INTRINSIC ULTRACONTRACTIVITY OF NON-SYMMETRIC DIFFUSION SEMIGROUPS IN BOUNDED DOMAINS

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Abstract. We extend the concept of intrinsic ultracontractivity to non-symmetric semigroups and prove the intrinsic ultracontractivity of the Dirichlet semigroups of non-symmetric second order elliptic operators in bounded Lipschitz domains.

1. Introduction

The notion of intrinsic ultracontractivity (IU in abbreviation), introduced in [10] for symmetric semigroups, is a very important concept and has been studied extensively. Although the concept of ultracontractivity has been extended to non-symmetric semigroups, (see, for instance, [21]), it seems that, up to now, no one has introduced the concept of intrinsic ultracontractivity for non-symmetric semigroups. In this paper, we plan to fill this gap and introduce the notion of intrinsic ultracontractivity for non-symmetric semigroups. We show that, under natural conditions, the Dirichlet semigroups of non-symmetric second order elliptic operators are intrinsic ultracontractive.

In the symmetric case, ultracontractivity and intrinsic ultracontractivity are connected to logarithmic Sobolev inequalities. The connection between logarithmic Sobolev inequalities and L^p to L^q bounds of semigroups was first discovered by Gross [12] in 1975. Davies and Simons [10] adapted Gross's approach to allow $q = \infty$ and therefore established the connection between logarithmic Sobolev inequalities and ultracontractivity. (For an updated survey on the subject of logarithmic Sobolev inequalities and contractive properties of semigroups, see [3] and [13].) In [4], Bañuelos proved the intrinsic ultracontractivity of killed Schrödinger semigroups on Hölder domains of order zero or uniformly Hölder domains of order $\alpha \in (0, 2)$, using a logarithmic Sobolev inequality characterization. In [6] and [7], Chen and Song extended the argument of [4] to prove the intrinsic ultracontractivity of the Schrödinger semigroup of killed symmetric stable processes in certain types of domains.

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In this paper, we will also use logarithmic Sobolev inequalities as a tool to establish the intrinsic ultracontractivity of non-symmetric semigroups. However, in the non-symmetric case, things are very delicate. One has to use a related symmetric semigroup as a bridge to make things work out. We show that, under some natural conditions the Dirichlet semigroup of a non-symmetric second order elliptic operator in a bounded Lipschitz domain is intrinsic ultracontractive.

To concentrate on the main ideas, we will not try to obtain the most general result in this paper. For simplicity, we will only deal with second order elliptic operators with smooth coefficients. The case of second order differential operators with measure-valued drifts and the case of non-local operators are considered in our papers [15] and [16], respectively.

This paper is organized as follows. In Section 2, we introduce the concept of ultracontractivity and intrinsic ultracontractivity for non-symmetric semigroups. Section 3 contains the proof of the intrinsic ultracontractivity for the Dirichlet semigroups of the non-symmetric second order elliptic operators in bounded Lipschitz domains. In Appendix, we prove some identities stated in Section 3.

Throughout this paper, we will use the following convention. The values of the constants $c_1, c_2, \dots$ may change from one appearance to another. In this paper, we use ":=" to denote a definition, which is read as "is defined to be".

2. Introduction to IU for Non-symmetric Semigroups

Suppose that E is a locally compact separable metric space and m is a positive finite measure on E such that $\mathrm{Supp}[m] = E$. Suppose that we are given two semigroups $\{P_t\}$ and $\{\hat{P}_t\}$ on $L^2(E,m)$ such that for any $t > 0$,

$$\int_E f(x) P_t g(x) m(dx) = \int_E g(x) \hat{P}_t f(x) m(dx).$$

We assume that there exists a family of continuous positive functions $\{p(t,\cdot,\cdot);\, t > 0\}$ on $E \times E$ such that for any $(t,x) \in (0,\infty) \times E$, we have

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy), \quad \hat{P}_t f(x) = \int_E p(t,y,x) f(y) m(dy).$$

Definition 2.1. The semigroups $\{P_t\}$ and $\{\hat{P}_t\}$ are said to be *ultracontractive* if, for any $t > 0$, there exists constant $c_t > 0$ such that

$$p(t,x,y) \le c_t \quad \text{for any } (x,y) \in E \times E.$$

For any operator A from $L^p(E,m)$ to $L^q(E,m)$, we will use $\|A\|_{L^q(E,m),L^p(E,m)}$ to denote the norm of A. When there is no danger of confusion, we will write $\|A\|_{q,p}$ for $\|A\|_{L^q(E,m),L^p(E,m)}$.

It is well-known that if $\{P_t\}$ and $\{\hat{P}_t\}$ are sub-Markov semigroups in the sense that

$$P_t 1(x) \le 1, \quad \hat{P}_t 1(x) \le 1$$

for all $t \ge 0$ and $x \in E$, then both of them are contractive semigroups on $L^2(E, m)$.

Proposition 2.2. *Suppose that $\{P_t\}$ and $\{\hat{P}_t\}$ are sub-Markov semigroups. Then $\{P_t\}$ and $\{\hat{P}_t\}$ are ultracontractive if and only if, for any $t > 0$, P_t and $\hat{P}_t$ are both bounded from $L^2(E, m)$ to $L^\infty(E, m)$.*

Proof. Suppose that $\{P_t\}$ and $\{\hat{P}_t\}$ are sub-Markov semigroups and that P_t and $\hat{P}_t$ are both bounded from $L^2(E, m)$ to $L^\infty(E, m)$. Then both $\|P_t\|_{\infty,2}$ and $\|\hat{P}_t\|_{\infty,2}$ are decreasing functions of t. Put

$$a_t = \max\{\|P_t\|_{\infty,2}, \|\hat{P}_t\|_{\infty,2}\}.$$

By taking adjoint, we know that

$$\|P_t\|_{2,1} = \|\hat{P}_t\|_{\infty,2}, \ \ \|\hat{P}_t\|_{2,1} = \|P_t\|_{\infty,2},$$

so we have

$$\begin{aligned} \|P_t\|_{\infty,1} &\le \|P_{t/2}\|_{\infty,2} \|P_{t/2}\|_{2,1} \le a_{t/2}^2, \\ \|\hat{P}_t\|_{\infty,1} &\le \|\hat{P}_{t/2}\|_{\infty,2} \|\hat{P}_{t/2}\|_{2,1} \le a_{t/2}^2. \end{aligned}$$

Therefore $\{P_t\}$ and $\{\hat{P}_t\}$ are ultracontractive.

Now suppose that, for any $t > 0$, we have

$$p(t, x, y) \le c_t \quad \text{ for any } (x, y) \in E \times E.$$

Then we have

$$\|P_t\|_{\infty,1} \le c_t, \ \ \|\hat{P}_t\|_{\infty,1} \le c_t.$$

Since $\{P_t\}$ and $\{\hat{P}_t\}$ are sub-Markov semigroups, we also have

$$\|P_t\|_{\infty,\infty} \le 1, \ \ \|\hat{P}_t\|_{\infty,\infty} \le 1,$$

and hence we can use interpolation to arrive at

$$\|P_t\|_{\infty,2} \le c_t^{1/2}, \ \ \|\hat{P}_t\|_{\infty,2} \le c_t^{1/2}.$$

□

To introduce the concept of intrinsic ultracontractivity, we further assume that

(a) $\{P_t\}$ and $\{\hat{P}_t\}$ are strongly continuous semigroups on $L^2(E, m)$;

(b) for each $t > 0$, $p(t, x, y)$ is bounded and strictly positive.

