Noncommutative Geometry
Lecture 3: Cyclic Cohomology

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Setup
\[\mathcal{A}\text{ is a unital algebra over } \mathbb{C}.\]

Definition (Hochschild Complex)

1. The space of \textit{n-cochains}, \( n \geq 0 \), is
\[C^n(\mathcal{A}) := \{(n + 1)\text{-linear forms } \varphi : \mathcal{A}^{n+1} \to \mathbb{C}\}, \quad n \geq 0.\]

2. The \textit{coboundary} \( b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A}) \) is given by
\[(b\varphi)(a^0, \ldots, a^{n+1}) = \sum_{0 \leq j \leq n} (-1)^j \varphi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n).\]
Lemma

We have $b^2 = 0$.

Definition

The cohomology of the complex $(C^\bullet(A), b)$ is called the Hochschild cohomology of $A$ and is denoted $HH^\bullet(A)$. 
Hochschild Cohomology

Example

Let \( C \) be a \( k \)-dimensional current on a compact manifold \( M \). Define a \( k \)-cochain on \( \mathcal{A} = C^\infty(M) \) by

\[
\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!} \langle C, f^0 df^1 \wedge \cdots \wedge df^k \rangle.
\]

Then \( b\varphi_C = 0 \). In fact, we have

Theorem (Hochschild-Kostant-Rosenberg, Connes)

There is an isomorphism,

\[
HH^k(M) \cong D'_k(M).
\]
Definition (Cyclic Cochains)

A cochain $\varphi \in C^n(A)$, $n \geq 0$, is cyclic when

$$\varphi(a^1, \cdots, a^n, a^0) = (-1)^n \varphi(a^0, \cdots, a^n) \quad \forall a^j \in A.$$ 

We denote by $C^n_\lambda(A)$ the space of cyclic $n$-cochains.

Example

Let $C$ be a $k$-dimensional current on a compact manifold $M$. We saw it defines a Hochschild cocycle, Then

$$C \text{ closed (i.e., } d^t C = 0) \implies \varphi_C \text{ cyclic.}$$
Lemma

\[ \varphi \text{ cyclic } \implies b\varphi \text{ cyclic.} \]

Definition

The cohomology of the sub-complex \((C^\bullet_{\lambda}(A), b)\) is called the cyclic cohomology of \(A\) and is denoted \(HC^\bullet(A)\).
Definition

Define $B : C^n(A) \rightarrow C^{n-1}(A)$ by

$$B = AB_0,$$

where $B_0 : C^n(A) \rightarrow C^{n-1}(A)$ and $A : C^{n-1}(A) \rightarrow C^{n-1}(A)$ are given by

$$B_0 \varphi(a^0, \ldots, a^{n-1}) = \varphi(1, a^0, \ldots, a^{n-1}) - (-1)^n \varphi(a^0, \ldots, a^{n-1}, 1),$$

$$A \psi(a^0, \ldots, a^{n-1}) = \sum (-1)^j(n-1) \psi(a^j, \ldots, a^{n-1}, a^0, \ldots, a^{j-1}).$$

Remark

1. If $\varphi$ is a cyclic cochain, then $B_0 \varphi = 0$, and hence $B \varphi = 0$.
2. An $n$-cochain $\varphi$ is cyclic if and only if $A \varphi = \frac{1}{n+1} \varphi$. 
Example

Let $C$ be a $k$-dimensional current on a compact manifold $M$. It defines a Hochschild cocycle,

$$\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!} \langle C, f^0 df^1 \wedge \cdots \wedge df^k \rangle.$$

We then have

$$B \varphi_C = \varphi_{d^t C}.$$

Lemma

We have

$$B^2 = 0 \quad \text{and} \quad bB + Bb = 0.$$
Definition (Even/Odd Cochains)

