A NEW SHORT PROOF OF THE LOCAL INDEX FORMULA AND SOME OF ITS APPLICATIONS

RAPHAËL PONGE

Abstract. We give a new short proof of the index formula of Atiyah and Singer based on combining Getzler's rescaling with Greiner's approach of the heat kernel asymptotics. As application we can easily compute the Connes-Moscovici cyclic cocycle of even and odd Dirac spectral triples, and then recover the Atiyah-Singer index formula (even case) and the Atiyah-Patodi-Singer spectral flow formula (odd case).

The Atiyah-Singer index Theorem ([AS1], [AS2]) gives a cohomological interpretation of the Fredholm index of an elliptic operator, but it reaches its true geometric content in the case Dirac operator for which the index is given by a local geometric formula. The local formula is somehow as important as the index theorem since, on the one hand, all the common geometric operators are locally Dirac operators ([ABP], [BGV], [LM], [Ro]) and, on the other hand, the local index formula is equivalent to the full index theorem ([ABP], [LM]). It was then attempted to bypass the index theorem to prove the local index formula. The first direct proofs were made by Patodi, Gilkey, Atiyah-Bott-Patodi partly by using invariant theory (see [ABP], [Gi]). Some years later Getzler ([Ge1], [Ge2]) and Bismut [Bi] gave purely analytic proofs, which led to many generalizations of the local index formula (see also [BGV], [Ro]).

The short proof of Getzler [Ge2] combines the Feynman-Kac representation of the heat kernel with an ingenious trick, the Getzler rescaling. In this paper we a give a new short proof of the local index formula for Dirac operators by combining Getzler rescaling with the (fairly standard) Greiner's approach of the heat kernel asymptotics ([Gr], [BGS]). Our proof is quite close to other proofs like those by Melrose [Me, pp. 295-327], Simon [CFKS, Chap. 12] and Taylor [Ta, Chap. 10], but here the justification of the convergence of the supertrace of the heat kernel, which is the key of the proof, follows from very elementary consideration on Getzler's orders (Lemma 2.7).

In fact, the proof yields a more general result, for it implies a differentiable version of the asymptotics for the supertrace of the heat kernel, which is hardly accessible by means of a probabilistic representation of the heat kernel as in [Ge2] (see Proposition 2.12).

In the second part of the paper we show how this enables us to compute the CM cyclic cocycle [CM] associated to a Dirac spectral triple, both in the even case (Theorem 4.1) and in the odd case (Theorem 5.1). Therefore we can bypass the use of Getzler's asymptotic pseudodifferential calculus [Ge1] of the previous approaches of the computation of the CM cocycle for Dirac spectral triples ([CM, Remark II.1]; see also [CH], [Le]).

Recall that the CM cocycle is important because it represents the cyclic cohomology Chern character of a spectral triple (i.e. a "noncommutative manifold") and is given by a formula which is local in the sense of noncommutative geometry ([CM]; see also Section 3). Thus it allows the local index formula to hold in a purely operator theoretic setting. For instance, the computation for Dirac spectral triples allows us to recover, in the even case, the local index formula of Atiyah-Singer and, in the odd case, the spectral flow formula of Atiyah-Patodi-Singer [APS] (*cf.* [CM, Remark II.1] and sections 4 and 5; see also [Ge3] for the odd case).

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The paper is organized as follows. In the first section we recall Greiner's approach of the heat kernel asymptotics following [Gr] and [BGS]. In Section 2 we prove the local index formula of Atiyah-Singer and in Section 3 we present the operator theoretic framework for the local index formula of [CM]. Then we compute the CM cocycle of Dirac spectral triples: the even case is treated in Section 4 and the odd case in Section 5.

1. Greiner's approach of the heat kernel asymptotics

In this section we recall Greiner's approach of the heat kernel asymptotics as in [Gr] and [BGS] (see also [Me, pp. 252-272] for an alternative point of view).

Here M^n is a manifold equipped with a smooth and strictly positive density, $\mathcal{E}$ a Hermitian vector bundle over M and Δ a second order elliptic differential operator on M acting on the sections of $\mathcal{E}$. In addition we assume that Δ with domain $C_c^\infty(M,\mathcal{E})$ is essentially selfadjoint and bounded from below on $L^2(M,\mathcal{E})$. Then by standard functional calculus we can define $e^{-t\Delta}$, $t \geq 0$, as a selfadjoint bounded operator on $L^2(M,\mathcal{E})$. In fact, $e^{-t\Delta}$ is smoothing for $t>0$ and so its distribution kernel $k_t(x,y)$ belongs to $C^\infty(M,\mathcal{E})\hat{\otimes}C^\infty(M,\mathcal{E}^*\otimes|\Lambda|(M))$ where $|\Lambda|(M)$ denotes the bundle of densities on M.

Recall that the heat semigroup allows us to invert the heat equation, in the sense that the operator

$$Q_0u(x,s) = \int_0^\infty e^{-s\Delta}u(x,t-s)dt, \qquad u \in C_c^\infty(M\times\mathbb{R},\mathcal{E}), \tag{1.1}$$

maps continuously into $C^0(\mathbb{R},L^2(M,\mathcal{E})) \subset \mathcal{D}'(M\times\mathbb{R},\mathcal{E})$ and satisfies

$$(\Delta+\partial_t)Q_0u = Q_0(\Delta+\partial_t)u = u \qquad \forall u \in C_c^\infty(M\times\mathbb{R},\mathcal{E}). \tag{1.2}$$

Notice that the operator Q_0 has the *Volterra property* in the sense of [Pi], i.e. it has a distribution kernel of the form $K_{Q_0}(x,y,t-s)$ where $K_{Q_0}(x,y,t)$ vanishes on the region $t<0$. In fact,

$$K_{Q_0}(x,y,t) = \begin{cases} k_t(x,y) & \text{if } t>0, \\ 0 & \text{if } t<0. \end{cases} \tag{1.3}$$

These equalities are the main motivation for using pseudodifferential techniques to study the heat kernel $k_t(x,y)$. The idea, which goes back to Hadamard [Ha], is to consider a class of ΨDO's, the Volterra ΨDO's ([Gr], [Pi], [BGS]), taking into account:

(i) The aforementioned Volterra property;

(ii) The parabolic homogeneity of the heat operator $\Delta+\partial_t$, i.e. the homogeneity with respect to the dilations $\lambda.(\xi,\tau) = (\lambda\xi,\lambda^2\tau)$, $(\xi,\tau)\in\mathbb{R}^{n+1}$, $\lambda \neq 0$.