Let L and $\hat{L}$ be the infinitesimal generators of the semigroups $\{P_t\}$ and $\{\hat{P}_t\}$ on $L^2(E, m)$, respectively. It follows from Jentzsch's Theorem (Theorem V.6.6 on page 337 of [22]) and the strong continuity of $\{P_t\}$ and $\{\hat{P}_t\}$ that the common value $\lambda_0 := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(\hat{L}))$ is an eigenvalue of multiplicity 1 for both L and $\hat{L}$, and that an eigenfunction ϕ_0 of L associated with λ_0 can be chosen to be strictly positive a.e. with $\|\phi_0\|_{L^2(E,m)} = 1$ and an eigenfunction ψ_0 of $\hat{L}$ associated with λ_0 can be chosen to be strictly positive with $\|\psi_0\|_{L^2(E,m)} = 1$. Thus for a.e. $x \in E$,

$$e^{\lambda_0 t}\phi_0(x) \;=\; \int_E p(t,x,z)\phi_0(z)m(dz), \quad e^{\lambda_0 t}\psi_0(x) \;=\; \int_E p(t,z,x)\psi_0(z)m(dz). \tag{2.1}$$

PROPOSITION 2.3. *$\phi_0(x)$ and $\psi_0(x)$ are strictly positive and continuous in E. Thus (2.1) is true for every $x \in E$.*

PROOF. By (2.1), we have

$$\phi_0(x) \;=\; e^{-\lambda_0} \int_E p(1,x,z)\phi_0(z)m(dz)$$

almost everywhere on E. Since $p(1,x,z)$ is bounded continuous and $m(E) < \infty$, the right hand side of the above equation is continuous by using the dominated convergence theorem and the fact $\|\phi_0\|_{L^2(E,m)} = 1$. Similarly, $e^{-\lambda_0} \int_E p(1,z,x)\psi_0(z)m(dz)$ is continuous. Thus there exist continuous versions of ϕ_0 and ψ_0, and (2.1) is true for every $x \in E$. Now the strict positivity of ϕ_0 and ψ_0 follow from the strict positivity of $p(1,\cdot,\cdot)$ and (2.1). $\Box$

Define, for any $(t,x,y) \in (0,\infty) \times E \times E$,

$$q(t,x,y) := \frac{e^{-\lambda_0 t}}{\phi_0(x)} p(t,x,y)\phi_0(y), \quad \hat{q}(t,x,y) := \frac{e^{-\lambda_0 t}}{\psi_0(y)} p(t,x,y)\psi_0(x).$$

Then it is easy to check that the operators $\{Q_t\}$ and $\{\hat{Q}_t\}$ defined by

$$Q_t f(x) := \int_E q(t,x,y)f(y)m(dy), \quad \hat{Q}_t f(x) := \int_E \hat{q}(t,y,x)f(y)m(dy)$$

form semigroups with $Q_t 1 = \hat{Q}_t 1 = 1$.

Define a function $\mu(x)$ by

$$\mu(x) := \frac{\phi_0(x)\psi_0(x)}{\displaystyle\int_E \phi_0(y)\psi_0(y)m(dy)}.$$

Then the measure $\mu(x)m(dx)$ is a probability measure on E. Put $M = \int_E \phi_0(y)\psi_0(y)m(dy)$. It is easy to see that $M \le 1$. For any $t > 0$ and any positive nonnegative functions f and

g on E, we have

$$\begin{aligned}
&\int_E g(x)\mu(x)Q_t f(x)m(dx)\\
&= \int_E g(x)\frac{e^{-\lambda_0 t}}{\phi_0(x)}\int_E p(t,x,y)\phi_0(y)f(y)m(dy)\frac{1}{M}\phi_0(x)\psi_0(x)m(dx)\\
&= \int_E g(x)\frac{e^{-\lambda_0 t}}{M}\psi_0(x)\int_E p(t,x,y)\phi_0(y)f(y)m(dy)m(dx)\\
&= \int_E \phi_0(y)f(y)\int_E \frac{e^{-\lambda_0 t}}{M}\psi_0(x)p(t,x,y)g(x)m(dx)m(dy)\\
&= \int_E \frac{1}{M}\phi_0(y)\psi_0(y)f(y)\frac{1}{\psi_0(y)}\int_E e^{-\lambda_0 t}\psi_0(x)p(t,x,y)g(x)m(dx)m(dy)\\
&= \int_E f(y)\mu(y)\hat{Q}_t g(y)m(dy),
\end{aligned}$$

Thus $\{Q_t\}$ and $\{\hat{Q}_t\}$ are dual semigroups on $L^2(E, \mu(x)m(dx))$.

By taking $g = 1$ in the display above, we see that μ is an invariant function of $\{Q_t\}$. Similarly, μ is also an invariant function of $\{\hat{Q}_t\}$.

DEFINITION 2.4. The semigroups $\{P_t\}$ and $\{\hat{P}_t\}$ are said to be *intrinsically ultracontractive* if, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p(t,x,y) \le c_t\phi_0(x)\psi_0(y) \quad \text{for all } (x,y) \in E \times E.$$

In Section 3, we will show that the Dirichlet semigroups of non-symmetric diffusions with smooth coefficients in bounded Lipschitz domains are intrinsic ultracontractive. One of the key steps in the argument of Section 3 is Lemma 3.1 which amounts to saying that ϕ_0 and ψ_0 are comparable. This comparability of the two eigenfunctions ϕ_0 and ψ_0 is not true in general. For instance, by using the Dirichlet heat kernel estimates in [14], one can easily see that for the semigroups $\{P^D_t\}$ and $\{\hat{P}^D_t\}$ defined before Lemma 5.5 in [15] with D being a bounded $C^{1,1}$ domain, ϕ_0 is comparable to $\delta_D(x)$ (the distance between x and ∂D) while ψ_0 is comparable with the constant function.

Since the density of Q_t with respect to $\mu(x)m(dx)$ is given by

$$\overline{q}(t,x,y) := \frac{Me^{-\lambda_0 t}p(t,x,y)}{\phi_0(x)\psi_0(y)},$$

it follows from Proposition 2.2 that $\{P_t\}$ and $\{\hat{P}_t\}$ are intrinsically ultracontractive if and only if the semigroups $\{Q_t\}$ and $\{\hat{Q}_t\}$ on $L^2(E, \mu(x)m(dx))$ are ultracontractive.

For the remainder of this section, we discuss some important consequences of the intrinsic ultracontractivity for non-symmetric semigroups. In particular, these results are used in our upcoming papers [15] and [16].

Intrinsic ultracontractivity implies the following lower bound on the density $p(t,x,y)$.

PROPOSITION 2.5. *Suppose that* $\{P_t\}$ *and* $\{\hat{P}_t\}$ *are intrinsically ultracontractive, that is, for any* $t > 0$, *there exists a constant* $c_t > 0$ *such that*

$$p(t, x, y) \le c_t \phi_0(x)\psi_0(y) \quad \text{for all } (x, y) \in E \times E.$$

Then, for any $t > 0$, *there exists a constant* $c'_t > 0$ *such that*

$$p(t, x, y) \ge c'_t \phi_0(x)\psi_0(y) \quad \text{for all } (x, y) \in E \times E.$$

PROOF. The idea of the proof comes from the proof of (iv)⇒(v) in Theorem 3.2 of [10].

Let K be a compact subset of E such that

$$\int_K \mu(x)m(dx) \ge 1 - \frac{e^{\lambda_0 t}}{2Mc_t}.$$

Then by Proposition 2.3, we obtain

$$\begin{aligned}
&e^{\lambda_0 t}\frac{1}{M}\phi_0(x) = \frac{1}{M}\int_E p(t, x, y)\phi_0(y)m(dy) \\
&\le \frac{1}{M}\int_{E\setminus K} c_t\phi_0(x)\psi_0(y)\phi_0(y)m(dy) + \frac{1}{M}\int_K p(t, x, y)\phi_0(y)m(dy) \\
&\le \frac{1}{2M}e^{\lambda_0 t}\phi_0(x) + \frac{1}{M}\int_K p(t, x, y)\phi_0(y)m(dy),
\end{aligned}$$

so that

$$\int_K p(t, x, y)\phi_0(y)m(dy) \ge \frac{1}{2}e^{\lambda_0 t}\phi_0(x) \quad \text{for all } x \in E. \tag{2.2}$$

Similarly, we also have

$$\int_K p(t, x, y)\psi_0(x)m(dx) \ge \frac{1}{2}e^{\lambda_0 t}\psi_0(y) \quad \text{for all } y \in E. \tag{2.3}$$

Note that by the strict positivity and continuity of $p(t, x, y)$ and Proposition 2.3, we have

$$J := \min\left\{\frac{p(t/3, x, y)}{\phi_0(x)\psi_0(y)}\,;\, x, y \in K\right\} > 0. \tag{2.4}$$

