ON THE ASYMPTOTIC COMPLETENESS OF THE VOLTERRA CALCULUS

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WITH AN APPENDIX BY H. MIKAYELYAN AND R. PONGE

Abstract. The Volterra calculus is a simple and powerful pseudodifferential tool for inverting parabolic equations and it has also found many applications in geometric analysis. On the other hand, an important property in the theory of pseudodifferential operators is the asymptotic completeness, which allows us to construct parametrices modulo smoothing operators. In this paper we present new and fairly elementary proofs the asymptotic completeness of the Volterra calculus.

Introduction

This paper deals with the asymptotic completeness of the Volterra calculus. Recall that the latter was invented in the early 70's by Piriou [Pi1] and Greiner [Gr] and consists in a modification of the classical ΨDO calculus in order to take into account two classical properties occurring in the context of parabolic equations: the Volterra property and the anisotropy with respect to the time variable (cf. Section 1). As a consequence the Volterra calculus proved to be a powerful tool for inverting parabolic equations (see Piriou ([Pi1], [Pi2])) and for deriving small heat kernel asymptotics for elliptic operators (see Greiner [Gr]).

Subsequently, the Volterra calculus has been extended to several other settings. In [BGS] Beals-Greiner-Stanton produced a version of the Volterra calculus for the hypoelliptic calculus on Heisenberg manifolds ([BG], [Ta]) and used it to derive the small time heat kernel asymptotics for the Kohn Laplacian on CR manifolds. Also, Melrose [Me] fit the Volterra calculus into the framework of his *b*-calculus on manifolds with boundary and used it to invert the heat equation with the purpose of producing a heat kernel proof of the Atiyah-Patodi-Singer index theorem [APS].

More recently, Buchholz-Schulze [BS], Krainer ([Kr1], [Kr2]) and Krainer-Schulze [KS] extended the Volterra calculus to the setting of the cone calculus

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of Schulze ([Sc1], [Sc2]) in order to solve general parabolic problems on manifold with conical singularities and to deal with large time asymptotics of solutions to parabolic problems on manifolds with boundary (by looking at the infinite time cylinder as a manifold with a conical singularity at time $t = \infty$; see [Kr1], [KS]). Furthermore, Mitrea [Mit] used a version of the Volterra calculus for studying parabolic equations with Dirichlet boundary conditions on Lipschitz domains and Mikayelyan [Mi2] dealt with parabolic problems on manifolds with edges via an extension of the Volterra calculus to the setting of Schulze's edge calculus ([Sc1], [Sc2]).

On the other hand, in [Po2] the approach to the heat kernel asymptotics of Greiner [Gr] was combined with the rescaling of Getzler [Ge] to produce a new short proof of the local index formula of Atiyah-Singer [AS]. The upshot is that this proof is as simple as Getzler short proof in [Ge] but, unlike the latter, it allows us to similarly compute the Connes-Moscovici cocycle [CM] for Dirac spectral triples. Furthermore, the pseudodifferential representation of the heat kernel provided by the Volterra calculus in [Gr] also gives an alternative to the construction by Seeley [Se] of pseudodifferential complex powers of (hypo)elliptic differential operators (*cf.* [Po1], [Po3]; see also [MSV]).

While most of the usual properties of the classical ΨDO calculus hold *verbatim* in the setting of the Volterra calculus, a more delicate issue is to check asymptotic completeness. This property allows us to construct parametrices for parabolic operators, but its standard proof cannot be carried through in the setting of the Volterra calculus. Indeed, at the level of symbols the Volterra property corresponds to analyticity with respect to the time covariable (see Section 1), but this property is not preserved by the cut-off arguments of the proof.

Since we cannot cut off Volterra symbols, Piriou [Pi1, pp. 82–88] proved the asymptotic completeness of the Volterra calculus by cutting off distribution kernels instead, which at this level does not harm the Volterra property, and by checking that under the Fourier transform we get an actual asymptotic expansion of symbols (see also [Me]). Recently, Krainer [Kr2, pp. 62–73] obtained a proof by making use of the kernel cut-off operator of Schulze ([Sc1], [Sc2]) and Mikayelyan [Mi1] produced another proof by combining translations in the time covariable with an induction process[1].

In this paper, we present somewhat simpler approaches. First, we show that we actually get a Volterra ΨDO by adding a suitable smoothing operator to the ΨDO provided by the standard proof of the asymptotic completeness of classical symbols (see Proposition 2.1).

[1]Despite that in [Mi1, p. 79] the induction hypothesis is not stated properly and there is a typo on line 14 the argument in the proof is correct.

Second, we deal with the asymptotic completeness of analytic Volterra symbols (see Proposition 3.3 and Proposition 3.6). This was the setting under consideration in [Kr2] and [Mi1], because this asymptotic completeness implies that of the Volterra calculus (see Section 3). Here our approach is inspired by the version of the Borel lemma for analytic functions on an angular sector (e.g. [AG, p. 63]).

This paper is organized as follows. In Section 1 we briefly review the main facts concerning the Volterra calculus. In Section 2 we present our first approach. In Section 3 we carry out our proofs of the asymptotic completeness of analytic symbols. Finally, in the appendix, written with Hayk Mikayelyan, we give alternative proofs of the asymptotic completeness of these analytic Volterra symbols by combining our approach with the use of translations in the time covariable from [Mi1]. In particular we remove the induction process used in that paper.

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1. Overview of the Volterra calculus

Throughout this paper U is an open subset of $\mathbb{R}^n$ and w denotes an even integer ≥ 2. Also, we let $\mathbb{C}_-$ denote the half-space $\mathbb{C}_- = \{\Im\tau < 0\} \subset \mathbb{C}$ with closure $\overline{\mathbb{C}}_- = \{\Im\tau \leq 0\}$.