Define

\[ C_{\text{even}}(A) := \bigoplus_{k \geq 0} C^{2k}(A) \]

\[ = \left\{ \varphi = (\varphi_0, \varphi_2, \cdots); \varphi_{2k} \in C^{2k}(A), \varphi_{2k} = 0 \text{ for large } k \right\}, \]

and

\[ C_{\text{odd}}(A) := \bigoplus_{k \geq 0} C^{2k+1}(A) \]

\[ = \left\{ \varphi = (\varphi_1, \varphi_3, \cdots); \varphi_{2k+1} \in C^{2k+1}(A), \varphi_{2k+1} = 0 \text{ for large } k \right\}. \]
Proposition

We have a 2-periodic complex,

\[ C^{\text{even}}(\mathcal{A}) \overset{b+B}{\leftrightarrow} C^{\text{odd}}(\mathcal{A}). \]

Definition

The cohomology of \((C^{\text{even}/\text{odd}}(\mathcal{A}), b + B)\) is called the periodic cyclic cohomology of \(\mathcal{A}\) and is denoted \(HC^{\text{even}/\text{odd}}(\mathcal{A})\).
Example

Let $C = C_0 + C_2 + \cdots$ be an even current on a compact manifold $M$. Then $C$ defines an even cochain,

$$\varphi_C = (\varphi_{C_0}, \varphi_{C_2}, \cdots),$$

$$\varphi_{C_{2k}}(f^0, f^1, \cdots, f^{2k}) = \frac{1}{(2k)!} \langle C_{2k}, f^0 df^1 \wedge \cdots \wedge df^{2k} \rangle.$$

Then

$$(b + B)\varphi_C = B\varphi_C = \varphi_{d^t C}.$$

$C$ closed $\implies$ $\varphi_C$ even cyclic cocycle.

Theorem (Connes)

*The map $C \rightarrow \varphi_C$ gives rise to isomorphisms,*

$$H_{\text{even/odd}}(M) \simeq HC_{\text{even/odd}}(C^\infty(M)).$$
Remark

Assume $M$ is oriented, Riemannian and has even dimension. Then the $\hat{A}$-form $\hat{A}(R^M)$ defines an even cyclic cocycle by

$$\varphi_{\hat{A}(R^M)} = \varphi_{\hat{A}(R^M)}^\vee,$$

i.e., $\varphi_{\hat{A}(R^M)} = (\varphi_0, \varphi_2, \cdots)$, with

$$\varphi_{2k}(f^0, f^1, \cdots, f^{2k}) = \frac{1}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^M).$$

Likewise the Pfaffian $\text{Pf}(R^M)$, the $L$-form $L(R^M)$ and the Todd form $\text{Td}(R^M)$ define even cyclic cocycles.
Morita Equivalence

**Definition**
Let \( \varphi \in C^n(A) \). The \( n \)-cochain \( \varphi \# \text{Tr} \) on \( M_q(A) = A \otimes M_q(\mathbb{C}) \) is defined by

\[
\varphi \# \text{Tr}(a^0 \otimes \mu^0, \ldots, a^n \otimes \mu^n) := \varphi(a^0, \ldots, a^n) \text{Tr} \left[ \mu^0 \mu^1 \cdots \mu^n \right]
\]

for all \( a^j \in A \) and \( \mu^j \in M_q(\mathbb{C}) \).

**Lemma**
We have

\[
b(\varphi \# \text{Tr}) = (b\varphi)\# \text{Tr}.
\]

**Theorem (Connes)**
*The map \( \varphi \rightarrow \varphi \# \text{Tr} \) gives rise to isomorphisms,*

\[
HC^\bullet(A) \cong HC^\bullet(M_q(A)).
\]
Example

Let $C$ be a $k$-dimensional current on a compact manifold $M$. For the cochain,

$$\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!} \langle C, f^0 df^1 \wedge \ldots \wedge df^k \rangle,$$

we have

$$\varphi_C \# \text{Tr}(a^0, a^1, \ldots, a^k) = \frac{1}{k!} \langle C, \text{Tr} \left[ a^0 da^1 \wedge \ldots \wedge da^k \right] \rangle$$

for all $a^j$ in $M_q(C^\infty(M)) = C^\infty(M, M_q(\mathbb{C}))$. 
Pairing with $K$-Theory ($\mathcal{A} = C^\infty(M)$)

Setup
- $M$ is a compact manifold.
- $E = \text{ran } e, e = e^2 \in M_q (C^\infty(M))$.
- $F^E$ is the curvature of the Grassmanian connection $\nabla^E$ of $E$.