Thus by the semigroup property and (2.4),

$$\begin{aligned}
p(t, x, y) &\ge \int_K\int_K p(t/3, x, z)p(t/3, z, w)p(t/3, w, y)m(dz)m(dw) \\
&\ge J\int_K\int_K p(t/3, x, z)\phi_0(z)\psi_0(w)p(t/3, w, y)m(dz)m(dw) \\
&= J\int_K p(t/3, x, z)\phi_0(z)m(dz)\int_K \psi_0(w)p(t/3, w, y)m(dw) \\
&\ge \frac{J}{4}e^{\frac{2}{3}\lambda_0 t}\phi_0(x)\psi_0(y).
\end{aligned}$$

In the last inequality above, we used (2.2) and (2.3). □

For simplicity, we will write $L^2(E,\mu(x)m(dx))$ as $L^2(E,\mu)$ from now on. The following result implies that, when $\{P_t\}$ and $\{\hat{P}_t\}$ are strongly continuous on $L^2(E,m)$, then $\{Q_t\}$ and $\{\hat{Q}_t\}$ are strongly continuous contraction semigroups on $L^2(E,\mu)$. Note that in the symmetric case, $\{Q_t\}$ and $\{\hat{Q}_t\}$ are automatically strongly continuous on $L^2(E,\mu)$. We also note that when $\{P_t\}$ and $\{\hat{P}_t\}$ are associated with a pair of dual right processes, by repeating the argument in Section 11.3 of [8], one can show that there are pair of right processes associated with the transition densities q and $\hat{q}$. These two right processes are duals of each other with respect to the measure $\mu(x)m(dx)$. Thus $\{Q_t\}$ and $\{\hat{Q}_t\}$ are strongly continuous on $L^2(E,\mu(x)m(dx))$; see, for instance, the second paragraph after Lemma 2.3 in [11]. But in general, the strong continuity of $\{Q_t\}$ and $\{\hat{Q}_t\}$ is not obvious. This is only one of the many indications that the non-symmetric case is much delicate to deal with.

PROPOSITION 2.6. *$\{Q_t\}$ and $\{\hat{Q}_t\}$ are strongly continuous contraction semigroups in $L^2(E,\mu)$.*

PROOF. The contraction property follows immediately from the fact that $\{Q_t\}$ and $\{\hat{Q}_t\}$ are Markov semigroups. For $f\in L^2(E,\mu)$, let

$$f_k := f1_{\{|f|\le k\}}, \qquad k\ge 1.$$

Since $\|\phi_0\|_{L^2(E,m)} = \|\psi_0\|_{L^2(E,m)} = 1$ and f_k is bounded, we have $\phi_0 f_k \in L^2(E,m)$, $\psi_0 f_k\in L^2(E,m)$ and $f_k\in L^2(E,\phi_0^2 m)\cap L^2(E,\psi_0^2 m)$. Moreover, for any $k\ge 1$ and $t>0$,

$$\begin{aligned}
&\|Q_t f_k - f_k\|^2_{L^2(E,\phi_0^2 m)}\\
&= \int_E\left(\int_E \frac{e^{-\lambda_0 t}}{\phi_0(x)} p(t,x,y)\phi_0(y)f_k(y)m(dy) - f_k(x)\right)^2\phi_0^2(x)m(dx)\\
&= e^{-2\lambda_0 t}\int_E\left(\int_E p(t,x,y)\phi_0(y)f_k(y)m(dy) - e^{\lambda_0 t}\phi_0(x)f_k(x)\right)^2 m(dx)\\
&= e^{-2\lambda_0 t}\|P_t(\phi_0 f_k) - e^{\lambda_0 t}\phi_0 f_k\|^2_{L^2(E,m)}\\
&\le 2e^{-2\lambda_0 t}\left(\|P_t(\phi_0 f_k)-\phi_0 f_k\|^2_{L^2(E,m)} + (1-e^{\lambda_0 t})^2\|\phi_0 f_k\|^2_{L^2(E,m)}\right).
\end{aligned}\tag{2.5}$$

Similarly, for any $k\ge 1$ and $t>0$,

$$\begin{aligned}
&\|\hat{Q}_t f_k - f_k\|^2_{L^2(E,\psi_0^2 m)}\\
&\le 2e^{-2\lambda_0 t}\left(\|\hat{P}_t(\psi_0 f_k)-\psi_0 f_k\|^2_{L^2(E,m)} + (1-e^{\lambda_0 t})^2\|\psi_0 f_k\|^2_{L^2(E,m)}\right).
\end{aligned}\tag{2.6}$$

Since $\{P_t\}$ and $\{\hat{P}_t\}$ are strongly continuous semigroups on $L^2(E,m)$, from (2.5) and (2.6) we have

$$\lim_{t\to 0}\|Q_t f_k - f_k\|^2_{L^2(E,\phi_0^2 m)} = \lim_{t\to 0}\|\hat{Q}_t f_k - f_k\|^2_{L^2(E,\psi_0^2 m)} = 0, \qquad k\ge 1. \tag{2.7}$$

On the other hand, since $\|Q_t f_k\|_\infty \le \|f_k\|_\infty \le k$ and $\|\hat{Q}_t f_k\|_\infty \le k$, we have

$$\int_E (Q_t f_k(x) - f_k(x))^2 \psi_0^2(x) m(dx) \;\le\; 4k^2, \qquad k \ge 1,\; t > 0, \tag{2.8}$$

and

$$\int_E (\hat{Q}_t f_k(x) - f_k(x))^2 \phi_0^2(x) m(dx) \;\le\; 4k^2, \qquad k \ge 1,\; t > 0. \tag{2.9}$$

Thus, by the Hölder inequality and (2.7) through (2.9), we obtain that

$$\begin{aligned}
&\limsup_{t\to 0} \|Q_t f_k - f_k\|^2_{L^2(E,\mu)} \\
&\quad = \frac{1}{M} \limsup_{t\to 0} \int_E (Q_t f_k(x) - f_k(x))\phi_0(x)(Q_t f_k(x) - f_k(x))\psi_0(x) m(dx) \\
&\quad \le \frac{1}{M} \limsup_{t\to 0} \|Q_t f_k - f_k\|_{L^2(E,\phi_0^2 m)} \left(\int_E (Q_t f_k(x) - f_k(x))^2 \psi_0^2(x) m(dx) \right)^{1/2} \\
&\quad \le \frac{2k}{M} \limsup_{t\to 0} \|Q_t f_k - f_k\|_{L^2(E,\phi_0^2 m)} \;=\; 0, \qquad k \ge 1.
\end{aligned}$$

and, similarly,

$$\limsup_{t\to 0} \|\hat{Q}_t f_k - f_k\|^2_{L^2(E,\mu)} \;=\; 0, \qquad k \ge 1.$$

Now, by the contraction property of $\{Q_t\}$ and $\{\hat{Q}_t\}$, we see that

$$\begin{aligned}
\|Q_t f - f\|_{L^2(E,\mu)} \;&\le\; \|Q_t(f - f_k)\|_{L^2(E,\mu)} + \|Q_t f_k - f_k\|_{L^2(E,\mu)} + \|f - f_k\|_{L^2(E,\mu)} \\
&\le\; 2\|f - f_k\|_{L^2(E,\mu)} + \|Q_t f_k - f_k\|_{L^2(E,\mu)}
\end{aligned}$$

and

$$\|\hat{Q}_t f - f\|_{L^2(E,\mu)} \;\le\; 2\|f - f_k\|_{L^2(E,\mu)} + \|\hat{Q}_t f_k - f_k\|_{L^2(E,\mu)}.$$

Therefore, for any $\varepsilon > 0$, we have

$$\begin{aligned}
\|Q_t f - f\|_{L^2(E,\mu)} \;&\le\; \frac{\varepsilon}{2} + \|Q_t f_k - f_k\|_{L^2(E,\mu)}, \\
\|\hat{Q}_t f - f\|_{L^2(E,\mu)} \;&\le\; \frac{\varepsilon}{2} + \|\hat{Q}_t f_k - f_k\|_{L^2(E,\mu)}
\end{aligned}$$

for large k. Thus $Q_t f(x)$ and $\hat{Q}_t f(x)$ converge to $f(x)$ in $L^2(E,\mu)$. □

The following result means that the intrinsic ultracontractivity of $\{P_t\}$ and $\{\hat{P}_t\}$ implies that the semigroups $\{Q_t\}$ and $\{\hat{Q}_t\}$ on $L^2(E,\mu)$ converge to equilibrium exponentially fast.