As alluded to in the introduction the Volterra calculus is a pseudodifferential calculus on $U \times \mathbb{R}$ which aims to take into account:

(i) The anisotropy of parabolic problems on $U \times \mathbb{R}$, i.e. their homogeneity with respect to the dilations of $\mathbb{R}^n \times \overline{\mathbb{C}}_-$ given by

$$\lambda.(\xi,\tau) = (\lambda\xi, \lambda^w\tau), \qquad \lambda \in \mathbb{R}\setminus 0. \tag{1.1}$$

(ii) The Volterra property, that is the fact for a continuous operator Q from $C_c^\infty(U_x \times \mathbb{R}_t)$ to $C^\infty(U_x \times \mathbb{R}_t)$ to have a distribution kernel of the form $k_Q(x,t;y,s) = K_Q(x,y,t-s)$, where $K_Q(x,y,t)$ vanishes in the region $U \times U \times \{t < 0\}$.

Definition 1.1. *$S_{\mathrm{v},m}(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of smooth functions $q_m(x,\xi,\tau)$ on $U_x \times (\mathbb{R}^{n+1}_{(\xi,\tau)} \setminus 0)$ such that $q_m(x,\xi,\tau)$ can be extended to a smooth function on $U_x \times [(\mathbb{R}^n_\xi \times \overline{\mathbb{C}}_{-,\tau}) \setminus 0]$ in such way to be analytic with respect to $\tau \in \mathbb{C}_-$ and to be homogeneous of degree m, i.e. $q_m(x,\lambda\xi,\lambda^2\tau) = \lambda^m q_m(x,\xi,\tau)$ for any $\lambda \in \mathbb{R}\setminus 0$.*

In fact, Definition 1.1 is intimately related to the Volterra property, since we have:

Lemma 1.2 ([BGS, Prop. 1.9]). *Any symbol $q(x,\xi,\tau) \in S_{\mathrm{v},m}(U \times \mathbb{R}^{n+1})$ can be extended into a unique distribution $g(x,\xi,\tau) \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n+1})$ in such way to be homogeneous with respect to the covariables (ξ,τ) and such that $\check{q}(x,y,t) := \mathcal{F}^{-1}_{(\xi,\tau)\to(y,t)}[g](x,y,t)$ vanishes for $t<0$.*

Next, we introduce the pseudo-norm on $\mathbb{R}^n \times \overline{\mathbb{C}}_-$ given by

$$\|\xi,\tau\| = (|\xi|^w + |\tau|)^{1/w}, \qquad (\xi,\tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-. \tag{1.2}$$

This pseudo-norm is homogeneous since $\|(\lambda\xi, \lambda^w\tau)\| = |\lambda|^w \|\xi,\tau\|$ for any $\lambda \in \mathbb{R}\setminus 0$. Also, for $(\xi,\tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-$ we have

$$2^{-1/w}(1+|\xi|+|\tau|)^{1/w} \leq 1 + \|\xi,\tau\| \leq 1 + |\xi| + |\tau|. \tag{1.3}$$

Definition 1.3. *$S^m_{\mathrm{v}}(U\times\mathbb{R}^{n+1})$, $m\in\mathbb{Z}$, consists of smooth functions $q(x,\xi,\tau)$ on $U_x \times \mathbb{R}^{n+1}_{(\xi,\tau)}$ which have an asymptotic expansion $q \sim \sum_{j\geq 0} q_{m-j}$, where $q_{m-j} \in S_{\mathrm{v},m-j}(U\times\mathbb{R}^{n+1})$ and $\sim$ means that, for any integer $N \geq 0$ and for any compact $K \subset U$, we have*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j<N} q_{m-j})(x,\xi,\tau)| \leq C_{NK\alpha\beta k} \|\xi,\tau\|^{m-|\beta|-wk-N}, \tag{1.4}$$

for $x \in K$ and for $(\xi,\tau) \in \mathbb{R}^{n+1}$ such that $\|\xi,\tau\| \geq 1$.

Remark 1.4. It follows from (1.3) and (1.4) that $S^m_{\mathrm{v}}(U\times\mathbb{R}^{n+1})$ is contained in the Hörmander's class $S^{m'}_{0,\frac{1}{w}}((U_x\times\mathbb{R}_t)\times\mathbb{R}^{n+1}_{(\xi,\tau)})$ with $m'=m$ if $m\geq 0$ and $m' = \frac{m}{w}$ otherwise. In fact, using Hörmander's Lemma (e.g. [Hö, Thm. 2.9], [Sh, Prop. 3.6]) one can even show that the asymptotic expansion in the sense of (1.4) coincides with that for standard symbols.

Definition 1.5. *$\Psi^m_{\mathrm{v}}(U\times\mathbb{R})$, $m\in\mathbb{Z}$, consists of continuous operators Q from $C^\infty_c(U\times\mathbb{R})$ to $C^\infty(U\times\mathbb{R})$ such that:*

(i) Q has the Volterra property;

(ii) Q is of the form $Q = q(x,D_x,D_t) + R$ with $q \in S^m_{\mathrm{v}}(U\times\mathbb{R}^{n+1})$ and R smoothing operator.

As it follows from Remark 1.4 the class $\Psi^*_{\mathrm{v}}(U\times\mathbb{R})$ is contained in the class of ΨDO's of type $(0,\frac{1}{w})$ on $U\times\mathbb{R}$. Therefore, once the asymptotic completeness is checked, all the standard properties of classical ΨDO's hold *verbatim* for Volterra ΨDO's: symbolic calculus, existence of parametrices for parabolic ΨDO's (i.e. those with an invertible principal symbol), invariance by diffeomorphisms which don't act on the time variable. In particular, the Volterra calculus makes sense on $M\times\mathbb{R}$ for any smooth manifold M.

On the other hand, the Volterra calculus has two important applications:

- *Inversion of parabolic operators* (Piriou ([Pi1], [Pi2])). Any parabolic differential operator on $U\times\mathbb{R}$, not only admits a parametrix, but has actually an inverse in the Volterra calculus. This makes use of the well known fact that if R is a smoothing operator which is properly supported and has the Volterra property, then the Levi series $\sum_{j\geq 1} R^j$ is convergent in the Fréchet space of smoothing operators. This result has been extended to several other settings (see [BGS], [Me], [BS], [Kr1], [Kr2], [KS], [Mi2], [Mit]).