In the sequel for $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ and $\lambda \neq 0$ we let g_λ be the tempered distribution defined by

$$\langle g_\lambda(\xi,\tau), u(\xi,\tau)\rangle = |\lambda|^{-(n+2)}\langle g(\xi,\tau), u(\lambda^{-1}\xi,\lambda^{-2}\tau)\rangle, \quad u \in \mathcal{S}(\mathbb{R}^{n+1}). \tag{1.4}$$

Definition 1.1. *A distribution $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ is parabolic homogeneous of degree m, $m \in \mathbb{Z}$, if for any $\lambda \neq 0$ we have $g_\lambda = \lambda^m g$.*

Let $\mathbb{C}_-$ denote the complex halfplane $\{\Im\tau > 0\}$ with closure $\bar{\mathbb{C}}_-$. Then:

Lemma 1.2 ([BGS, Prop. 1.9])**.** *Let $q(\xi,\tau) \in C^\infty((\mathbb{R}^n\times\mathbb{R})\setminus 0)$ be a parabolic homogeneous symbol of degree m such that:*

(i) q extends to a continuous function on $(\mathbb{R}^n\times\bar{\mathbb{C}}_-)\setminus 0$ in such way to be holomorphic in the last variable when the latter is restricted to $\mathbb{C}_-$.

Then there is a unique $g \in \mathcal{S}'(\mathbb{R}^{n+1})$ agreeing with q on $\mathbb{R}^{n+1}\setminus 0$ so that:

(ii) g is homogeneous of degree m;

(iii) The inverse Fourier transform $\check{g}(x,t)$ vanishes for $t<0$.

Remark 1.3. If we take $m \leq -(n+2)$ the result fails in general for symbols not satisfying (i).

Let U be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDO's on $U \times \mathbb{R}^{n+1} \setminus 0$ as follows.

Definition 1.4. *$S_{\mathrm{v}}^m(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists in smooth functions $q(x, \xi, \tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q \sim \sum_{j \geq 0} q_{m-j}$ where:*

- $q_l \in C^\infty(U \times [(\mathbb{R}^n \times \mathbb{R}) \setminus 0])$ is a homogeneous Volterra symbol of degree l, i.e. q_l is parabolic homogeneous of degree l and satisfies the property (i) in Lemma 1.2 with respect to the last $n+1$ variables;

- The sign $\sim$ means that, for any integer N and any compact $K \subset U$, there is a constant $C_{NK\alpha\beta k} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j<N} q_{m-j})(x, \xi, \tau)| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{1/2})^{m-N-|\beta|-2k}, \tag{1.5}$$

for $x \in K$ and $|\xi| + |\tau|^{\frac{1}{2}} > 1$.

Definition 1.5. *$\Psi_{\mathrm{v}}^m(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists in continuous operators Q from $C_c^\infty(U_x \times \mathbb{R}_t)$ to $C^\infty(U_x \times \mathbb{R}_t)$ such that:*

(i) Q has the Volterra property;

(ii) $Q = q(x, D_x, D_t) + R$ for some symbol q in $S_{\mathrm{v}}^m(U \times \mathbb{R})$ and some smoothing operator R.

In the sequel if Q is a Volterra ΨDO we let $K_Q(x, y, t-s)$ denote its distribution kernel, so that the distribution $K_Q(x, y, t)$ vanishes for $t < 0$.

Example 1.6. Let P be a differential operator of order 2 on U and let $p_2(x, \xi)$ denote the principal symbol of P. Then the heat operator $P + \partial_t$ is a Volterra ΨDO of order 2 with principal symbol $p_2(x, \xi) + i\tau$.

Other examples of Volterra ΨDO's are given by the homogeneous operators as in below.

Definition 1.7. *Let $q_m(x, \xi, \tau) \in C^\infty(U \times (\mathbb{R}^{n+1} \setminus 0))$ be a homogeneous Volterra symbol of order m and let $g_m \in C^\infty(U) \hat{\otimes} \mathcal{S}'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 1.2. Then:*

- $\check{q}_m(x, y, t)$ is the inverse Fourier transform of $g_m(x, \xi, \tau)$ in the last $n+1$ variables;

- $q_m(x, D_x, D_t)$ is the operator with kernel $\check{q}_m(x, y-x, t)$.

Proposition 1.8 ([Gr], [Pi], [BGS]). *The following properties hold.*

1) Composition. *Let $Q_j \in \Psi_{\mathrm{v}}^{m_j}(U \times \mathbb{R})$, $j = 1, 2$, have symbol q_j and suppose that Q_1 or Q_2 is properly supported. Then $Q_1 Q_2$ belongs to $\Psi_{\mathrm{v}}^{m_1+m_2}(U \times \mathbb{R})$ and has symbol $q_1 \# q_2 \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha q_1 D_\xi^\alpha q_2$.*

2) Parametrices. *An operator $Q \in \Psi_{\mathrm{v}}^m(U \times \mathbb{R})$ admits a parametrix in $\Psi_{\mathrm{v}}^{-m}(U \times \mathbb{R})$ if, and only if, its principal symbol is nowhere vanishing on $U \times [(\mathbb{R}^n \times \bar{\mathbb{C}}_- \setminus 0)]$.*

3) Invariance. *Let $\phi : U \to V$ be a diffeomorphism onto another open subset V of $\mathbb{R}^n$ and let Q be a Volterra ΨDO on $U \times \mathbb{R}$ of order m. Then $Q = (\phi \oplus \mathrm{id}_{\mathsf{R}})_* Q$ is a Volterra ΨDO on $V \times \mathbb{R}$ of order m.*

In addition to the above standard properties there is the one below which shows the relevance of Volterra ΨDO's for deriving small times asymptotics.