Theorem 2.7. *Suppose that $\{P_t\}$ and $\{\hat{P}_t\}$ are intrinsically ultracontractive. Then there exist positive constants c and ν such that*

$$\left| \frac{M e^{-\lambda_0 t} p(t,x,y)}{\phi_0(x)\psi_0(y)} - 1 \right| \le c e^{-\nu t}, \qquad (t,x,y) \in (1,\infty) \times E \times E. \tag{2.10}$$

Proof. The argument in this proof is very much similar to that used in the proof of Theorem 4 in [20]. We can not directly use proof of Theorem 3 in [20] in the present situation since we have to work with L^2 spaces instead of the space of bounded continuous functions. Let $\overline{L}$ and $\overline{\hat{L}}$ be the generators of $\{Q_t\}$ and $\{\hat{Q}_t\}$ in $L^2(E,\mu)$. Then $0 = \sup \mathrm{Re}(\sigma(\overline{L})) = \sup \mathrm{Re}(\sigma(\overline{\hat{L}}))$ and 1 is a positive eigenfunction of both $\overline{L}$ and $\overline{\hat{L}}$ corresponding to the eigenvalue 0. It follows the intrinsic ultracontractive assumption, Proposition 2.2 and Proposition 2.5 that for any $t > 0$, $\overline{q}(t,x,y)$ is bounded and strictly positive. Applying Jentzsch's Theorem and the strong continuity of $\{Q_t\}$ and $\{\hat{Q}_t\}$ (Proposition 2.6), we know that the eigenvalue 0 is of multiplicity 1. By the Riesz-Schauder theory of compact operators, it follows that $L^2(E,\mu) = N \otimes R$, where $N = \{c; c \in \mathbf{R}\}$ and $\hat{Q}_t$ leaves N and R invariant (see Section 6.6 of [5]). Since $1 = \sup \mathrm{Re}(\sigma(\hat{Q}_t))$ and the nonzero eigenvectors of a compact operator is isolated, it follows that there exist positive constants c_1 and ν such that

$$\|\hat{Q}_t|_R\|_{L^2(E,\mu)} \leq c_1 e^{-\nu t}, \quad t > 0.$$

By the above decomposition of $L^2(E,\mu)$, it follows that any $f \in L^2(E,\mu)$ can be written as $f = c_f + \psi_f$, where $\psi_f \in R$. Thus

$$\|\hat{Q}_t f - c_f\|_{L^2(E,\mu)} = \|\hat{Q}_t \psi_f\|_{L^2(E,\mu)} \leq c_1 e^{-\nu t} \|\psi_f\|_{L^2(E,\mu)}. \tag{2.11}$$

We now identify c_f. Since $\mu(x)m(dx)$ is a probability measure on E, we have

$$\begin{aligned} 0 &= \lim_{t\to\infty} \int_E (\hat{Q}_t f(x) - c_f(x))\mu(x)m(dx) \\ &= \lim_{t\to\infty} \int_E f(x) Q_t 1(x)\mu(x)m(dx) - c_f = \int_E f(x)\mu(x)m(dx) - c_f. \end{aligned}$$

Thus $c_f = \int_E f(x)\mu(x)m(dx)$ and $|c_f| \leq \|f\|_{L^2(E,\mu)}$. Hence

$$\|\psi_f\|_{L^2(E,\mu)} \leq \|f\|_{L^2(E,\mu)} + |c_f| \leq 2\|f\|_{L^2(E,\mu)}.$$

Therefore, it follows from (2.11) that for all $t > 0$,

$$\|\hat{Q}_t f - c_f\|_{L^2(E,\mu)} \leq 2c_1 e^{-\nu t} \|f\|_{L^2(E,\mu)}. \tag{2.12}$$

Since for $t > 1/2$ we have

$$\overline{q}(t,x,y) = \int_E \overline{q}(1/2,x,z)\overline{q}(t-1/2,z,y)\mu(z)m(dz) = \hat{Q}_{t-1/2} f_x(y), \tag{2.13}$$

with $f_x(z) = \overline{q}(1/2,x,z)$, we obtain

$$c_{f_x} = \int_E \overline{q}(1/2,x,z)\mu(z)m(dz) = 1. \tag{2.14}$$

Let $c_2^2 = \sup_{x\in E}\int_E \overline{q}^2(1/2,x,z)\mu(z)m(dz)$. From (2.12), (2.13) and (2.14) we obtain for any $t > 1/2$,

$$\begin{aligned}
&\sup_{x\in E}\int_E |\overline{q}(t,x,y)-1|^2\mu(y)m(dy) = \sup_{x\in E}\int_E |\hat{Q}_{t-\frac{1}{2}} f_x(y) - c_{f_x}|^2\mu(y)m(dy)\\
&\quad\le \sup_{\|f\|_{L^2(E,\mu)}\le c_2}\int_E |\hat{Q}_{t-\frac{1}{2}} f(y) - c_f|^2\mu(y)m(dy) \le \left(2c_1c_2e^{-\nu(t-\frac{1}{2})}\right)^2.
\end{aligned}$$

Thus for any $t > 1/2$, there exist $c_3 > 0$ such that

$$\begin{aligned}
&\sup_{x\in E}\int_E |\overline{q}(t,x,z)-1|^2\mu(z)m(dz) \le c_3e^{-2\nu t},\\
&\sup_{y\in E}\int_E |\overline{q}(t,z,y)-1|^2\mu(z)m(dz) \le c_3e^{-2\nu t}.
\end{aligned} \tag{2.15}$$

By the semigroup property of $\overline{q}(t,x,y)$, we have

$$\begin{aligned}
\overline{q}(t,x,y)-1 &= \int_E \overline{q}(t/2,x,z)\overline{q}(t/2,z,y)\mu(z)m(dz) - 1\\
&= \int_E \overline{q}(t/2,x,z)\overline{q}(t/2,z,y)\mu(z)m(dz) - \int_E \overline{q}(t/2,x,z)\mu(z)m(dz)\\
&\quad - \int_E \overline{q}(t/2,z,y)\mu(z)m(dz) + \int_E \mu(z)m(dz)\\
&= \int_E \left(\overline{q}(t/2,x,z)-1\right)\left(\overline{q}(t/2,z,y)-1\right)\mu(z)m(dz).
\end{aligned}$$

Therefore, from (2.15), we obtain for any $t > 1$,

$$\begin{aligned}
&\sup_{(x,y)\in E\times E} |\overline{q}(t,x,y)-1|^2\\
&\le \left(\sup_{x\in E}\int_E |\overline{q}(t/2,x,z)-1|^2\mu(z)m(dz)\right)\left(\sup_{y\in E}\int_E |\overline{q}(t/2,z,y)-1|^2\mu(z)m(dz)\right)\\
&\le c_3^2e^{-2\nu t}.
\end{aligned}$$

□

In the remainder of this section we assume that the semigroups $\{P_t\}$ and $\{\hat{P}_t\}$ are associated with two dual Hunt processes X and $\hat{X}$, respectively. We are going to use SH^+ to denote the family of nonnegative superharmonic functions of X, or equivalently, the family of excessive functions of X. For any $h \in SH^+$, we use P_h^x to denote the law of the h-conditioned process X and use E_h^x to denote the expectation with respect to P_h^x. The following result gives some important consequences of intrinsic ultracontractivity.