- *Heat kernel asymptotics* (Greiner [Gr]). Let P be differential operator of order w on a compact Riemannian manifold M and assume that the principal symbol of P is positive definite. Then we can relate the heat kernel $k_t(x,y)$ of P to the the distribution kernel of $(P+\partial_t)^{-1}$ so that, as the latter is a Volterra ΨDO, we can derive the asymptotics for $k_t(x,x)$ as $t\rightarrow 0^+$ in terms of the symbol of $(P+\partial_t)^{-1}$. As alluded to in the introduction this approach to the heat kernel asymptotics has been extended to the setting of the hypoelliptic calculus on Heisenberg manifolds (see [BGS]) and has been used for proving the local index formula of Atiyah-Singer (cf. [Po2]) and for constructing complex powers of (hypo)elliptic operators (cf. [Po1], [Po3]; see also [MSV]).

2. Asymptotic completeness of the Volterra calculus

Here we give our first proof of the asymptotic completeness of the Volterra calculus. More precisely, we shall prove:

Proposition 2.1. *Given $q_{m-j}\in S_{\mathrm{v},m-j}(U\times\mathbb{R}^{n+1})$, $j=0,1,\ldots$, there always exists $Q\in\Psi_{\mathrm{v}}^m(U\times\mathbb{R})$ with symbol $q\sim\sum_{j\geq 0} q_{m-j}$.*

Proof. For $\epsilon>0$ and $(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}$ let $c_\epsilon(\xi,\tau)=1-\phi(\epsilon\|\xi,\tau\|)$, where $\phi(u)\in C_c^\infty([0,\infty))$ is such that $\phi(u)=1$ near $u=0$. Then similar arguments as those in the standard proof of the asymptotic completeness of symbols (e.g. [Hö, Thm. 2.7], [Sh, Prop. 2.5]) show that for any $\epsilon\geq 1$ and for any compact $K\subset U$ we have

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k[c_\epsilon(\xi,\tau)q_{m-j}(x,\xi,\tau)]|\leq C_{jK\alpha\beta k}\epsilon(1+\|\xi,\tau\|)^{m+1-j-|\beta|-wk}, \tag{2.1}$$

for $(x,\xi,\tau)\in K\times\mathbb{R}^n\times\mathbb{R}$ and where the constant $C_{jK\alpha\beta k}$ does depend on ϵ.

Next, given $(K_j)_{j\geq 0}$ an increasing compact exhaustion of U the estimates (2.1) allows us to find numbers $\epsilon_j\geq 1$, $j=0,1,\ldots$, such that

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k[c_{\epsilon_j}(\xi,\tau)q_{m-j}(x,\xi,\tau)]|\leq 2^{-j}(1+\|\xi,\tau\|)^{m+1-j-|\beta|-wk}, \tag{2.2}$$

for $l+|\alpha|+\beta|+k<j$ and $(x,\xi,\tau)\in K_l\in\mathbb{R}^n\times\mathbb{R}$. Therefore, the series $\sum_{j\geq 0} c_{\epsilon_j}q_{m-j}$ converges in $C^\infty(U\times\mathbb{R}^{n+1})$ to some function q. Moreover, the estimates (2.2) also imply that $q\sim\sum_{j\geq 0} q_{m-j}$. Thus, q is in $S_{\mathrm{v}}^m(U\times\mathbb{R}^{n+1})$.

Nevertheless, the operator $q(x,D_x,D_t)$ needs not have the Volterra property, since the cut-off functions $c_{\epsilon_j}(\xi,\tau)$ kill the analyticity of $q_{m-j}(x,\xi,\tau)$ with respect to τ. Thus, we need to construct a smoothing operator R such that $q(x,D_x,D_t)+R$ has the Volterra property.

First, as the Fourier transform relates the decay at infinity to the behavior at the origin of the Fourier transform, the estimates (1.4) imply that for any integer N the distribution $\check{q}(x,y,t)-\sum_{j\leq J}\check{q}_{m-j}(x,y,t)$ is in $C^N(U_x\times\mathbb{R}^n_y\times\mathbb{R}_t)$ as soon as J is large enough. As $\check{q}_{m-j}(x,y,t)$ vanishes for $t<0$ it follows that for every integer $l\geq 0$ the limit $\lim_{t\to 0^-}\partial_t^l\check{q}(.,.,t)$ exists in $C^N(U\times\mathbb{R}^n)$ for any $N\geq l$, hence exists in $C^\infty(U\times\mathbb{R}^n)$.

Now, using a version of the Borel lemma with coefficients in the Fréchet space $C^\infty(U\times\mathbb{R}^n)$ we can construct a smooth function $R(x,y,t)$ on $U\times\mathbb{R}^n\times\mathbb{R}$ such that for any integer $l\in\mathbb{N}$ we have $\partial_t^l R(.,.,0)=\lim_{t\to 0^-}\partial_t^l\check{q}(.,.,t)$ in $C^\infty(U\times\mathbb{R}^n)$. Then on $U\times\mathbb{R}^n\times\mathbb{R}$ we define

$$R_1(x,y,t)=(1-\chi(t))(\check{q}(x,y,t)-R(x,y,t)), \tag{2.3}$$

where $\chi(t)$ denotes the characteristic function of the interval $[0,\infty)$. In fact, $R_1(x,y,t)$ is a smooth function on $U\times\mathbb{R}^n\times\mathbb{R}$. Indeed, $R_1(x,y,t)$ is obviously smooth for $t\neq 0$ and, as $\partial_t^l R_1(.,.,t)=0$ for $t>0$ and as we have $\lim_{t\to 0^-}\partial_t^l R_1(x,y,t)=0$ in $C^\infty(U\times\mathbb{R}^n)$, we see that $R_1(x,y,t)$ is also smooth near $t=0$.

