N.
It is well-known that the spectral measure at the root is semi-circle.

Cords to infinity: $\sigma^2 > 0$

- The weights are interpreted as lengths of edges. Think of v and ∞_v as being infinitely far apart, but with infinitely many parallel edges between.
- Distance $=$ resistance on electric networks, and resistance $= 1/$conductance
- The conductance of each parallel edge is zero; however, their collective effective conductance is σ and the effective resistance is $1/\sigma$.
- Identifying all edges with small conductance to one single point we get that

$$\lim_{\varepsilon \searrow 0} \lim_{n\to\infty} \sum_{j=1}^{n} |a_{1j}|^2 1_{\{|a_{1j}|^2 \le \varepsilon\}} = 0.$$

Schur complement formula

Corollary (J. 2014): For $z \in \mathbb{C}_+$, $\quad R_{jj}(z) \overset{d}{=} (A_\infty - zI)^{-1}_{11}$ satisfies

$$R_{00}(z) \overset{d}{=} -\left(z + \sigma^2 R_{11}(z) + \sum_{j\geq 2} a_j^2 R_{jj}(z)\right)^{-1}$$

where $\{a_j\}$ are arrivals of an independent Poisson(ν) process.

- Random conductance model / Poisson-Dirichlet boundary
- Connections with free probability
- NonHermitian case: "circular laws"
- Spectral edge

Thanks for your attention!

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