Theorem 2.8. *Suppose that $\{P_t\}$ and $\{\hat{P}_t\}$ are intrinsically ultracontractive and that $\lambda_0 < 0$.*

(1) *If ζ_h stands for the lifetime of the h-conditioned process X, then*

$$\sup_{x\in E, h\in SH^+} E^x_h(\zeta_h) < \infty.$$

(2) *For any $h \in SH^+$, we have*

$$\lim_{t\uparrow\infty} e^{-\lambda_0 t} P^x_h(\zeta_h > t) = \frac{\phi_0(x)}{Mh(x)} \int_E \psi_0(y)h(y)m(dy).$$

In particular,

$$\lim_{t\uparrow\infty} \frac{1}{t} \log P^x_h(\zeta_h > t) = \lambda_0.$$

Proof. (1) For any $h \in SH^+$, it follows from Proposition 2.5 that there exists a constant $c_1 > 0$ such that

$$h(x) \geq \int_E p(1,x,y)h(y)m(dy) \geq c_1\phi_0(x) \int_E \psi_0(y)h(y)m(dy), \quad x \in E.$$

Therefore

$$\sup_{x\in E, h\in SH^+} \frac{\phi_0(x)}{h(x)} \int_E \psi_0(y)h(y)m(dy) \leq c_1^{-1} < \infty.$$

By Theorem 2.7 we know there exists a constant $c_2 > 0$ such that

$$\begin{aligned}
\sup_{x\in E, h\in SH^+} E^x_h(\zeta_h) &= \sup_{x\in E, h\in SH^+} \frac{1}{h(x)} \int_0^\infty \int_E p(t,x,y)h(y)m(dy)dt \\
&\leq \sup_{x\in E, h\in SH^+} \left(\frac{1}{h(x)} \int_0^1 \int_E p(t,x,y)h(y)m(dy)dt + \frac{1}{h(x)} \int_1^\infty \int_E p(t,x,y)h(y)m(dy)dt \right) \\
&\leq 1 + c_2 \int_1^\infty e^{\lambda_0 t} dt \sup_{x\in E, h\in SH^+} \frac{\phi_0(x)}{h(x)} \int_E \psi_0(y)h(y)m(dy) < \infty.
\end{aligned}$$

(2) By Theorem 2.7 we have

$$\begin{aligned}
\lim_{t\uparrow\infty} e^{-\lambda_0 t} P^x_h(\zeta_h > t) &= \lim_{t\uparrow\infty} e^{-\lambda_0 t} \frac{1}{h(x)} \int_E p(t,x,y)h(y)m(dy) \\
&= \frac{\phi_0(x)}{Mh(x)} \int_E \psi_0(y)h(y)m(dy).
\end{aligned}$$

□

3. IU for Non-symmetric Diffusion Semigroups

In this section we assume that $a_{ij}(x), b_i(x), i,j = 1,\ldots,d$, and $c(x)$ are bounded C^∞ functions on $\mathbf{R}^d$. We will also assume that the functions $\partial b_i/\partial x_i$, $i = 1,\ldots,d$, are bounded and that the matrix $(a_{ij}(x))$ is symmetric and uniformly elliptic, that is, there is a positive number λ such that

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \lambda \|\xi\|^2 \quad \text{for all } \xi \in \mathbf{R}^d.$$

In this section we assume that L is a second order differential operator

$$L=\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)-\sum_{i=1}^{d}b_i\frac{\partial}{\partial x_i}-c.$$

The formal adjoint of L is given by

$$\hat{L}=:\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{d}b_i\frac{\partial}{\partial x_i}-\left(c-\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i}\right).$$

In this section, we will always assume that D is a bounded domain in $\mathbf{R}^d$. Let $p(t,x,y)$ be the Dirichlet heat kernel of the operator L in D. For any $t>0$, define

$$P_tf(x):=\int_D p(t,x,y)f(y)dy,\quad \hat{P}_tf(x):=\int_D p(t,y,x)f(y)dy.$$

Then $\{P_t\}$ and $\{\hat{P}_t\}$ are both strongly continuous semigroups in $L^2(D,dx)$. The generator of the semigroup $\{P_t\}$ is $L|_D$ with zero Dirichlet boundary condition and the generator of the semigroup $\{\hat{P}_t\}$ is $\hat{L}|_D$ with zero Dirichlet boundary condition. By definition, we have

$$\int_D f(x)P_tg(x)dx=\int_D g(x)\hat{P}_tf(x)dx.$$

The bilinear form associated with $\{P_t\}$ and $\{\hat{P}_t\}$ is given by $(\mathcal{E},H_0^1(D))$, where

$$\mathcal{E}(u,v):=\sum_{i,j=1}^{d}\int_D a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx+\sum_{i=1}^{d}\int_D b_iv\frac{\partial u}{\partial x_i}dx+\int_D cuvdx,\quad u,v\in H_0^1(D).$$

If we assume that c is a nonnegative function, then there is a diffusion process X with generator L and $\{P_t\}$ is the semigroup of X^D, the process obtained by killing the process X upon exiting D. If we further assume that

$$c(x)-\sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i}(x)\geq 0,\quad x\in\mathbf{R}^d,$$

then there is a diffusion process $\hat{X}$ with generator $\hat{L}$ and $\{\hat{P}_t\}$ is the semigroup of $\hat{X}^D$, the process obtained by killing the process $\hat{X}$ upon exiting D, and the bilinear form $(\mathcal{E},H_0^1(D))$ is a Dirichlet form in the sense of [17].

It follows from Jentzsch's Theorem (Theorem V.6.6 on page 337 of [22]) that the common value $\lambda_0:=\sup\mathrm{Re}(\sigma(L|_D))=\sup\mathrm{Re}(\sigma(\hat{L}|_D))$ is an eigenvalue of multiplicity 1 for both $L|_D$ and $\hat{L}|_D$, and that an eigenfunction ϕ_0 of $L|_D$ associated with λ_0 can be chosen to be strictly positive with $\|\phi_0\|_{L^2(D,dx)}=1$ and an eigenfunction ψ_0 of $\hat{L}|_D$ associated with λ_0 can be chosen to be strictly positive with $\|\psi_0\|_{L^2(D,dx)}=1$. It is well-known that ϕ_0 and ψ_0 are C^∞ in D.

Define, for any $(t,x,y)\in(0,\infty)\times D\times D$,

$$q(t,x,y):=\frac{e^{-\lambda_0t}}{\phi_0(x)}p(t,x,y)\phi_0(y),\quad \hat{q}(t,x,y):=\frac{e^{-\lambda_0t}}{\psi_0(y)}p(t,x,y)\psi_0(x).$$

Then it is easy to check that the operators $\{Q_t\}$ and $\{\hat{Q}_t\}$ defined by

$$Q_t f(x) := \int_D q(t,x,y)f(y)dy, \quad \hat{Q}_t f(x) := \int_D \hat{q}(t,y,x)f(y)dy$$

form Markov semigroups on D.

Define a function $\mu(x)$ by

$$\mu(x) := \frac{\phi_0(x)\psi_0(x)}{\int_D \phi_0(y)\psi_0(y)dy}.$$

Then the measure $\mu(x)dx$ is a probability measure on D. From Section 2 we know that $\{Q_t\}$ and $\{\hat{Q}_t\}$ are dual semigroups on $L^2(D,\mu)$ and that μ is an invariant function for both $\{Q_t\}$ and $\{\hat{Q}_t\}$.

The generators of $\{Q_t\}$ and $\{\hat{Q}_t\}$ are given respectively by

$$\frac{1}{\phi_0}(L|_D - \lambda_0)(\phi_0 f) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial f}{\partial x_j}\right) - \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i} + \frac{2}{\phi_0}\sum_{i,j=1}^d a_{ij}\frac{\partial \phi_0}{\partial x_j}\frac{\partial f}{\partial x_i}$$

$$\frac{1}{\psi_0}(\hat{L}|_D - \lambda_0)(\psi_0 f) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial f}{\partial x_j}\right) + \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i} + \frac{2}{\psi_0}\sum_{i,j=1}^d a_{ij}\frac{\partial \psi_0}{\partial x_j}\frac{\partial f}{\partial x_i}.$$

Put

$$\mathcal{F} := \left\{ f \in L^2(D,\mu); f\phi_0, f\psi_0 \in H_0^1(D) \right\},$$

and define a bilinear form $\mathcal{Q}$ on $\mathcal{F}$ by

$$\mathcal{Q}(f,g) := \frac{1}{M}\mathcal{E}(f\phi_0, g\psi_0) + \lambda_0 \int_D \mu(x)f(x)g(x)dx, \quad f,g \in \mathcal{F}. \tag{3.1}$$

It is obvious that $C_c^\infty(D)$ is contained in $\mathcal{F}$. It can be checked by elementary calculations (see the appendix for a proof) that for any bounded $f \in C^1(D) \cap \mathcal{F}$

$$\mathcal{Q}(f,f) = \int_D \mu(x) \sum_{i,j=1}^d \frac{\partial f(x)}{\partial x_i} a_{ij}(x) \frac{\partial f(x)}{\partial x_j} dx. \tag{3.2}$$

We are going to use $(\tilde{\mathcal{E}}, H_0^1(D))$ to denote the symmetric part of $(\mathcal{E}, H_0^1(D))$:

$$\tilde{\mathcal{E}}(u,v) := \frac{1}{2}\left(\mathcal{E}(u,v) + \mathcal{E}(v,u)\right), \quad u,v \in H_0^1(D).$$

Then $(\tilde{\mathcal{E}}, H_0^1(D))$ is a symmetric bilinear form on $L^2(D)$ and its generator is given by

$$\tilde{L}|_D = \frac{L|_D + \hat{L}|_D}{2} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) - c + \frac{1}{2}\sum_{i=1}^d \frac{\partial b_i}{\partial x_i}.$$

Let $(\tilde{P}_t)$ be the semigroup associated with the form $(\tilde{\mathcal{E}}, H_0^1(D))$. Then $(\tilde{P}_t)$ has a strictly positive continuous transition density $\tilde{p}(t,x,y)$ with respect to the Lebesgue measure on D.