Finally, let $Q: C_c^\infty(U\times\mathbb{R}^{n+1})\to C^\infty(U\times\mathbb{R}^{n+1})$ be the operator with distribution kernel

$$\begin{aligned} K_Q(x,y,t-s) &= \chi(t-s)(\check{q}(x,x-y,t-s)-R(x,x-y,t-s)),\\ &= \check{q}(x,x-y,t-s)-R(x,x-y,t-s)-R_1(x,x-y,t-s). \end{aligned} \tag{2.4}$$

Then Q has the Volterra property and differs from $q(x,D_x,D_t)$ by a smoothing operator, so is a Volterra ΨDO with symbol $q\sim\sum_{j\geq 0}q_{m-j}$. □

3. Asymptotic completeness of analytic Volterra symbols

Using a different approach, partly inspired by the proof of the Borel lemma for analytic functions on an angular sector (see [AG, p. 63]), we will now prove the asymptotic completeness of the analytic Volterra symbols below.

Definition 3.1. *$S^m_{\mathrm{v,a}}(U\times\mathbb{R}^{n+1})$, $m\in\mathbb{Z}$, consists of smooth functions $q(x,\xi,\tau)$ on $U_x\times\mathbb{R}^{n+1}_{(\xi,\tau)}$ such that:*

(i) $q(x,\xi,\tau)$ extends to a smooth function on $U_x\times\mathbb{R}^n_\xi\times\overline{\mathbb{C}}_{-,\tau}$ in such way to be analytic with respect to $\tau\in\mathbb{C}_-$;

(ii) We have $q\sim_{\mathrm{a}}\sum_{j\geq 0}q_{m-j}$, $q_{m-j}\in S_{\mathrm{v},m-j}(U\times\mathbb{R}^{n+1})$, in the sense that, for any integer $N\geq 0$ and for any compact $K\subset U$, we have

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k(q-\sum_{j<N}q_{m-j})(x,\xi,\tau)|\leq C_{NK\alpha\beta k}\|\xi,\tau\|^{m-|\beta|-wk-N}, \tag{3.1}$$

for $x\in K$ and for $(\xi,\tau)\in\mathbb{R}^n\times\overline{\mathbb{C}}_-$ such that $\|\xi,\tau\|\geq 1$.

In fact, by the Paley-Wiener theorem if $q(x,\xi,\tau) \in S^m_{\mathrm{v,a}}(U \times \mathbb{R}^{n+1})$ then $\check{q}(x,y,t) = 0$ for $t < 0$. Thus, the operator $q(x,D_x,D_t)$ is already a Volterra ΨDO since its distribution kernel is $\check{q}(x,x-y,s-t)$. Thus, the asymptotic completeness of analytic Volterra symbols implies the asymptotic completeness of the Volterra calculus.

Next, consider the homogeneous symbol $\rho(\xi,\tau) \in S_{\mathrm{v},-1}(\mathbb{R}^{n+1})$ given by

$$\rho(\xi,\tau) = (|\xi|^p + i\tau)^{-1/w}, \qquad (\xi,\tau) \in (\mathbb{R}^n \times \overline{\mathbb{C}}_-) \setminus 0, \tag{3.2}$$

where in order to define the w'th root we use the continuous determination of the argument on $\mathbb{C} \setminus [0,-\infty)$ with values in $(-\pi,\pi)$, so that $\rho(\xi,\tau)$ takes values in $\Omega = \{z \in \mathbb{C} \setminus 0;\ |\arg z| \leq \frac{\pi}{2w}\}$. Moreover, as $\rho(\xi,\tau)$ never vanishes on $(\mathbb{R}^n \times \overline{\mathbb{C}}_-) \setminus 0$ and is homogeneous of degree -1 there exists $C_\rho > 0$ such that for $(\xi,\tau) \in (\mathbb{R}^n \times \overline{\mathbb{C}}_-) \setminus 0$ we have

$$C_\rho^{-1} \|\xi,\tau\|^{-1} \leq \rho(x,\xi) \leq C_\rho \|\xi,\tau\|^{-1}. \tag{3.3}$$

Now, for any integer N we have $z^N e^{-z} \to 0$ as $z \in \Omega$ goes to infinity. Therefore, for any $\epsilon > 0$ we define a smooth function on $\mathbb{R}^n \times \overline{\mathbb{C}}_-$ by letting

$$a_\epsilon(0,0) = 0 \quad \text{and} \quad a_\epsilon(\xi,\tau) = e^{-\epsilon\rho(\xi,\tau)} \text{ for } (\xi,\tau) \neq 0. \tag{3.4}$$

Notice that $a_\epsilon(\xi,\tau)$ is analytic with respect to $\tau \in \mathbb{C}_-$. In fact, we have:

Lemma 3.2. *1) a_ϵ is in $S^0_{\mathrm{v,a}}(\mathbb{R}^{n+1})$ and we have $a_\epsilon \sim_{\mathrm{a}} \sum_{j\geq 0} \frac{\epsilon^j}{j!}\rho^j$.*

2) For any $\epsilon \geq 1$ and any integer $N \geq 0$ we have

$$|\partial_\xi^\beta \partial_\tau^k a_\epsilon(\xi,\tau)| \leq C_{\beta k} \epsilon^{-1} \|\xi,\tau\|^{1-|\beta|-wj}, \qquad \|\xi,\tau\| \geq 1, \tag{3.5}$$

$$|\partial_\xi^\beta \partial_\tau^k a_\epsilon(\xi,\tau)| \leq C_{N\beta k} \epsilon^{-1} \|\xi,\tau\|^{N}, \qquad \|\xi,\tau\| \leq 1, \tag{3.6}$$

where the constants $C_{\beta j}$ and $C_{N\beta j}$ are independent of ϵ.