Lemma 1.9 ([Gr, Chap. I], [BGS, Thm. 4.5]). *Let $Q \in \Psi_{\mathrm{v}}^m(U \times \mathbb{R})$ have symbol $q \sim \sum q_{m-j}$. Then the following asymptotics holds in $C^\infty(U)$,*

$$K_Q(x, y, t) \sim_{t \to 0^+} t^{-(\frac{n}{2} + [\frac{m}{2}] + 1)} \sum_{l \geq 0} t^l \check{q}_{2[\frac{m}{2}] - 2l}(x, 0, 1), \tag{1.6}$$

where the notation $\check{q}_k$ has the same meaning as in Definition 1.7.

Proof. As the Fourier transform relates the decay at infinity to the behavior at the origin of the Fourier transform the distribution $\check{q} - \sum_{j\leq J} \check{q}_{m-j}$ lies in $C^N(U_x \times \mathbb{R}^n_y \times \mathbb{R}_t)$ as soon as J is large enough. Since $Q - q(x, D_x, D_t)$ is smoothing it follows that $R_J(x,t) = K_Q(x,x,t) - \sum_{j\leq J} \check{q}_{m-j}(x,0,t)$ is of class C^N. As $R_J(x,y,t) = 0$ for $t < 0$ we get $\partial_t^l R_J(x,0) = 0$ for $l = 0, 1, \ldots, N$, so that $R_J(.,t)$ is a $\mathrm{O}(t^N)$ in $C^N(U)$ as $t \rightarrow 0^+$. It follows that in $C^\infty(U)$ we have the asymptotics $K_Q(x,x,t) \sim_{t\rightarrow 0^+} \sum \check{q}_{m-j}(x,0,t)$.

Now, $(\check{q}_{m-j})_\lambda = |\lambda|^{-(n+2)}(q_{m-j,\lambda^{-1}})^\vee = |\lambda|^{-(n+2)}\lambda^{j-m}\check{q}_{m-j}$ for any $\lambda \neq 0$. So letting $\lambda = \sqrt{t}$, $t > 0$, yields $\check{q}_{m-j}(x,0,t) = t^{\frac{j-n-m}{2}-1}\check{q}_{m-j}(x,0,1)$, while for $\lambda = -1$ we get $\check{q}_{m-j}(x,0,1) = -q_{m-j}(x,0,1) = 0$ whenever $m-j$ is odd. Thus,

$$K_Q(x,x,t) \sim_{t\rightarrow 0^+} \sum_{m-j \text{ even}} t^{\frac{j-n-m}{2}-1}\check{q}_{m-j}(x,0,1), \tag{1.7}$$

that is $K_Q(x,x,t) \sim_{t\rightarrow 0^+} t^{-(\frac{n}{2}+[\frac{m}{2}]+1)} \sum_{l\geq 0} t^l \check{q}_{2[\frac{m}{2}]-l}(x,0,1)$. ☐

The invariance property in Proposition 1.8 allows us to define Volterra ΨDO's on $M \times \mathbb{R}$ acting on the sections of the vector bundle $\mathcal{E}$. Then all the preceding properties hold *verbatim* in this context. In particular the heat operator $\Delta + \partial_t$ has a parametrix Q in $\Psi_{\mathrm{v}}^{-2}(M, \times\mathbb{R}, \mathcal{E})$. In fact, comparing the operator (1.1) with any Volterra parametrix for $\Delta + \partial_t$ allows us to prove:

Theorem 1.10 ([Gr], [Pi], [BGS, pp. 363-362])**.** *The differential operator $\Delta + \partial_t$ is invertible and its inverse $(\Delta + \partial_t)^{-1}$ is a Volterra ΨDO of order -2.*

Combining this with Lemma 1.9 gives the heat kernel asymptotics below.

Theorem 1.11 ([Gr, Thm. 1.6.1])**.** *In $C^\infty(M, |\Lambda|(M) \otimes \operatorname{End}\mathcal{E})$ we have:*

$$k_t(x,x) \sim_{t\rightarrow 0^+} t^{\frac{-n}{2}} \sum_{l\geq 0} t^l a_l(\Delta)(x), \qquad a_l(\Delta)(x) = \check{q}_{-2-2l}(x,0,1), \tag{1.8}$$

where the equality on the right-hand side shows how to compute the densities $a_l(\Delta)(x)$'s in local trivializing coordinates by means of the symbol $q \sim \sum q_{-2-j}$ of any Volterra parametrix for $\Delta + \partial_t$.

This approach to the heat kernel asymptotics present several advantages. First, as Theorem 1.11 is a purely local statement we can easily localize the heat kernel asymptotics. In fact, given a Volterra parametrix Q for $\Delta + \partial_t$ in some local trivializing coordinates around $x_0 \in M$, comparing the asymptotics (1.6) and (1.8) we get

$$k_t(x_0,x_0) = K_Q(x_0,x_0,t) + \mathrm{O}(t^\infty) \qquad \text{as } t \rightarrow 0^+. \tag{1.9}$$

Therefore in order to determine the heat kernel asymptotics (1.8) at x_0 we only need a Volterra parametrix for $\Delta + \partial_t$ near x_0.

Second, we have a genuine asymptotics with respect to the C^∞-topology, which can be differentiated as follows.

Proposition 1.12. *Let $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$ be a differential operator of order m and let $h_t(x,y)$ denote the distribution kernel of $Pe^{-t\Delta}$. Then in $C^\infty(M, |\Lambda| \otimes \operatorname{End}\mathcal{E})$ we have*

$$h_t(x,x) \sim_{t\rightarrow 0^+} t^{[\frac{m}{2}]-\frac{n}{2}} \sum_{l\geq 0} t^l b_l(x), \qquad b_l(x) = \check{r}_{2[\frac{m}{2}]-2-2l}(x,0,1), \tag{1.10}$$

where the equality on the right-hand side gives a formula for computing the densities $b_l(x)$'s in local trivializing coordinates using the symbol $r \sim \sum r_{m-2-j}$ of $R = P(\Delta + \partial_t)^{-1}$ (or of $R = PQ$ where Q is any Volterra parametrix for $\Delta + \partial_t$).