Let $\tilde{\lambda}_0 = \sup \sigma(\tilde{L}|_D)$. Then $\tilde{\lambda}_0$ is an eigenvalue of $\tilde{L}|_D$ of multiplicity 1. Let φ_0 be the positive eigenfunction of $\tilde{L}|_D$ corresponding to $\tilde{\lambda}_0$ such that $\int_D \varphi_0^2(x)dx = 1$.

Define, for any $(t,x,y) \in (0,\infty) \times D \times D$,

$$\tilde{q}(t,x,y) := \frac{e^{-\tilde{\lambda}_0 t}}{\varphi_0(x)} \tilde{p}(t,x,y)\varphi_0(y).$$

Then the semigroup $\{\tilde{Q}_t\}$ defined by

$$\tilde{Q}_t f(x) := \int_D \tilde{q}(t,x,y)f(y)dy$$

is a strongly continuous symmetric Markov semigroup on $L^2(D,\varphi_0^2)$.

Let $(\tilde{\mathcal{Q}}, D(\tilde{\mathcal{Q}}))$ be the Dirichlet form on $L^2(D,\varphi_0^2)$ associated with $\{\tilde{Q}_t\}$. Then it follows from [4] and [6] that

$$D(\tilde{\mathcal{Q}}) = \left\{ f \in L^2(D,\varphi_0^2); f\varphi_0 \in H_0^1(D) \right\}$$

and that

$$\tilde{\mathcal{Q}}(f,f) = \tilde{\mathcal{E}}(f\varphi_0, f\varphi_0) + \tilde{\lambda}_0 \int_D \varphi_0^2(x) f^2(x)dx, \quad f \in D(\tilde{\mathcal{Q}}). \tag{3.3}$$

We also know from [3] and [4] that for any $f \in D(\tilde{\mathcal{Q}})$,

$$\tilde{\mathcal{Q}}(f,f) = \int_D \varphi_0^2(x) \sum_{i,j=1}^d \frac{\partial f(x)}{\partial x_i} a_{ij}(x) \frac{\partial f(x)}{\partial x_j} dx. \tag{3.4}$$

In the remainder of this section we will always assume that D is a bounded Lipschitz domain. Then we have the following

Lemma 3.1. *The functions ϕ_0, ψ_0 and φ_0 are comparable, that is, there exists constants $c_1, c_2 \geq 1$ such that for all $x \in D$,*

$$c_1^{-1}\phi_0(x) \leq \varphi_0(x) \leq c_1\phi_0(x), \quad c_2^{-1}\psi_0(x) \leq \varphi_0(x) \leq c_2\psi_0(x).$$

Proof. Take a positive constant λ such that

$$c(x) + \lambda \geq 0, \quad c(x) + \lambda - \sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(x) \geq 0 \quad \text{for all } x \in \mathbf{R}^d.$$

The functions

$$G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)dt, \quad \tilde{G}_\lambda(x,y) = \int_0^\infty e^{-\lambda t} \tilde{p}(t,x,y)dt$$

are finite off the diagonal of $D \times D$. It follows from [2] that both G_λ and $\tilde{G}_\lambda$ are comparable to the Green function of the operator

$$\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j} \right)$$

with zero Dirichlet boundary condition on ∂D. Our assertion now follows easily from Theorem 1.5 in [18]. $\square$

Using this lemma, one can easily check that $\{Q_t\}$ and $\{\hat{Q}_t\}$ are strongly continuous semigroups in $L^2(D, \mu)$.

LEMMA 3.2. $\mathcal{F} \cap L^\infty(D, \mu) = \mathcal{D}(\tilde{\mathcal{Q}}) \cap L^\infty(D, \varphi_0^2)$.

PROOF. If $f \in \mathcal{D}(\tilde{\mathcal{Q}}) \cap L^\infty(D, \varphi_0^2)$, then $f\varphi_0 \in H_0^1(D)$ and f is bounded. Since

$$\begin{aligned}\nabla(f\phi_0) &= \nabla(f\varphi_0 \cdot \frac{\phi_0}{\varphi_0}) = \frac{\phi_0}{\varphi_0}\nabla(f\varphi_0) + f\varphi_0\nabla(\frac{\phi_0}{\varphi_0}) \\ &= \frac{\phi_0}{\varphi_0}\nabla(f\varphi_0) + f\nabla\phi_0 - f \cdot \frac{\phi_0}{\varphi_0}\nabla\varphi_0,\end{aligned}$$

we know by the previous lemma that $f\phi_0 \in H_0^1(D)$. Similarly we also have $f\psi_0 \in H_0^1(D)$. Therefore we know that $f \in \mathcal{F} \cap L^\infty(D, \mu)$.

Now suppose that $f \in \mathcal{F} \cap L^\infty(D, \mu)$. Then $f\phi_0, f\psi_0 \in H_0^1(D)$. Since

$$\begin{aligned}\nabla(f\varphi_0) &= \nabla(f\phi_0 \cdot \frac{\varphi_0}{\phi_0}) = \frac{\varphi_0}{\phi_0}\nabla(f\phi_0) + f\phi_0\nabla(\frac{\varphi_0}{\phi_0}) \\ &= \frac{\varphi_0}{\phi_0}\nabla(f\phi_0) + f\nabla\varphi_0 - f \cdot \frac{\varphi_0}{\phi_0}\nabla\phi_0,\end{aligned}$$

we know by the previous lemma that $f\varphi_0 \in H_0^1(D)$. Therefore $f \in D(\tilde{\mathcal{Q}}) \cap L^\infty(D, \varphi_0^2)$. $\square$

Now, combining this lemma above with (3.2) and (3.4), we immediately arrive at the following

LEMMA 3.3. *There exists a constant $c > 1$ such that for any bounded $f \in C^1(D) \cap \mathcal{F}$,*

$$\frac{1}{c}\tilde{\mathcal{Q}}(f, f) \leq \mathcal{Q}(f, f) \leq c\tilde{\mathcal{Q}}(f, f). \tag{3.5}$$

By following the argument in the proofs of Theorem 4.6 and Theorem 5.2 in [6] (see also the proof of Theorem 1 in [4]), we can get the following result.

LEMMA 3.4. *For any $\epsilon > 0$ and bounded $f \in D(\tilde{\mathcal{Q}})$, we have*

$$\int_D \varphi_0^2 f^2 \log|f| dx \leq \epsilon\tilde{\mathcal{Q}}(f, f) + \beta(\epsilon)\|f\|_{L^2(D,\varphi_0^2)}^2 + \|f\|_{L^2(D,\varphi_0^2)}^2 \log\|f\|_{L^2(D,\varphi_0^2)}$$

with

$$\beta(\epsilon) = \begin{cases} -c_1 \log\epsilon + c_2, & \epsilon \leq 1 \\ c_1 + c_2, & \epsilon > 1 \end{cases}$$

for some constant $c_1, c_2 > 0$.

PROOF. We omit the details. $\square$

Combining the result above with (3.1) and Lemma 3.3, we can easily get the following

LEMMA 3.5. *For any $\epsilon > 0$ and bounded $f \in C^1(D) \cap \mathcal{F}$, we have*

$$\int_D \mu f^2 \log |f| dx \le \epsilon \mathcal{Q}(f,f) + \beta(\epsilon) \|f\|^2_{L^2(D,\mu)} + \|f\|^2_{L^2(D,\mu)} \log \|f\|_{L^2(D,\mu)} \tag{3.6}$$

with

$$\beta(\epsilon) = \begin{cases} -c_1 \log \epsilon + c_2, & \epsilon \le 1 \\ c_1 + c_2, & \epsilon > 1 \end{cases} \tag{3.7}$$

for some constant $c_1, c_2 > 0$.