Proof. First, if $\|\xi,\tau\| \geq 1$ then by (3.3) we have $\rho(\xi,\tau) \leq C_\rho$, and so we get:

$$|a_\epsilon(x,\tau) - \sum_{j<J} \frac{\epsilon^j}{j!}\rho(\xi,\tau)^j| \leq |\rho(x,\xi)|^J \sum_{j\geq J} \frac{\epsilon^j}{j!} C_\rho^{j-J} \leq C_{\epsilon J}\|\xi,\tau\|^{-J}. \tag{3.7}$$

On the other hand, an easy induction shows that for any multi-order β and any integer j the function $\partial_\xi^\beta \partial_\tau^k a_\epsilon(\xi,\tau)$ is a linear combination of terms of the form $\epsilon^l \eta_{\beta k l}(\xi,\tau) e^{-\epsilon\rho(\xi,\tau)}$, where l is an integer $\leq j$ and $\eta_{\beta k l}(\xi,\tau)$ is homogeneous of degree $-(|\beta|+wk)-l$ and does not depend on ϵ. In particular, as $\epsilon \geq 1$ and as for any $N \geq 0$ the function $z^N e^{-z}$ is bounded on Ω, we get

$$\begin{aligned} &\|\xi,\tau\|^{-N}\epsilon^l |\eta_{\beta k l}(\xi,\tau) e^{-\epsilon\rho(\xi,\tau)}| = \\ &\quad \epsilon^{-(N+1)}.\|\xi,\tau\|^{-N} |\eta_{\beta k l}(\xi,\tau)\rho(\xi,\tau)^{-(N+l+1)}|.|(\epsilon\rho(\xi,\tau))^{N+l+1} e^{-\epsilon\rho(\xi,\tau)}|, \\ &\qquad \leq C_{\beta k l N} \epsilon^{-1} \|\xi,\tau\|^{1+|\beta|+wk}, \end{aligned} \tag{3.8}$$

where the constant $C_{\beta k l N}$ does not depend on ϵ. Then by setting $N = 0$ we obtain (3.5) and by taking N large enough we get (3.6).

Finally, thanks to the Hörmander Lemma ([Hö, Thm. 2.9], [Sh, Prop. 3.6]) the estimates (3.5)–(3.7) are enough to show that $a_\epsilon \sim_{\mathrm{a}} \sum_{j\geq 0} \frac{\epsilon^j}{j!}\rho^j$. In particular, the symbol a_ϵ belong to $S^0_{\mathrm{v}}(\mathbb{R}^{n+1})$. □

Proposition 3.3. *For $j = 0, 1, 2, \ldots$ let $q_{m-j} \in S_{\mathrm{v},m-j}(U \times \mathbb{R}^{n+1})$. Then there exists $q \in S^m_{\mathrm{v},a}(U \times \mathbb{R}^{n+1})$ such that $q \sim_{\mathrm{a}} \sum_{j\geq 0} q_{m-j}$. In particular, the operator $q(x, D_x, D_t)$ is a Volterra ΨDO with symbol $q \sim \sum_{j\geq 0} q_{m-j}$.*

Proof. We seek for numbers $\epsilon_j \geq 1$ and symbols $r_{m-j} \in S_{\mathrm{v},m-j}(U \times \mathbb{R}^{n+1})$, $j = 0, 1, \ldots$, such that:

(i) The series $\sum_{j\geq 0} a_{\epsilon_j}(\xi,\tau) r_{m-j}(x,\xi,\tau)$ converges in $C^\infty(U \times \mathbb{R}^n \times \overline{\mathbb{C}}_-)$ to some function $q(x,\xi,\tau)$ which is analytic with respect to $\tau \in \overline{\mathbb{C}}_-$;

(ii) We have $q \sim_{\mathrm{a}} \sum_{j\geq 0} a_{\epsilon_j} r_{m-j}$.

Notice that by Lemma 3.2 the function $a_{\epsilon_j}(\xi,\tau) r_{m-j}(x,\xi,\tau)$ is smooth on $U \times \mathbb{R}^n \times \overline{\mathbb{C}}_-$ and analytic with respect to $\tau \in \mathbb{C}_-$, so that (i) makes sense. Also, Lemma 3.2 implies that $a_\epsilon r_{m-j} \sim_{\mathrm{a}} \sum_{k\geq 0} \frac{\epsilon_j^k}{k!}\rho^k r_{m-j}$. Therefore, if (ii) holds then we obtain

$$(3.9) \qquad q \sim_{\mathrm{a}} \sum_{j\geq 0} a_\epsilon r_{m-j} \sim_{\mathrm{a}} \sum_{j,l\geq 0} \frac{\epsilon_j^l}{l!}\rho^l r_{m-j}.$$

Thus, we would have $q \sim_{\mathrm{a}} \sum_{j\geq 0} q_{m-j}$ if, and only if, for $j = 0, 1, \ldots$ we have

$$(3.10) \qquad q_{m-j} = r_{m-j} + \epsilon_{j-1}\rho r_{m-j+1} + \ldots + \frac{\epsilon_0^j}{j!}\rho^j r_m, \qquad j \geq 0.$$

By an easy induction these equalities allow us to uniquely determine r_{m-j} in terms of $q_m, \ldots, q_{m-j}$ and $\epsilon_0, \ldots, \epsilon_{j-1}$ only, so that r_{m-j} does not depend on ϵ_l for $l \geq 0$. Therefore, using (3.5) and (3.6) we see that for any compact $K \subset U$ we have

$$(3.11) \quad |\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k [a_{\epsilon_j} r_{m-j}](x,\xi,\tau)| \leq C_{K\alpha\beta kj}\epsilon_j^{-1}(1 + \|\xi,\tau\|)^{m-j+1-|\beta|-wk},$$

for $x \in K$ and for $(\xi,\tau) \times \mathbb{R}^n \times \overline{\mathbb{C}}_-$.

Now, let $(K_j)_{j\geq 0}$ be an increasing exhaustion of U by compact subsets. Then thanks to (3.11) we can choose the sequence $(\epsilon_j)_{j\geq 0}$ in such way that we have

$$(3.12) \qquad |\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k [a_{\epsilon_j} r_{m-j}](x,\xi,\tau)]| \leq 2^{-j}(1 + \|\xi,\tau\|)^{m-j+1-|\beta|-wk},$$

for $l + |\beta| + k \leq j$ and $(x,\xi,\tau) \in K_l \times \mathbb{R}^n \times \overline{\mathbb{C}}_-$.

It follows from (3.12) that the series $\sum_{j\geq 0} a_{\epsilon_j}(\xi,\tau) r_{m-j}(x,\xi,\tau)$ converges in $C^\infty(U \times \mathbb{R}^n \times \overline{\mathbb{C}}_-)$ to some function $q(x,\xi,\tau)$. This function is furthermore is analytic with respect to $\tau \in \mathbb{C}_-$ since each term $a_{\epsilon_j}(\xi,\tau)r_{m-j}(x,\xi,\tau)$ in the series is.