Lemma
1. $F^E = e(de)^2 = e(de)^2 e$.
2. $\text{Ch}(F^E) = \sum \frac{(-1)^k}{k!} \text{Tr} \left[ e(de)^k \right]$. 
Let $C = C_0 + C_2 + \cdots$ be a closed even current on $M$ with associated even cocycle $\varphi_C = (\varphi_{C_0}, \varphi_{C_2}, \cdots)$. Then

$$\langle C, E \rangle = \sum (-1)^k \frac{(2k)!}{k!} (\varphi_{C_{2k}} \not\equiv \text{Tr}) (e, e, \cdots e),$$

$$= \varphi_{C_0}(e) + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} (\varphi_{C_{2k}} \not\equiv \text{Tr}) \left( e - \frac{1}{2}, e, \cdots e \right).$$
Setup

$\mathcal{A}$ is a unital algebra over $\mathbb{C}$.

Definition

A cochain $\varphi \in C^n(\mathcal{A})$, $n \geq 1$, is normalized when

$$\varphi(a^0, a^1, \cdots, a^n) = 0$$

whenever $a^j = 1$ for some $j \geq 1$.

Lemma

Any class in $HC_{\text{even}}(\mathcal{A})$ contains a normalized representative.
Pairing with $K$-Theory

Example

Let $C$ be a $k$-dimensional current on a compact manifold $M$ with associated cochain,

$$\varphi_C(f^0, f^1, \ldots, f^k) = \frac{1}{k!} \langle C, f^0 df^1 \wedge \cdots \wedge df^k \rangle.$$ 

Then $\varphi_C$ is a normalized cochain.
Definition

Let $\varphi = (\varphi_0, \varphi_2, \cdots)$ be an even cyclic cocycle and let $\mathcal{E} = eA^q$, $e = e^2 \in M_q(A)$, a finitely generated projective module. The pairing of $\varphi$ and $\mathcal{E}$ is

$$
\langle \varphi, \mathcal{E} \rangle := \varphi_0(e) + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} (\varphi_{2k} \# \text{Tr}) \left( e - \frac{1}{2}, e, \cdots e \right).
$$

Theorem (Connes)

The above pairing descends to a bilinear pairing,

$$
\langle \cdot, \cdot \rangle : HC^{\text{even}}(A) \times K_0(A) \longrightarrow \mathbb{C}.
$$
Example

Let $C = C_0 + C_2 + \cdots$ be a closed even current on a compact manifold $M$ and let $E = \text{ran } e$, $e = e^2 \in M_q (C^\infty(M))$, so that $\mathcal{E} = C^\infty(M, E) \cong C^\infty(M)^q$. Then

$$\langle \varphi_C, \mathcal{E} \rangle = \langle C, E \rangle.$$
The Atiyah-Singer Index Theorem

Example

Assume $M$ is spin, oriented, Riemannian and has even dimension.

1. For $C = \hat{A}(R^M)^\vee$ and the Dirac operator,

$$\langle \varphi \hat{A}(R^M), \mathcal{E} \rangle = \langle \hat{A}(R^M)^\vee, E \rangle \quad \text{and} \quad \text{ind}_D[\mathcal{E}] = \text{ind}_D[E].$$

2. By the $K$-theoretic version Atiyah-Singer Index Theorem explained in Lecture 2,

$$\text{ind}_D[E] = (2i\pi)^{-\frac{n}{2}} \langle \hat{A}(R^M)^\vee, E \rangle.$$ 

3. The Atiyah-Singer Index Theorem then can be further restated as

**Theorem**

$$\text{ind}_D[\mathcal{E}] = (2i\pi)^{-\frac{n}{2}} \langle \varphi \hat{A}(R^M), \mathcal{E} \rangle \quad \forall \mathcal{E} \in K_0(C^\infty(M)).$$