Proof. As $h_t(x,y) = P_x k_t(x,y) = P_x K_{(\Delta+\partial_t)^{-1}}(x,y,t) = K_{P(\Delta+\partial_t)^{-1}}(x,y,t)$ the result follows by applying Lemma 1.9 to $P(\Delta+\partial_t)^{-1}$ (or to PQ where Q is any Volterra parametrix for $\Delta+\partial_t$). ☐

Finally, in local trivializing coordinates the densities $a_j(\Delta)(x)$'s can be explicitly computed in terms of the symbol $p = p_2 + p_1 + p_0$ of Δ. To see this let $q \sim \sum q_{-2-j}$ be the symbol of a Volterra parametrix Q for $\Delta + \partial_t$. As $q\#p \sim q(p + i\tau) + \sum \frac{1}{\alpha!}\partial_\xi^\alpha q D_x^\alpha p \sim 1$ we get $q_{-2} = (p_2 + i\tau)^{-1}$ and

$$q_{-2-j} = -\Big(\sum_{k+l+|\alpha|=j} \frac{1}{\alpha!}\partial_\xi^\alpha q_{-2-k} D_x^\alpha p_{2-l}\Big)(p_2 + i\tau)^{-1}, \quad j \geq 1. \tag{1.11}$$

Therefore, combining with (1.8) we deduce that, as in [Gi], the densities $a_j(\Delta)(x)$'s are universal polynomials in the the jets at x_0 of the symbol of Δ with coefficients depending smoothly on its principal symbol. Similarly, in local trivializing coordinates the densities $b_l(x)$'s in (1.10) can be expressed universal polynomials in the the jets at x_0 of the symbols of Δ and P with coefficients depending smoothly on the principal symbol of Δ.

2. The local index formula of Atiyah and Singer

In this section we shall give a new proof the local index formula of Atiyah and Singer ([AS1], [AS2]) by using Greiner's approach of the heat kernel asymptotics.

Let (M^n, g) be an even dimensional compact Riemannian spin manifold with spin bundle $\not{S}$ and let $\mathcal{E}$ denote a Hermitian vector bundle over M equipped with an unitary connection $\nabla^{\mathcal{E}}$ with curvature $F^{\mathcal{E}}$. Since n is even $\operatorname{End}\not{S}$ is as a bundle of algebras over M isomorphic to the Clifford bundle $\mathrm{Cl}(M)$, whose fiber $\mathrm{Cl}_x(M)$ at $x \in M$ is the complex algebra generated by 1 and T_x^*M with relations

$$\xi.\eta + \eta.\xi = -2\langle \xi, \eta\rangle, \qquad \xi, \eta \in T_x^*M. \tag{2.1}$$

Recall that he quantization map $c : \Lambda T_{\mathbb{C}}^*M \to \mathrm{Cl}(M)$ and the symbol map $\sigma = c^{-1}$ satisfy

$$\sigma(c(\xi)c(\eta)) = \xi \wedge \eta - \xi \llcorner \eta, \qquad \xi \in T_{\mathbb{C}}^*M, \quad \eta \in \Lambda T_{\mathbb{C}}^*M, \tag{2.2}$$

where $\llcorner$ is the interior product. Therefore, for ξ and η in $\Lambda T_{\mathbb{C}}^*M$ we have

$$\sigma(c(\xi^{(i)})c(\eta^{(j)})) = \xi^{(i)} \wedge \eta^{(j)} \mod \Lambda^{i+j-2}T_{\mathbb{C}}^*M, \tag{2.3}$$

where $\zeta^{(l)}$ denotes the component in $\Lambda^l T_{\mathbb{C}}^*M$ of $\zeta \in \Lambda T_{\mathbb{C}}^*M$. Thus the $\mathbb{Z}_2$-grading on $\Lambda T_{\mathbb{C}}^*M$ given by the parity of forms induces a $\mathbb{Z}_2$-grading $\not{S} = \not{S}^+ \otimes \not{S}^-$ on the spin bundle. Furthermore, if $e_1, \ldots, e_n$ is an orthonormal frame for T_xM and we regard $c(dx^{i_1}) \cdots c(dx^{i_k})$, $i_1 < \ldots < i_k$, as an endomorphism of $\not{S}_x$ then

$$\operatorname{Str}_x c(e^{i_1}) \cdots c(e^{i_k}) = \begin{cases} 0 & \text{if } k \neq n, \\ (-2i)^{\frac{n}{2}} & \text{if } k = n. \end{cases} \tag{2.4}$$

Let $\nabla^{\not{S}\otimes\mathcal{E}} = \nabla^{\not{S}} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}}$ be the connection on $\not{S} \otimes \mathcal{E}$, where $\nabla^{\not{S}}$ denotes the Levi-Civita connection lifted to the spin bundle. Then the Dirac operator $\not{D}_{\mathcal{E}}$ acting on the sections of $\not{S} \otimes \mathcal{E}$ is given by the composition

$$C^\infty(M, \not{S} \otimes \mathcal{E}) \xrightarrow{\nabla^{\not{S}\otimes\mathcal{E}}} C^\infty(M, T^*M \otimes \not{S} \otimes \mathcal{E}) \xrightarrow{c\otimes 1} C^\infty(M, \not{S} \otimes \mathcal{E}). \tag{2.5}$$

It is odd with respect to the $\mathbb{Z}_2$-grading $\not{S} \otimes \mathcal{E} = (\not{S}^+ \otimes \mathcal{E}) \oplus (\not{S}^- \otimes \mathcal{E})$, i.e. it can be written in the form

$$\not{D}_{\mathcal{E}} = \begin{pmatrix} 0 & D_{\mathcal{E}}^+ \\ D_{\mathcal{E}}^- & 0 \end{pmatrix}, \qquad \not{D}_{\mathcal{E}}^\pm : C^\infty(M, \not{S}^\mp \otimes \mathcal{E}) \to C^\infty(M, \not{S}^\pm \otimes \mathcal{E}). \tag{2.6}$$

Moreover, by the Lichnerowicz formula ([BGV], [LM], [Ro]) we have

$$\not{D}_{\mathcal{E}}^2 = (\nabla_i^{\not{S}\otimes\mathcal{E}})^*\nabla_i^{\not{S}\otimes\mathcal{E}} + \mathcal{F}^{\mathcal{E}} + \frac{\kappa^M}{4}, \tag{2.7}$$

where κ^M denotes the scalar curvature of M and $\mathcal{F}^{\mathcal{E}}$ the curvature $F^{\mathcal{E}}$ lifted to $\not{S} \otimes \mathcal{E}$, i.e. $F^{\mathcal{E}} = \frac{1}{2} c(e^k) c(e^l) F^{\mathcal{E}}(e_k, e_l)$ for any local orthonormal tangent frame $e_1, \ldots, e_n$. It follows that $\not{D}_{\mathcal{E}}$ and $\not{D}_{\mathcal{E}}^{\pm}$ are elliptic, hence are Fredholm.