On the other hand, the estimates (3.12) also imply that $q \sim_{\mathrm{a}} \sum_{j\geq 0} a_{\epsilon_j} r_{m-j}$, which in view of (3.9) and (3.10) yields $q \sim_{\mathrm{a}} \sum_{j\geq 0} q_{m-j}$. In particular, the function q belongs to $S^m_{\mathrm{v},a}(U \times \mathbb{R}^{n+1})$. □

This approach also allows us to deal with the asymptotic completeness of non-polyhomogeneous analytic Volterra symbols. These symbols can be defined as follows.

Definition 3.4. *$S^m_{\|\mathrm{v}}(U\times\mathbb{R}^{n+1})$, $m \in \mathbb{R}$, consists of smooth functions $q(x,\xi,\tau)$ on $U \times \mathbb{R}^{n+1}_{(\xi,\tau)}$ which can be extended to a smooth function on $U \times \mathbb{R}^n \times \overline{\mathbb{C}}_-$ in such way that:*

(i) $q(x,\xi,\tau)$ is analytic with respect to $\tau \in \mathbb{C}_-$;

(ii) For any compact $K \subset U$ we have

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k q(x,\xi,\tau)| \leq C_{K\alpha\beta k}(1+\|\xi,\tau\|)^{m-|\beta|-w|k}, \tag{3.13}$$

for $x \in K$ and $(\xi,\tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-$.

Remark 3.5. Any symbol in $S_{\mathrm{v,a}}(U \times \mathbb{R}^{n+1})$ is contained in $S^m_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$. Furthermore, if $q \in S^{m_j}_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$ where $m_j \downarrow -\infty$ as $m_j \to \infty$ then we have $q \sim_{\mathrm{a}} \sum_{j\geq 0}$ if, and only if, for any integer $N \geq 0$ the symbol $q - \sum_{j\leq J} q_j$ is $S^{-N}_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$ as soon as J is large enough.

Proposition 3.6. *Let $q_j \in S^{m_j}_{\mathrm{v}\|}(U \times \mathbb{R}^{n+1})$, $j = 0,1,\ldots$, where $m_j \downarrow -\infty$ as $j \to \infty$. Then there exists $q \in S^{m_0}_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$ such that $q \sim_{\mathrm{a}} \sum_{j\geq 0} q_j$.*

Proof. First, we can always assume $m_j - 1 \leq m_{j+1}$ for any $j \geq 0$, possibly by replacing the sequence $(q_j)_{j\geq 0}$ by the sequence $(q_{j,l})$, which is indexed by couples $(j,l) \in \mathbb{N}^2$ such that $0 \leq j \leq m_j - m_{j+1}$ and is given by

$$q_{j,0} = q_j \quad \text{and} \quad q_{j,l} = 0 \text{ for } 1 \leq l \leq m_j - m_{j+1}. \tag{3.14}$$

This has the effect to insert finitely many zero terms of order $\geq m_{j+1}$ into the sequence $(q_j)_{j\geq 0}$, so does not affect the class of symbols that are asymptotic to $\sum_{j\geq 0} q_j$.

Bearing this assumption in mind we now seek for numbers $\epsilon_j \geq 1$ and symbols $r_j \in S^{m_j}$, $j = 0,1,..$, such that:

(i) The series $\sum_{j\geq 0} a_{\epsilon_j}(\xi,\tau)r_j(x,\xi,\tau)$ converges in $C^\infty(U \times \mathbb{R}^n \times \overline{\mathbb{C}}_-)$ to a function $q(x,\xi,\tau)$ which is analytic with respect to $\tau \in \mathbb{C}_-$;

(ii) We have $q \sim_{\mathrm{a}} \sum_{j\geq 0} a_{\epsilon_j} r_j$.

As in (3.10) the condition (ii) would imply that $q \sim_{\mathrm{a}} \sum_{j\geq 0} q_j$ if we choose the symbols r_j in such way that for $j = 0, 1, \ldots$ we have

$$q_j = \sum_{m_{j+1} < m_k - l \leq m_j} \frac{\epsilon_k^l}{l!} \rho^l r_k = r_j + \sum_{\substack{m_{j+1} < m_k - l \leq m_j, \\ k > j}} \frac{\epsilon_k^l}{l!} \rho^l r_k, \tag{3.15}$$

where the second equality holds because $m_j - 1 \leq m_{j+1}$. This uniquely determines r_j in terms of $q_0, \ldots, q_j$ and $\epsilon_0, \ldots, \epsilon_{j-1}$ only. Therefore, along the same lines as that of the proof of Proposition 3.3 we can find numbers $\epsilon_j \geq 1$ such that (i) and (ii) hold. Then thanks to (3.15) we have $q \sim_{\mathrm{a}} \sum_{j\geq 0} q_j$. □

Remark 3.7. For some authors (Buchholz-Schulze [BS], Krainer ([Kr1],[Kr2]), Krainer-Schulze [KS], Mikayelyan [Mi2]) the Volterra ΨDO's are defined as those coming from analytic Volterra symbols only. Since Proposition 3.3 implies that any Volterra ΨDO in the sense of Definition 1.5 coincides up to a smoothing operator with the quantification of an analytic Volterra symbol, it follows that the two possible definitions are actually equivalent.

Appendix by H. Mikayelyan and R. Ponge

In this appendix we present alternative proofs of the asymptotic completeness of the analytic Volterra symbols by combining the use of translations in the time covariable from [Mi1] with some of the ideas from Section 3. In particular we remove the induction process used in [Mi1].

In the sequel given a symbol q on $U \times \mathbb{R}^n \times \overline{\mathbb{C}}_-$ for any $T > 0$ we let

$$q^{(T)}(x, \xi, \tau) = q(x, \xi, \tau - iT). \tag{A.1}$$

We shall first deal with non-polyhomogeneous symbols, which is the setting under consideration in [Mi1].

