Continuity and estimates for the Liouville heat kernel

Sebastian Andres

joint work with Naotaka Kajino

University of Bonn

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Liouville Quantum Gravity

- Construct a random metric on a 2-dim. manifold $D \subseteq \mathbb{R}^2$ of the form

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Liouville Brownian motion – Idea

Aim: Analyse "Brownian motion $(\mathcal{B}_t)$ on (D, g_X)".

Pretend X were a continuous Gaussian field. Then,

- $g_X = e^{\gamma X(x)}\, g_{\mathrm{Euc}}$,
- $\mathrm{vol}_{g_X}(dx) = e^{\gamma X(x)}\, dx$,
- $\nabla_{g_X} u(x) = e^{-\gamma X(x)} \nabla u(x)$,
- $\int_D |\nabla_{g_X} u|^2_{g_X} \,\mathrm{vol}_{g_X}(dx) = \int_D |\nabla u|^2\, dx$.

Hence,

$$\Delta_{g_X} u = e^{-\gamma X(x)} \Delta u$$

and

$$\mathcal{B}_t = B_{F_t^{-1}}, \qquad F_t := \int_0^t e^{\gamma X(B_s)}\, ds, \qquad t \geq 0,$$

where B is a standard Brownian motion on D.

Problem: How to make sense of $e^{\gamma X(x)}\, dx$ and F_t if X is the GFF?

Gaussian free field

Let D be bounded. The **(continuum) GFF** X is the **Gaussian Hilbert space** associated with the Sobolev space $H_0^1(D)$, which is the closure of $C_0^\infty(D)$ w.r.t. the Dirichlet inner product

$$(f,g)_\nabla = \int_D \nabla f \cdot \nabla g \, dx,$$

i.e. $\{(X,f)_\nabla\}_{f\in H_0^1(D)}$ is a family of centered Gaussian random variables on $(\Omega,\mathcal{A},\mathbb{P})$ satisfying

$$\mathbb{E}\big[(X,f)_\nabla\,(X,g)_\nabla\big] = (f,g)_\nabla = \big(f,(-\Delta)g\big)_{L^2}.$$

The covariance function is given by the standard Green function g_D on D,

$$\mathbb{E}\big[X(x)\,X(y)\big] = g_D(x,y).$$

Note that: $g_D(x,y) \sim \log \frac{1}{|x-y|}$ as $|x-y| \to 0$.

Irregularity of the GFF

Formally,

$$X = \sum_n \alpha_n f_n,$$

where (α_n) i.i.d. $\mathcal{N}(0,1)$ and (f_n) ONB of $H_0^1(D)$. Since

$$\Big\| \sum_{n=1}^{N} \alpha_n f_n \Big\|_\nabla^2 = \sum_{n=1}^{N} \alpha_n^2 \to \infty \qquad \text{as } N \to \infty,$$

the GFF cannot be defined pointwise as a random element in $H_0^1(D)$, but it can be identified as **random distribution**.

Massive Gaussian free field

From now on: $D = \mathbb{R}^2$.

The **massive GFF** X is the **Gaussian Hilbert space** associated with the Sobolev space $\mathcal{H}^1_m$, which is the closure of $C_0^\infty(\mathbb{R}^2)$ w.r.t. the Dirichlet inner product

$$(f,g)_m = m^2 (f,g)_{L^2} + (f,g)_\nabla, \qquad m > 0 \text{ **mass**},$$

i.e. $\{(X,f)_m\}_{f \in \mathcal{H}^1_m}$ is a family of centered Gaussian random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying

$$\mathbb{E}\big[(X,f)_m\,(X,g)_m\big] = (f,g)_m = \big(f, (m^2 - \Delta)g\big)_{L^2}.$$

The covariance function is given by the massive Green function g_{m^2} associated with $m^2 - \Delta$,

$$\mathbb{E}\big[X(x)\,X(y)\big] = g_{m^2}(x,y).$$

n-regularized Liouville Brownian motion

- By a cutoff-procedure define a sequence (X_n) of continuous centered Gaussian fields with

$$\mathbb{E}\big[X_n(x)\,X_n(y)\big] \uparrow g_{m^2}(x,y) \qquad \text{as } n \uparrow \infty.$$

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- Let $M_{n,\gamma}$ be the volume measure

$$M_{n,\gamma}(dx) = e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx,$$

associated with the metric $e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]} g_{\mathrm{Euc}}$.