Theorem 2.1 ([AS1], [AS2])**.** *We have:*

$$\operatorname{ind} \not{D}_{\mathcal{E}}^{+} = (2i\pi)^{-\frac{n}{2}} \int_M [\hat{A}(R_M) \wedge \operatorname{Ch}(F^{\mathcal{E}})]^{(n)}, \tag{2.8}$$

where $\hat{A}(R^M) = \det^{\frac{1}{2}}\left(\frac{R^M/2}{\sinh(R^M/2)}\right)$ *is the total $\hat{A}$-form of the Riemann curvature and* $\operatorname{Ch}(F^{\mathcal{E}}) = \operatorname{Tr} \exp(-F^{\mathcal{E}})$ *the total Chern form of the curvature $F^{\mathcal{E}}$.*

In fact, by the McKean-Singer formula $\operatorname{ind} \not{D}_{\mathcal{E}}^{+} = \operatorname{Str} e^{-t \not{D}_{\mathcal{E}}^2}$ for any $t > 0$. Therefore the index formula follows from:

Theorem 2.2. *In $C^\infty(M, |\Lambda|(M))$ we have*

$$\operatorname{Str}_x k_t(x,x) = [\hat{A}(R_M) \wedge \operatorname{Ch}(F^{\mathcal{E}})]^{(n)} + \mathrm{O}(t) \qquad \text{as } t \to 0^+. \tag{2.9}$$

This theorem, also called local index theorem, was first proved by Patodi, Gilkey and Atiyah-Bott-Patodi ([ABP], [Gi]), and then in a purely analytic fashion by Getzler ([Ge1], [Ge2]) and Bismut [Bi] (see also [BGV], [Ro]). Moreover, as it is a purely local statement it holds *verbatim* for (geometric) Dirac operators acting on a Clifford bundle. Thus it allows us to recover, on the one hand, the Gauss-Bonnet, signature and Riemann-Roch theorems ([ABP], [BGV], [LM], [Ro]) and, on the other hand, the full index theorem of Atiyah-Singer ([ABP], [LM]).

The short proof of Getzler [Ge2] combines the Feynman-Kac representation of the heat kernel with an ingenious trick, the Getzler rescaling. We can alternatively prove Theorem 2.2 by combining Getzler rescaling with Greiner's approach of the heat kernel asymptotics as follows.

Proof of Theorem 2.2. First, the Greiner approach allows us to easily localize the problem (compare [Ge2]). Indeed, thanks to Theorem 1.11 $\operatorname{Str}_x k_t(x,x)$ admits an asymptotics in $C^\infty(M, |\Lambda|(M))$ as $t \to 0^+$. Thus, it is enough to prove (2.9) at a point $x_0 \in M$. Furthermore, to reach this aim we know from (1.9) that we only need a Volterra parametrix for $\not{D}_{\mathcal{E}}^2 + \partial_t$ in local trivializing coordinates centered at x_0. Therefore, using normal coordinates centered at x_0 and a trivialization of the tangent bundle by means of a synchronous frame $e_1, \ldots, e_n$ such that $e_j = \partial_j$ at $x = 0$ we may replace $\not{D}_{\mathcal{E}}$ by a Dirac operator $\not{D}$ on $\mathbb{R}^n$ acting on the trivial bundle with fiber $\not{S}_n \otimes \mathbb{C}^p$, where $\not{S}_n$ denotes the spin bundle of $\mathbb{R}^n$. Then we have

$$k_t(0,0) = K_Q(0,0,t) + \mathrm{O}(t^\infty) \qquad \text{as } t \to 0^+. \tag{2.10}$$

Second, as pointed out in [ABP] (see also [BGV], [Ro]) choosing normal coordinates and a synchronous tangent frame makes the metric g and the coefficients $\omega_{ikl} = \langle \nabla_i^{LC} e_k, e_l \rangle$ of the Levi-Civita connection have behaviors near $x = 0$ of the form

$$g_{ij}(x) = \delta_{ij} + \mathrm{O}(|x|^2), \qquad \omega_{ikl}(x) = -\frac{1}{2} R^M_{ijkl}(0) x^j + \mathrm{O}(|x|^2), \tag{2.11}$$

where $R^M_{ijkl}(0) = \langle R^M(0)(\partial_i, \partial_j)\partial_k, \partial_l \rangle$. Then using (2.4) and (2.10) we get

$$\operatorname{Str} k_t(0,0) = (-2i)^{\frac{n}{2}} \sigma \otimes \operatorname{Tr}_{\mathbf{C}^p} [K_Q(0,0,t)]^{(n)} + \mathrm{O}(t^\infty). \tag{2.12}$$

Thus we are reduced to prove the convergence of $\sigma[K_Q(0,0,t)]^{(n)}$ as $t \to 0^+$ and to identify its limit.

Now, recall that the Getzler rescaling [Ge2] assigns the following degrees:

$$\deg \partial_j = \frac{1}{2} \deg \partial_t = \deg c(dx^j) = -\deg x^j = 1, \tag{2.13}$$

while $\deg B = 0$ for any $B \in M_p(\mathbb{C})$. It can define a filtration of Volterra ΨDO's with coefficients in $\operatorname{End}(\$_n \otimes \mathbb{C}^p) \simeq \mathrm{Cl}(\mathbb{R}^n) \otimes M_p(\mathbb{C})$ as follows.