Lemma A.1 (Krainer ([Kr1], [Kr2])). *If $q \in S^m_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{R}$, then the symbol $q^{(T)}$ is in $S^m_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$ and we have $q^{(T)} \sim_{\mathrm{a}} \sum_{l\geq 0} \frac{(-iT)^l}{l!} \partial_\tau^l q$.*

Proof. Since $T > 0$ we have $|\tau| \leq |\tau - iT| \leq |\tau| + T$ for any $\tau \in \overline{\mathbb{C}}_-$. Therefore, for any $(\xi, \tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-$ we have

$$1 + \|\xi, \tau\| \leq 1 + \|\xi, \tau - iT\| \leq (1 + T)^{1/w}(1 + \|\xi, \tau\|). \tag{A.2}$$

If we combine these inequalities with a Taylor formula about $\tau = 0$ then for any compact $K \subset U$ we get

$$
\begin{aligned}
\text{(A.3)} \quad & |\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k [q^{(T)} - \sum_{l=0}^{N} \frac{(-iT)^l}{l!} \partial_\tau^l q](x,\xi,\tau)| \\
& \qquad \leq C_{TNK\alpha\beta k} \int_0^1 (1 + \|\xi, \tau - isT\|)^{m-|\beta|-kw-N-1} ds, \\
& \qquad\qquad \leq C_{TNK\alpha\beta k} (1 + \|\xi,\tau\|)^{m-|\beta|-kw-N-1},
\end{aligned}
$$

for $x \in K$ and $(\xi,\tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-$. Hence $q^{(T)} \sim_{\mathrm{a}} \sum_{l\geq 0} \frac{(-iT)^l}{l!} \partial_\tau^l q$. In particular, the function q belongs to $S^m_{\|\mathrm{v}}(U \times \mathbb{R}^n \times \overline{\mathbb{C}}_-)$. □

Lemma A.2 (Mikayelyan [Mi1])**.** *Let $q \in S^m_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$ with $m \leq -1$. Then for any compact $K \subset U$ we have*

$$
\text{(A.4)} \quad |\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k q^{(T)}(x,\xi,\tau)| \leq C_{K\alpha\beta k} (1+T)^{-1/w} (1 + \|\xi,\tau\|)^{m+1-|\beta|-w|k},
$$

for $x \in K$ and $(\xi,\tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-$ and where the constant $C_{K\alpha\beta k}$ does not depend on T.

Proof. Since $m \leq -1$ and since for $\tau \in \overline{\mathbb{C}}_-$ we have $|\tau - iT| \geq \sup(T, |\tau|)$, we see that for $(\xi,\tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-$ we get:

$$
\text{(A.5)} \qquad (1 + \|\xi, \tau - iT\|)^{-1} \leq (1+T)^{-1/w},
$$

$$
\text{(A.6)} \qquad (1 + \|\xi, \tau - iT\|)^{m+1-|\beta|-kw} \leq (1 + \|\xi,\tau\|)^{m+1-|\beta|-kw}.
$$

Therefore, for any compact $K \subset U$ we have

$$
\text{(A.7)} \quad
\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k q^{(T)}(x,\xi,\tau)| & \leq C_{K\alpha\beta k} (1 + \|\xi, \tau - iT\|)^{-1+m+1-|\beta|-wk}, \\
& \leq C_{K\alpha\beta k} (1+T)^{-1/w} (1 + \|\xi,\tau\|)^{m+1-|\beta|-wk},
\end{aligned}
$$

for $x \in K$ and $(\xi,\tau) \in \mathbb{R}^n \times \overline{\mathbb{C}}_-$ and where the constants $C_{K\alpha\beta k}$ do not depend on T. □

We can now give a second proof of Proposition 3.6.

Second proof of Proposition 3.6. Here we let $q_j \in S^{m_j}_{\mathrm{v}\|}(U \times \mathbb{R}^{n+1})$, $j = 0, 1, \ldots$, with $m_j \downarrow -\infty$ as $j \to \infty$ and we look for $q \in S^{m_0}_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$ such that $q \sim_{\mathrm{a}} \sum_{j\geq 0} q_j$.

First, as in the first proof of Proposition 3.6 we can assume $m_j - w \leq m_{j+1}$, possibly by replacing the sequence $(q_j)_{j\geq 0}$ by the sequence $(q_{j,l})$ which is indexed by the couples $(j,l) \in \mathbb{N}^2$ such that $0 \leq j \leq w^{-1}(m_j - m_{j+1})$ and is given by $q_{j,0} = q_j$ if $l = 0$ and $q_{j,l} = 0$ if $1 \leq l \leq m_j - m_{j+1}$.

Bearing this assumption in mind we now seek for numbers $T_j > 0$ and symbols $r_j \in S^{m_j}_{\|\mathrm{v}}(U \times \mathbb{R}^{n+1})$, $j = 0, 1, 2, ..$, such that:

(i) The series $\sum_{j=0}^{\infty} r_j^{(T_j)}(x,\xi,\tau)$ converges in $C^\infty(U\times\mathbb{R}^n\times\overline{\mathbb{C}}_-)$ to some symbol $q(x,\xi,\tau)$ which is analytic with respect to $\tau\in\mathbb{C}_-$;

(ii) We have $q\sim_{\mathrm{a}}\sum_{j\geq 0} r_j^{(T_j)}$.

Assuming that (i) and (ii) hold, using Lemma A.1 we get

$$q\sim_{\mathrm{a}}\sum_{j\geq 0} r_j^{(T_j)}\sim_{\mathrm{a}}\sum_{j,l\geq 0}\frac{(-iT_j)^l}{l!}\partial_\tau^l r_j. \tag{A.8}$$

Therefore, we would have $q\sim_{\mathrm{a}}\sum_{j\geq 0} q_j$ if for $j=0,1,\ldots$ we have

$$q_j=\sum_{m_{j+1}<m_{j'}-lw\leq m_j}\frac{(-iT_{j'})^l}{l!}\partial_\tau^l r_{j'}=r_j+\sum_{\substack{m_{j+1}<m_{j'}-lw\leq m_j,\\ j'<j}}\frac{(-iT_{j'})^l}{l!}\partial_\tau^l r_{j'}, \tag{A.9}$$

where the second equality holds because $m_j-w\leq m_{j+1}$.