PROOF. If (3.6) is true for $f \in C^1(D) \cap \mathcal{F}$ with $|f| \le 1$, then (3.6) is true for every bounded $f \in C^1(D) \cap \mathcal{F}$ by applying to $f/\|f\|_\infty$. Thus we will assume that $f \in C^1(D) \cap \mathcal{F}$ with $|f| \le 1$.

We know from (3.1), Lemma 3.1 and Lemma 3.3 that there exists a constant $L > 1$ such that

$$\frac{1}{L} \varphi_0^2 \le \mu \le L \varphi_0^2 \quad \text{and} \quad \frac{1}{L} \tilde{\mathcal{Q}}(f,f) \le \mathcal{Q}(f,f) \le L \tilde{\mathcal{Q}}(f,f). \tag{3.8}$$

Since $\log |f| \le 0$, from (3.8) we have

$$\int_D \mu f^2 \log |f| dx \le \frac{1}{L} \int_D \varphi_0^2 f^2 \log |f| dx.$$

Now, applying the previous lemma and (3.8), it follows that the above is bounded by

$$\begin{aligned} &\frac{1}{L} \epsilon \tilde{\mathcal{Q}}(f,f) + \frac{1}{L} \beta(\epsilon) \|f\|^2_{L^2(D,\varphi_0^2)} + \frac{1}{L} \|f\|^2_{L^2(D,\varphi_0^2)} \log \|f\|_{L^2(D,\varphi_0^2)} \\ &\quad \le \epsilon \mathcal{Q}(f,f) + (\beta(\epsilon) + \log L) \|f\|^2_{L^2(D,\mu)} + \|f\|^2_{L^2(D,\mu)} \log \|f\|_{L^2(D,\mu)}. \end{aligned}$$

□

LEMMA 3.6. *For any $p \in (2, \infty)$, $\epsilon > 0$ and bounded nonnegative $g \in C^1(D) \cap \mathcal{F}$, we have*

$$\int_D \mu(x) g^p(x) \log g(x) dx \le \epsilon \mathcal{Q}(g, g^{p-1}) + 2\beta(\epsilon) p^{-1} \|g\|^p_{L^p(D,\mu)} + \|g\|^p_{L^p(D,\mu)} \log \|g\|_{L^p(D,\mu)},$$

$$\int_D \mu(x) g^p(x) \log g(x) dx \le \epsilon \mathcal{Q}(g^{p-1}, g) + 2\beta(\epsilon) p^{-1} \|g\|^p_{L^p(D,\mu)} + \|g\|^p_{L^p(D,\mu)} \log \|g\|_{L^p(D,\mu)},$$

where $\beta(\epsilon)$ is the function defined in (3.7)*.*

PROOF. It is well-known that if g is a bounded nonnegative function in $\mathcal{D}(\tilde{\mathcal{Q}})$, then $g^{p/2}$ and g^{p-1} are also in $\mathcal{D}(\tilde{\mathcal{Q}})$. Thus it follows from Lemma 3.2 that if f is a bounded nonnegative function in $C^1(D) \cap \mathcal{F}$, then, for $p > 2$, $f^{p/2}$ and f^{p-1} are also in $C^1(D) \cap \mathcal{F}$. By using elementary calculations, one can check that for any bounded nonnegative function $f \in C^1(D) \cap \mathcal{F}$,

$$\mathcal{Q}(f^{p/2}, f^{p/2}) = \frac{p^2}{4(p-1)} \mathcal{Q}(f, f^{p-1}) = \frac{p^2}{4(p-1)} \mathcal{Q}(f^{p-1}, f) \ge 0. \tag{3.9}$$

(See the appendix for a proof.) Putting $f = g^{p/2}$ in (3.6) we get

$$\frac{p}{2}\int_D \mu(x)g^p(x)\log g(x)dx \le \epsilon\mathcal{Q}(g^{p/2}, g^{p/2}) + \beta(\epsilon)\|g\|^p_{L^p(D,\mu)} + \frac{p}{2}\|g\|^p_{L^p(D,\mu)}\log\|g\|_{L^p(D,\mu)}.$$

Therefore we have by (3.9) that

$$\int_D \mu(x)g^p(x)\log g(x)dx \le \frac{\epsilon p}{2(p-1)}\mathcal{Q}(g, g^{p-1}) + 2\beta(\epsilon)p^{-1}\|g\|^p_{L^p(D,\mu)} + \|g\|^p_{L^p(D,\mu)}\log\|g\|_{L^p(D,\mu)},$$

$$\int_D \mu(x)g^p(x)\log g(x)dx \le \frac{\epsilon p}{2(p-1)}\mathcal{Q}(g^{p-1}, g) + 2\beta(\epsilon)p^{-1}\|g\|^p_{L^p(D,\mu)} + \|g\|^p_{L^p(D,\mu)}\log\|g\|_{L^p(D,\mu)}.$$

Now the desired assertions follows immediately from the inequalities above. □

It is clear that for the function $\beta(\epsilon)$ defined in (3.7)

$$M(t) = \frac{1}{t}\int_0^t \beta(\epsilon)d\epsilon \tag{3.10}$$

is finite for all $t > 0$. Now we can state the main result of this paper.

THEOREM 3.7. *For any $t > 0$, we have*

$$\max\{\|Q_t\|_{L^\infty(D,\mu),L^2(D,\mu)}, \|\hat{Q}_t\|_{L^\infty(D,\mu),L^2(D,\mu)}\} \le e^{M(t)},$$

where $M(t)$ is the function defined in (3.10).

PROOF. For any $t > 0$ and $p > 2$, put

$$\epsilon(p) = \frac{2t}{p}, \quad \Gamma(p) = \frac{2\beta(\epsilon(p))}{p},$$

where $\beta(\epsilon)$ is the function defined in (3.7). Then we have

$$M(t) = \int_2^\infty \frac{2\beta(\epsilon(p))}{p^2}dp = \int_2^\infty \frac{\Gamma(p)}{p}dp.$$

It follows from Lemma 3.6 that for all $p \in (2, \infty)$ and all bounded nonnegative functions $f \in C^1(D) \cap \mathcal{F}$ we have

$$\begin{aligned}\int_D &\mu(x)f^p(x)\log f(x)dx \\ &\le \epsilon(p)\mathcal{Q}(f, f^{p-1}) + \Gamma(p)\|f\|^p_{L^p(D,\mu)} + \|f\|^p_{L^p(D,\mu)}\log\|f\|_{L^p(D,\mu)},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\int_D &\mu(x)f^p(x)\log f(x)dx \\ &\le \epsilon(p)\mathcal{Q}(f^{p-1}, f) + \Gamma(p)\|f\|^p_{L^p(D,\mu)} + \|f\|^p_{L^p(D,\mu)}\log\|f\|_{L^p(D,\mu)}.\end{aligned} \tag{3.12}$$

Note that for any bounded function f, we have

$$\nabla(\psi_0 Q_t f) = e^{-\lambda_0 t}\nabla(\frac{\psi_0}{\phi_0}P_t(\phi_0 f)) = e^{-\lambda_0 t}\frac{\psi_0}{\phi_0}\nabla(P_t(\phi_0 f)) + (Q_t f)\cdot\nabla\psi_0 - (Q_t f)\cdot\frac{\psi_0}{\phi_0}\nabla\phi_0,$$

and

$$\nabla(\phi_0(x)\hat{Q}_t f) = e^{-\lambda_0 t}\nabla(\frac{\phi_0}{\psi_0}\hat{P}_t(\psi_0 f)) = e^{-\lambda_0 t}\frac{\phi_0}{\psi_0}\nabla(\hat{P}_t(\psi_0 f)) + (\hat{Q}_t f)\cdot\nabla\phi_0 - (\hat{Q}_t f)\cdot\frac{\phi_0}{\psi_0}\nabla\psi_0.$$

Thus one can easily see that for any bounded nonnegative function f, $Q_t f$, $\hat{Q}_t f$ are both bounded nonnegative functions in $\mathcal{F}$. It is well-known that if g is a bounded nonnegative function in $\mathcal{D}(\tilde{\mathcal{Q}})$, then g^{p-1} is also in $\mathcal{D}(\tilde{\mathcal{Q}})$. Now, using Lemma 3.2, we can see that $(Q_t f)^{p-1}$ and $(\hat{Q}_t f)^{p-1}$ are both bounded nonnegative functions in $\mathcal{F}$. From the fact that $a_{i,j}(x), b_i(x)$ and $c(x)$ are smooth, we have $Q_t f$, $\hat{Q}_t f$ are in $C^1(D)$. Moreover, $Q_t f$ and $\hat{Q}_t f$ are in the domains of the generators of $\{Q_t\}$ and $\{\hat{Q}_t\}$, respectively. Thus $\phi_0 Q_t f$ and $\psi_0 \hat{Q}_t f$ are in the domains of the generators of $\{P_t\}$ and $\{\hat{P}_t\}$, respectively. Now we can repeat the proof of Theorem 2.2.7 of [9] (both for $Q_t f$ and $\hat{Q}_t f$) to arrive at the desired conclusion. We omit the details. □

REMARK 3.8. The observation that one can easily modify the argument in the symmetric case to get hypercontractivity from the inequalities (3.11) and (3.12) was first pointed out in [19].