Let $Q \in \Psi_{\mathrm{v}}^*(\mathbb{R}^n \times \mathbb{R}, \$_n \otimes \mathbb{C}^p)$ have symbol $q(x,\xi,\tau) \sim \sum_{k \leq m'} q_k(x,\xi,\tau)$. Then taking components in each subspace $\Lambda^j T^*_{\mathbf{C}}\mathbb{R}^n(n)$ and then using Taylor expansions at $x=0$ gives formal expansions

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j,k} \sigma[q_k(x,\xi,\tau)]^{(j)} \sim \sum_{j,k,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_k(0,\xi,\tau)]^{(j)}. \tag{2.14}$$

According to (2.13) the symbol $\frac{x^\alpha}{\alpha!}\partial_x^\alpha \sigma[q_k(0,\xi,\tau)]^{(j)}$ is Getzler homogeneous of degree $k+j-|\alpha|$. Therefore, we can expand $\sigma[q(x,\xi,\tau)]$ as

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j \geq 0} q_{(m-j)}(x,\xi,\tau), \qquad q_{(m)} \neq 0, \tag{2.15}$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m-j$.

Definition 2.3. *Using (2.15) we make the following definitions:*

- *The integer m is the Getzler order of Q,*
- *The symbol $q_{(m)}$ is the principal Getzler homogeneous symbol of Q,*
- *The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ is the model operator of Q.*

Remark 2.4. The model operator $Q_{(m)}$ is well defined according to definition 1.7.

Remark 2.5. By construction we always have Getzler order $\leq$ order $+n$, but this is not an equality in general.

Example 2.6. Let $A = A_i dx^i$ is the connection one-form on $\mathbb{C}^p$. Then by (2.11) the covariant derivative $\nabla_i = \partial_i + \frac{1}{4}\omega_{ikl}(x)c(e^k)c(e^l) + A_i$ on $\$_n \otimes \mathbb{C}^p$ has Getzler order 1 and model operator

$$\nabla_{i(1)} = \partial_i - \frac{1}{4} R^M_{ij}(0) x^j, \qquad R^M_{ij}(0) = R^M_{ijkl}(0) dx^k \wedge dx^l. \tag{2.16}$$

The interest to introduce Getzler orders stems from the following.

Lemma 2.7. *Let $Q \in \Psi_{\mathrm{v}}^*(\mathbb{R}^n \times \mathbb{R}, S_n \otimes \mathbb{C}^p)$ have Getzler order m and model operator $Q_{(m)}$. Then as $t \to 0^+$ we have:*

- $\sigma[K_Q(0,0,t)]^{(j)} = \mathrm{O}(t^{\frac{j-m-n-1}{2}})$ *if $m-j$ is odd;*
- $\sigma[K_Q(0,0,t)]^{(j)} = t^{\frac{j-m-n}{2}-1} K_{Q_{(m)}}(0,0,1)^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}})$ *if $m-j$ is even.*

In particular for $m=-2$ we get

$$\sigma[K_Q(0,0,t)]^{(n)} = K_{Q_{(-2)}}(0,0,1)^{(n)} + \mathrm{O}(t). \tag{2.17}$$

Proof. Let $q(x,\xi,\tau) \sim \sum q_k(x,\xi,\tau)$ be the symbol of Q and let $q_{(m)}(x,\xi,\tau)$ be the principal Getzler homogeneous symbol. By Lemma 1.9 we have

$$\sigma[K_Q(0,0,t)]^{(j)} \sim_{t\to 0^+} \sum t^{-\frac{n+2+m-j}{2}} \sigma[\check{q}_k(0,0,1)]^{(j)}, \tag{2.18}$$

and we know that $\check{q}_k(0,0,1) = 0$ if k is odd. Also, the symbol $\sigma[q_k(0,\xi,\tau)]^{(j)}$ is Getzler homogeneous of degree $k+j$, so it must be zero if $k+j > m$ since otherwise Q would not have Getzler order m. Hence:

- $\sigma[K_Q(0,0,t)]^{(j)} = \mathrm{O}(t^{\frac{j-m-n+1}{2}})$ if $m-j$ is odd;
- $\sigma[K_Q(0,0,t)]^{(j)} = t^{\frac{j-m-n}{2}-1} \sigma[\check{q}_{m-j}(0,0,1)]^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}})$ if $m-j$ is even.

On the other hand, notice that the symbol $\sigma[q_{(m)}(0,\xi,\tau)]^{(j)}$ is equal to

$$\sum_{k+j-|\alpha|=m} \left(\frac{x^\alpha}{\alpha!}\partial_x^\alpha \sigma[q_k(0,\xi,\tau)]^{(j)}\right)_{x=0} = \sigma[q_{m-j}(0,\xi,\tau)]^{(j)}. \tag{2.19}$$

Thus $\sigma[\check{q}_{m-j}(0,0,1)]^{(j)} = K_{Q_{(m)}}(0,0,1)^{(j)}$. Hence the lemma. $\square$

In the sequel we say that a symbol or a ΨDO is $\mathrm{O}_{\mathrm{G}}(m)$ if it has Getzler order $\leq m$.