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- The n-regularized Liouville Brownian motion is defined as

$$\mathcal{B}^n_t = B_{(F^n)^{-1}_t}, \qquad F^n_t := \int_0^t e^{\gamma X_n(B_s) - \frac{\gamma^2}{2}\mathbb{E}[X_n(B_s)^2]}\,ds,$$

where $(B, \{P_x\}_{x\in\mathbb{R}^2})$ is a standard BM and F^n is strictly increasing.

Liouville Measure

- By the theory of **Gaussian multiplicative chaos** (Kahane '85), $\mathbb{P}$-a.s.

$$M_{n,\gamma} = e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\, dx \rightharpoonup M_\gamma, \qquad \gamma \in [0,2),$$

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- For $\gamma \in (0,2)$, $M_\gamma \perp dx$ and M_γ is concentrated on the set K_γ of γ-thick points of the field X.

$$x \in \mathbb{R}^2 \text{ is } \gamma\text{-thick} \;:\Longleftrightarrow \liminf_{r\to 0} \frac{A(x,r)}{\log \frac{1}{r}} \geq \sqrt{\frac{\gamma}{\pi}},$$

  where

$$A(x,r) = \frac{1}{|B(x,r)|}\int_{B(x,r)} X(y)\,dy.$$

Definition of Liouville Brownian motion

Theorem (Garban, Rhodes, Vargas (2013))

Let $\gamma \in [0,2)$. Then, $\mathbb{P}$-a.s. the following hold:

i) *For all $x \in \mathbb{R}^2$, F^n converges to some F in P_x-probability in $C([0,\infty),\mathbb{R})$.*

ii) *For all $x \in \mathbb{R}^2$, P_x-a.s., F is strictly increasing and satisfies $\lim_{t\to\infty} F_t = \infty$.*

iii) *The functional F is the unique PCAF with Revuz measure M_γ.*

The process $(\mathcal{B}, \{P_x\}_{x\in\mathbb{R}^2})$, $\mathbb{P}$-a.s. defined by

$$\mathcal{B}_t := B_{F_t^{-1}}, \qquad t \geq 0,$$

is called the (massive) **Liouville Brownian motion** (LBM).

A similar result has been obtained by N. Berestycki (2013).

Properties of the LBM

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- $\mathrm{Leb}[t \geq 0 : \mathcal{B}_t \notin K_\gamma] = 0$, K_γ set of γ-thick points.
- For $t \geq 0$, P_t is absolutely continuous w.r.t. M_γ, so there exists the **Liouville heat kernel** $p_t(\cdot, \cdot)$ such that

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- The intrinsic metric $d_{\mathcal{B}}$ associated with the LBM is identically zero, which indicates that

$$\lim_{t \to 0} t \log p_t(x, y) = -\frac{d_{\mathcal{B}}(x, y)^2}{2} = 0.$$

That is, irregular off-diagonal behaviour of $p_t(\cdot, \cdot)$ is expected.

Continuity and on-diagonal upper bounds on $p_t(x,y)$

Theorem (A., Kajino (2014))

i) *The Liouville heat kernel $p_t(x,y)$ is jointly continuous on $(0,\infty)\times\mathbb{R}^2\times\mathbb{R}^2$.*

ii) *For any bounded $U\subset\mathbb{R}^2$ and $\beta>(\gamma+2)^2/2>2$, $\mathbb{P}$-a.s., there exists $c_i=c_i(X,\gamma,U,\beta)$ such that for all $t<\frac{1}{2}$, $x,y\in U$,*

$$p_t(x,y)\le c_1\,t^{-1}\log(t^{-1})\exp\Big(-c_2(|x-y|^\beta/t)^{1/(\beta-1)}\Big).$$