Now, we set $T_j=1$ for all indices such that $m_j>-1$. Since the equalities (A.9) uniquely determine r_j in terms of $q_0,\ldots,q_j$ and $T_0,\ldots,T_{j-1}$ only, when $m_j\geq -1$ Lemma A.2 implies that for any compact $K\subset U$ we have

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k r_j^{(T_j)}(x,\xi,\tau)|\leq C_{K\alpha\beta kj}(1+T_j)^{-1/w}(1+\|\xi,\tau\|)^{m_j+1-|\beta|-wk}, \tag{A.10}$$

for $x\in K$ and $(\xi,\tau)\in\mathbb{R}^n\times\overline{\mathbb{C}}_-$. Then by similar arguments as those of the proof of Proposition 3.3 we can construct a sequence of positive numbers $(T_j)_{m_j\leq -1}$ converging fast enough to ∞ such that the condition (i) and (ii) above hold. Then (A.9) implies that $q\sim_{\mathrm{a}}\sum_{j\geq 0} q_j$. □

Next, we deal with the polyhomogeneous case.

Lemma A.3. *Let $q\in S_{\mathrm{v},m}(U\times\mathbb{R}^{n+1})$. Then:*

1) $q^{(T)}$ belongs to $S^m_{\mathrm{v},a}(U\times\mathbb{R}^{n+1})$ and we have $q^{(T)}\sim_{\mathrm{a}}\sum_{l\geq 0}\frac{(-iT)^l}{l!}\partial_\tau^l q$.

2) If $m\leq -1$ and $T\geq 1$ then for any compact $K\subset U$ we have

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k q^{(T)}(x,\xi,\tau)|\leq C_{K\alpha\beta k}T^{-1/w}(1+\|\xi,\tau\|)^{m+1-|\beta|-wk}, \tag{A.11}$$

for $(x,\xi,\tau)\in K\times\mathbb{R}^n\times\overline{\mathbb{C}}_-$ and where the constant $C_{K\alpha\beta k}$ does not depend on T.

Proof. First, as T is positive $\overline{\mathbb{C}}_- - iT$ is contained in $\mathbb{C}_-$. Since $q(x,\xi,\tau)$ is smooth and analytic with respect to τ on $U\times\mathbb{R}^n\times\mathbb{C}_-$, it follows that $q^{(T)}(x,\xi,\tau)$ is smooth on $U\times\mathbb{R}^n\times\overline{\mathbb{C}}_-$ and is analytic with respect to $\tau\in\mathbb{C}_-$.

Next, as in (A.3) by combining a Taylor formula with (A.2) we get

$$\text{(A.12)}\quad \begin{aligned}|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k(q^{(T)}-\sum_{l\leq N}\frac{(-iT)^l}{l!}\partial_\tau^l q)(x,\xi,\tau)| \\ \leq C_{TNK\alpha\beta k}\int_0^1\|\xi,\tau-isT\|^{m-|\beta|-kw-N-1}ds, \\ \leq C_{TNK\alpha\beta k}\|\xi,\tau\|^{m-|\beta|-kw-N-1},\end{aligned}$$

for $x\in K$ and for $(\xi,\tau)\in\mathbb{R}^n\times\overline{\mathbb{C}}_-$ such that $\|\xi,\tau\|\geq 1$. Thus $q^{(T)}$ is asymptotic to $\sum_{l\geq 0}\frac{(-iT)^l}{l!}\partial_\tau^l q$ in the sense of (3.1). Hence $q^{(T)}$ belongs to $S^m_{\mathrm{v},a}(U\times\mathbb{R}^{n+1})$.

On the other hand, as in (A.5) since $|\tau-iT|\geq T$ we have

$$\text{(A.13)}\qquad \|\xi,\tau-iT\|^{-1}\leq T^{-1/w}.$$

Moreover, if $T\geq 1$ then for $(\xi,\tau)\in\mathbb{R}^n\times\overline{\mathbb{C}}_-$ we also get

$$\text{(A.14)}\quad \|\xi,\tau-iT\|\geq[|\xi|^w+(|\Re\tau|^2+(|\Im\tau|+T)^2)^{1/2}]^{1/w}\geq C_w(1+\|\xi,\tau\|).$$

Therefore, for any $T\geq 1$ and for any compact $K\subset U$ we have

$$\text{(A.15)}\quad \begin{aligned}|\partial_x^\alpha\partial_\xi^\beta\partial_\tau^k q^{(T)}(x,\xi,\tau)| &\leq C_{K\alpha\beta k}\|\xi,\tau-iT\|^{-1+m+1-|\beta|-wk}, \\ &\leq C_{K\alpha\beta k}T^{-1/w}(1+\|\xi,\tau\|)^{m+1-|\beta|-wk},\end{aligned}$$

for $x\in K$ and $(\xi,\tau)\in\mathbb{R}^n\times\overline{\mathbb{C}}_-$ and where the constant $C_{K\alpha\beta k}$ does not depend on T. □

Second proof of Proposition 3.3. For $j=0,1,\ldots$ let $q\in S_{\mathrm{v},m-j}(U\times\mathbb{R}^{n+1})$. Then, provided that we make use of Lemma A.3 instead of Lemma A.1 and Lemma A.2, similar arguments as those of the second proof of Proposition 3.6 show that we can find numbers $T_j\geq 1$ and symbols $r_{m-j}\in S_{\mathrm{v},m-j}(U\times\mathbb{R}^{n+1})$, $j=0,1,\ldots$, such that:

(i) The series $\sum_{j\geq 0}r_{m-j}^{(T_j)}$ converges in $C^\infty(U\times\mathbb{R}^n\times\overline{\mathbb{C}}_-)$ to some symbol $q(x,\xi,\tau)$ which is analytic with respect to $\tau\in\mathbb{C}_-$;

(ii) We have $q\sim_{\mathrm{a}}\sum_{j\geq 0}r_{m-j}^{(T_j)}\sim_{\mathrm{a}}\sum_{j\geq 0}q_{m-j}$.

Hence the proposition. □

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