Lemma 2.8. *For $j=1,2$ let $Q_j \in \Psi^*_{\mathrm{v}}(\mathbb{R}^n \times \mathbb{R}, \mathrm{End}(\not{S}_n \otimes \mathbb{C}^p)$ have Getzler order m_j and model operator $Q_{(m_j)}$ and assume either Q_1 or Q_2 properly supported. Then we have:*

$$Q_1 Q_2 = c[Q_{(m_1)} Q_{(m_2)}] + \mathrm{O}_G(m_1+m_2-1). \tag{2.20}$$

Proof. Let q_j be the symbol of Q_j and let $q_{(m_j)}$ be its principal Getzler homogeneous symbol. By Proposition 1.8 the operator Q_1Q_2 has symbol $q_1 \# q_2$. Moreover for N large enough $q_1\#q_2 - \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\partial_\xi^\alpha q_1 . D_x^\alpha q_2$ has order $< m_1+m_2-n$, so has Getzler order $< m_1+m_2$. As $\partial_\xi^\alpha q_1 . D_x^\alpha q_2 - c[\partial_\xi^\alpha q_{(m_1)} \wedge D_x^\alpha f_{(m_2)}]$ has Getzler order $\leq m_1+m_2-|\alpha|-1$ it follows that for N large enough,

$$q_1\#q_2 = \sum_{|\alpha|\leq N} \frac{1}{\alpha!} c(\partial_\xi^\alpha q_{m_1} \wedge D_x^\alpha q_{m_2}) + \mathrm{O}_G(m_1+m_2-1). \tag{2.21}$$

On the other hand, $\sum \frac{1}{\alpha!}\partial_\xi^\alpha q_{(m_1)} \wedge D_x^\alpha q_{(m_2)}$ is exactly the symbol of $Q_{(m_1)}Q_{(m_2)}$ since $q_{(m_2)}(x,\xi,\tau)$ is polynomial in x and thus the sum is finite. Therefore taking N large enough in (2.21) shows that the symbols of Q_1Q_2 and $Q_{(m_1)}Q_{(m_2)}$ coincide modulo a symbol of Getzler order $\leq m_1+m_2-1$. $\square$

Recall that by the Lichnerowicz formula (2.7) we have

$$\not{D}_{\mathcal{E}}^2 = -g^{ij}(\nabla_i\nabla_j - \Gamma_{ij}^k \nabla_k) + \frac{1}{2} c(e^i)c(e^j)F(e_i,e_j) + \frac{\kappa}{4}, \tag{2.22}$$

where the Γ_{ij}^k's are the Christoffel symbols of the metric. Thus combining Lemma 2.8 with (2.11) and (2.16) shows that $\not{D}^2$ has Getzler order 2 and its model operator is

$$\begin{aligned} \not{D}_{(2)}^2 &= -\delta_{ij}\nabla_{i(1)}\nabla_{j(1)} + \frac{1}{2}F^{\mathcal{E}}(\partial_k,\partial_l)(0)dx^k \wedge dx^l \\ &= H_R + F^{\mathcal{E}}(0), \qquad H_R = -\sum_{i=1}^n (\partial_i - \frac{1}{4}R_{ij}^M(0)x^j)^2. \end{aligned} \tag{2.23}$$

Lemma 2.9. *Let Q be a Volterra parametrix for $\not{D}^2 + \partial_t$. Then:*

1) Q has Getzler order 2 and its model operator is $(H_R + F^{\mathcal{E}}(0) + \partial_t)^{-1}$.

2) We have

$$K_{(H_R+F^{\mathcal{E}}(0)+\partial_t)^{-1}}(x,0,t) = G_R(x,t) \wedge e^{-tF^{\mathcal{E}}(0)}, \tag{2.24}$$

where $G_R(x,t)$ is the fundamental solution of $H_R+\partial_t$, i.e. the unique distribution such that $(H_R + F^{\mathcal{E}}(0)+\partial_t)G_R(x,t) = \delta(x,t)$.

3) As $t \to 0^+$ we have

$$\sigma[K_Q(0,0,t)]^{(2j)} = t^{j-\frac{n}{2}}[G_R(0,1) \wedge e^{-F^{\mathcal{E}}(0)}]^{(2j)} + \mathrm{O}(t^{j-\frac{n}{2}+1}). \tag{2.25}$$

Proof. Note that 3) follows by combining 1) and 2) with Lemma 2.7, so we only have to prove the first two assertions. Let $p(x,\xi) = \sum p_j(x,\xi)$ be the symbol of $\not{D}^2$ and let $q \sim \sum q_{-2-j}$ denote the symbol of Q. As $\not{D}^2$ is elliptic and has Getzler order 2 we have $p_{(2)}(0,\xi)^{(0)} = p_2(0,\xi) \neq 0$. Hence $q_{-2} = (p_2 + i\tau)^{-1}$ has Getzler order -2. It then follows from (1.11) that each symbol q_{-2-j} has Getzler order ≤ -2. Hence Q has Getzler order -2.

On the other hand, $(\not{D}^2+\partial_t)Q - 1$ is smoothing, so by Lemma 2.8 the operator $(H_R+F^{\mathcal{E}}(0) + \partial_t)Q_{(-2)} - 1$ has Getzler order ≤ -1. As the latter is Getzler homogeneous of degree 0 it must be zero. Hence $Q_{(-2)} = (H_R + F^{\mathcal{E}}(0)+\partial_t)^{-1}$, so that we have

$$(H_{R,x} + F^{\mathcal{E}}(0) + \partial_t)K_{Q_{(-2)}}(x,y,t-s) = \delta(x-y,t-s). \tag{2.26}$$

Now, setting $y = 0$ and $s = 0$ in (2.26) shows that $G_{R,F}(x,t) = K_{Q_{(-2)}}(x,0,t)$ is the fundamental solution of $H_R + F(0) + \partial_t$. In fact, if we let $G_R(x,t)$ be the fundamental solution of $H_R + F(0) + \partial_t$ then $G_{R,F}(x,t) = G_R(x,t) \wedge e^{-tF^{\mathcal{E}}(0)}$. Thus $K_{Q_{(-2)}}(x,0,t) = G_R(x,t) \wedge e^{-tF^{\mathcal{E}}(0)}$. □

At this stage remark that H_R is the harmonic oscillator associated to the antisymmetric matrix $R^M(0) = (R^M_{ij}(0))$. Therefore we can make use of a version of the Melher formula ([GJ], [Ge2]) to obtain:

Lemma 2.10. *The fundamental solution $G_R(x,t)$ of $H_R + \partial_t$ is*

$$\chi(t)(4\pi t)^{-\frac{n}{2}} \det{}^{\frac{1}{2}}\Big(\frac{tR^M(0)/2}{\sinh(tR^M(0)/2)}\Big) \exp\Big(-\frac{1}{4t}\langle \frac{tR^M(0)/2}{\tanh(tR^M(0)/2)}x, x\rangle\Big), \tag{2.27}$$

where $\chi(t)$ is the characteristic function of $(0, +\infty)$.