Related result by Maillard, Rhodes, Vargas, Zeitouni (2014):

- Continuity of the Liouville heat kernel on the torus $\mathbb{T}$.
- For any $\delta>0$ there exists $\beta_\delta>0$ such that for $t>0$ and $x,y\in\mathbb{T}$.

$$p_t(x,y)\le c_1\,t^{-(1+\delta)}\exp\Big(-c_2(|x-y|^{\beta_\delta}/t)^{1/(\beta_\delta-1)}\Big),$$

$$p_t(x,y)\ge\exp\Big(-t^{-\frac{1}{1+\gamma^2/4-\delta}}\Big),\quad\forall t\in(0,t_0]\text{ with }t_0=t_0(X,\gamma,x,y).$$

On-diagonal lower bound and spectral dimension

Theorem (A., Kajino (2014))

Let $\gamma \in [0,2)$. Then, $\mathbb{P}$-a.s., for M_γ-a.e. $x \in \mathbb{R}^2$ there exist $c_3 = c_3(X,\gamma)$ and $t_0 = t_0(X,\gamma,x)$ such that

$$p_t(x,x) \geq c_3\, t^{-1} \left(\log(t^{-1})\right)^{-\eta}, \qquad \forall t \in (0, t_0],$$

for some explicit constant $\eta > 0$ ($\eta = 34$ would be enough).

Corollary (A., Kajino (2014))

Let $\gamma \in [0,2)$. Then, $\mathbb{P}$-a.s., for M_γ-a.e. $x \in \mathbb{R}^2$,

$$\lim_{t \downarrow 0} \frac{2 \log p_t(x,x)}{-\log t} = 2.$$

Rhodes, Vargas (2013): $$\lim_{y \to x} \int_0^\infty e^{-\lambda t} t^\alpha p_t(x,y)\, dt \begin{cases} < \infty, & \text{if } \alpha > 0, \\ = \infty, & \text{if } \alpha = 0. \end{cases}$$

Volume and exit time estimates

Lemma

Let $\varepsilon > 0$, $\alpha_1 := \frac{1}{2}(\gamma+2)^2$ *and* $\alpha_2 := 2(1-\frac{\gamma}{2})^2$. *Then,* $\mathbb{P}$*-a.s., for any* $R > 0$ *there exists* $c_i = c_i(X, \gamma, R, \varepsilon) > 0$, *such that*

$$c_4\, r^{\alpha_1+\varepsilon} \le M_\gamma(B(x,r)) \le c_5\, r^{\alpha_2-\varepsilon}, \qquad \forall x \in B(0,R),\ r \in (0,1).$$

Let $\tau_U := \inf\{s \ge 0 :\ \mathcal{B}_s \notin U\}$ be the first exit time from an open set U.

Proposition

For any $\beta > \alpha_1$ *and* $R \ge 1$, $\mathbb{P}$*-a.s., there exist* $c_i = c_i(X, \gamma, R, \beta)$ *such that*

$$P_x[\tau_{B(x,r)} \le t] \le c_6 \exp\left(-c_7 (r^\beta/t)^{1/(\beta-1)}\right),$$

for all $t > 0$, $x \in B(0,R)$ *and* $r \in (0,1]$.

The Dirichlet heat kernel

Consider the heat kernel $p_t^U(x,y)$ of the LBM killed upon exiting a bounded open set U.

- Strong Feller property of the resolvent.
- Faber-Krahn inequality, which implies a Nash inequality and on-diagonal bounds (ultracontractivity).
- By a general result by Davies ('89)

$$p_t^U(x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y)$$

converges uniformly on $[s_0, \infty) \times U \times U$ for all $s_0 > 0$, which implies continuity of $p_t^U(x,y)$.

- In combination with the exit time estimate a general result by Grigor'yan, Hu and Lau (2010) gives the off-diagonal bound.