Now we arrive at the main result of this paper.

THEOREM 3.9. *The semigroups $\{P_t\}$ and $\{\hat{P}_t\}$ are intrinsically ultracontractive.*

PROOF. This follows easily from Theorem 3.7 and the observation made after Definition 2.4. □

4. Appendix

In this section, we give the proofs of (3.2) and (3.9). Note that, unlike [19], we use no integration by parts argument in the proofs of lemmas below.

LEMMA 4.1. *For any bounded nonnegative $f \in C^1(D) \cap \mathcal{F}$ and $p \geq 2$,*

$$\sum_{i,j=1}^{d} \int_D a_{ij} \left[f^{p-1} \left(\phi_0 \frac{\partial f}{\partial x_i} \frac{\partial \psi_0}{\partial x_j} + (p-1) \psi_0 \frac{\partial \phi_0}{\partial x_i} \frac{\partial f}{\partial x_j} \right) + f^p \frac{\partial \phi_0}{\partial x_i} \frac{\partial \psi_0}{\partial x_j} \right] dx$$
$$+ \sum_{i=1}^{d} \int_D b_i \left(f^{p-1} \psi_0 \phi_0 \frac{\partial f}{\partial x_i} + f^p \psi_0 \frac{\partial \phi_0}{\partial x_i} \right) dx + \int_D c \phi_0 \psi_0 f^p dx + \lambda_0 \int_D \phi_0 \psi_0 f^p dx \;=\; 0.$$

PROOF. Note that we have

$$\mathcal{Q}(1, f^p) = \frac{1}{M} \mathcal{E}(\phi_0, f^p \psi_0) + \lambda_0 \int_D \mu f^p dx = -\frac{\lambda_0}{M} \int_D \phi_0 f^p \psi_0 dx + \lambda_0 \int_D \mu f^p dx = 0$$

and

$$\mathcal{Q}(f^p, 1) = \frac{1}{M} \mathcal{E}(f^p \phi_0, \psi_0) + \lambda_0 \int_D \mu f^p dx = -\frac{\lambda_0}{M} \int_D f^p \phi_0 \psi_0 dx + \lambda_0 \int_D \mu f^p dx = 0.$$

Thus

$$
\begin{aligned}
0 &= M\mathcal{Q}(f^p,1)+M(p-1)\mathcal{Q}(1,f^p)\\
&= \mathcal{E}(f^p\phi_0,\psi_0)+(p-1)\mathcal{E}(\phi_0,f^p\psi_0)+Mp\lambda_0\int_D \mu f^p dx\\
&= \sum_{i,j=1}^d\int_D a_{ij}\frac{\partial}{\partial x_i}\left(f^p\phi_0\right)\frac{\partial\psi_0}{\partial x_j}dx+(p-1)\sum_{i,j=1}^d\int_D a_{ij}\frac{\partial\phi_0}{\partial x_i}\frac{\partial}{\partial x_j}\left(f^p\psi_0\right)dx\\
&\quad+\sum_{i=1}^d\int_D b_i\psi_0\frac{\partial}{\partial x_i}\left(f^p\phi_0\right)dx+(p-1)\sum_{i=1}^d\int_D b_i f^p\psi_0\frac{\partial\phi_0}{\partial x_i}dx\\
&\quad+p\int_D c\phi_0\psi_0 f^p dx+Mp\lambda_0\int_D\mu f^p dx\\
&= \sum_{i,j=1}^d\int_D a_{ij}\left[f^{p-1}\left(p\phi_0\frac{\partial f}{\partial x_i}\frac{\partial\psi_0}{\partial x_j}+(p-1)p\psi_0\frac{\partial\phi_0}{\partial x_i}\frac{\partial f}{\partial x_j}\right)+pf^p\frac{\partial\phi_0}{\partial x_i}\frac{\partial\psi_0}{\partial x_j}\right]dx\\
&\quad+\sum_{i=1}^d\int_D b_i\left(pf^{p-1}\psi_0\phi_0\frac{\partial f}{\partial x_i}+pf^p\psi_0\frac{\partial\phi_0}{\partial x_i}\right)dx\\
&\quad+p\int_D c\phi_0\psi_0 f^p dx+p\lambda_0\int_D\phi_0\psi_0 f^p dx.
\end{aligned}
$$

□

Lemma 4.2. *For any bounded nonnegative $f\in C^1(D)\cap\mathcal{F}$ and $p\geq 2$,*

$$
\begin{aligned}
\mathcal{Q}(f,f^{p-1}) &= \int_D\mu(x)\sum_{i,j=1}^d\frac{\partial f(x)}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}\left(f^{p-1}(x)\right)dx\\
&= (p-1)\int_D\mu(x)f^{p-2}(x)\sum_{i,j=1}^d\frac{\partial f(x)}{\partial x_i}a_{ij}(x)\frac{\partial f(x)}{\partial x_j}dx. \qquad (4.1)
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\mathcal{Q}(f,f^{p-1})=\frac{1}{M}\mathcal{E}(f\phi_0,f^{p-1}\psi_0)+\lambda_0\int_D\mu(x)f^p(x)dx\\
&=\frac{1}{M}\sum_{i,j=1}^d\int_D a_{ij}\frac{\partial}{\partial x_i}\left(f\phi_0\right)\frac{\partial}{\partial x_j}\left(f^{p-1}\psi_0\right)dx+\frac{1}{M}\sum_{i=1}^d\int_D b_i f^{p-1}\psi_0\frac{\partial}{\partial x_i}\left(f\phi_0\right)dx\\
&\ +\frac{1}{M}\int_D c\phi_0\psi_0 f^p dx+\lambda_0\int_D\mu f^p dx.
\end{aligned}
$$

Now, using the product rule, we get

$$\begin{aligned}\mathcal{Q}(f, f^{p-1}) &= \sum_{i,j=1}^{d} \int_D a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \left(f^{p-1}\right) \mu(x) dx \\ &+ \frac{1}{M} \sum_{i,j=1}^{d} \int_D a_{ij} \left[\phi_0 f^{p-1} \frac{\partial f}{\partial x_i} \frac{\partial \psi_0}{\partial x_j} + (p-1) f^{p-1} \psi_0 \frac{\partial \phi_0}{\partial x_i} \frac{\partial f}{\partial x_j} + f^p \frac{\partial \phi_0}{\partial x_i} \frac{\partial \psi_0}{\partial x_j}\right] dx \\ &+ \frac{1}{M} \sum_{i=1}^{d} \int_D b_i \left(f^{p-1} \psi_0 \phi_0 \frac{\partial f}{\partial x_i} + f^p \psi_0 \frac{\partial \phi_0}{\partial x_i}\right) dx + \frac{1}{M} \int_D c \phi_0 \psi_0 f^p dx \\ &+ \frac{\lambda_0}{M} \int_D \phi_0 \psi_0 f^p dx.\end{aligned}$$

Applying Lemma 4.1 to the above equation, we arrive at the conclusion of the lemma. □

It is easy to see that for $p = 2$ the proofs of Lemma 4.1 and Lemma 4.2 work for any bounded $f \in C^1(D) \cap \mathcal{F}$ and in this way we arrive at (3.2). By taking $f = g^{p/2}$ in (3.2), we get the first equality in (3.9) from (4.1). The proof of the other equality in (3.9) is similar.

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