Proof. Let $a \in \mathbb{R}$ and let H_a denote the harmonic oscillator $-\frac{d}{dx^2} + \frac{1}{4}a^2x^2$ on $\mathbb{R}$. Then the fundamental solution of $H_a + \partial_t$ is $G_a(x,t) = \chi(t)S_a(x,t)$, where

$$S_a(x,t) = (4\pi t)^{-\frac{1}{2}}\Big(\frac{at}{\sinh at}\Big)^{\frac{1}{2}} \exp\Big(-\frac{1}{4t}x^2\frac{at}{\tanh at}\Big), \quad t > 0. \tag{2.28}$$

In fact $(H + \partial_t)S_a = 0$ on $\mathbb{R} \times (0, +\infty)$ and $S(.,t) \to \delta$ in $\mathcal{S}'(\mathbb{R})$, since on compact sets $\hat{S}_{x\to\xi}(\xi,t) = \cosh^{-\frac{1}{2}}(at)\exp(-\xi^2 t\frac{\tanh at}{at})$ converges to 1. Hence $(H + \partial_t)k_a = \chi' G(.,0) + \chi(H + \partial_t)G = \delta$.

More generally, if A is a real $n \times n$ antisymmetric matrix and we let $B = -A^2$, then the fundamental solution of $-\sum \partial_j^2 + \frac{1}{4}B_{jk}x^jx^k + \partial_t$ on $\mathbb{R}^n \times \mathbb{R}$ is

$$G_A(x,t) = \chi(t)(4\pi t)^{-\frac{n}{2}} \det{}^{\frac{1}{2}}\Big(\frac{iAt}{\sinh(iAt)}\Big) \exp\Big(-\frac{1}{4t}\langle\frac{iAt}{\tanh(iAt)}x, x\rangle\Big). \tag{2.29}$$

The passage from the formula for G_a to the one for G_A uses $O(n)$-invariance and in particular invariance under rotations in the (x_j, x_k)-plane, $j < k$. Thus G_A is also the fundamental solution for $-\sum(\partial_j - \frac{i}{2}A_{jk}x^j)^2 + \partial_t$.

Now, the r.h.s. in (2.29) is analytic with respect to A and $R^M(0)$ is an antisymmetric matrix made out of 2-forms which commute with other forms. Therefore the formula for G_A with A replaced by $-iR^M(0)/2$ gives the fundamental solution of $H_R + \partial_t$. □

Finally, combining the formula for $G_R(x,t)$ with Lemma 2.9 and (2.12) we get

$$\operatorname{Str} k_t(0,0) = (2i\pi)^{-\frac{n}{2}}[\hat{A}(R^M(0)) \wedge \operatorname{Ch}(F^{\mathcal{E}}(0))]^{(n)} + \mathrm{O}(t) \quad \text{as } t \to 0^+. \tag{2.30}$$

This completes the proof of Theorem 2.2 and of the Atiyah-Singer index formula. □

The main new feature in the previous proof is the use of Lemma 2.7 which, by very elementary considerations on Getzler orders, shows that the convergence of the supertrace of the heat kernel is a consequence of a general fact about Volterra ΨDO's. It also gives a differentiable version of Theorem 2.2 as follows.

In the sequel we abbreviate by *synchronous normal coordinates centered at* $x_0 \in M$ the data of normal coordinates centered at x_0 and of a trivialization of the tangent bundle TM by means of a synchronous frame as in the proof of Theorem 2.2.

Definition 2.11. *We say that $Q \in \Psi^*_{\mathrm{v}}(M \times \mathbb{R}, \not{S} \otimes \mathcal{E})$ has Getzler order m if it has Getzler order m in synchronous normal coordinates centered at any $x_0 \in M$.*

Proposition 2.12. *Let $\not{P}$ be a differential operator on M acting on $\not{S} \otimes \mathcal{E}$ whose Getzler order is equal to m and let $h_t(x,y)$ denote the kernel of $\not{P}e^{-t\not{D}^2_{\mathcal{E}}}$. Then as $t \to 0^+$ we have an asymptotics in $C^\infty(M, |\Lambda|(M))$ of the form:*

- $\operatorname{Str}_x h_t(x,x) = \mathrm{O}(t^{\frac{-m+1}{2}})$ *if m is odd;*

- $\mathrm{Str}_x\, h_t(x,x) = t^{\frac{-m}{2}} B_0(\not{D}^2_{\mathcal{E}}, \not{P})(x) + \mathrm{O}(t^{\frac{-m}{2}+1})$ *if m is even, where in synchronous normal coordinates centered at x_0 and with $\not{P}_{(m)}$ denoting the model operator of $\not{P}$ we have* $B_0(\not{D}^2_{\mathcal{E}}, \not{P})(0) = (-2i\pi)^{\frac{n}{2}}[(\not{P}_{(m)} G_R)(0,1) \wedge \mathrm{Ch}(F^{\mathcal{E}}(0))]^{(n)}$.

Proof. As in the proof of Proposition 1.12 we have $h_t(x,y) = K_{\not{P}(\not{D}^2_{\mathcal{E}}+\partial_t)^{-1}}(x,y,t)$. Notice that by Lemma 2.8 and Lemma 2.9 in synchronous normal coordinates $\not{P}(\not{D}^2_{\mathcal{E}} + \partial_t)^{-1}$ has Getzler order $m-2$ and its model operator is $Q_{(m-2)} = \not{P}_{(m)}(H_R + F^{\mathcal{E}} + \partial_t)^{-1}$. Thus $K_{Q_{(m-2)}}(x,0,t) = \not{P}_{(m)x} K_{(H_R+F^{\mathcal{E}}+\partial_t)^{-1}}(x,0,t) = (\not{P}_{(m)} G_R)(x,t) \wedge e^{-tF^{\mathcal{E}}(0)}$. Then the proposition follows by applying Proposition 1.12 and Lemma 2.7. □

3. The local index formula in noncommutative geometry

In this section we recall the operator theoretic framework for the local index formula ([Co], [CM]; see also [Hi]). This uses two main tools, spectral triples and cyclic cohomology.

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where the involutive unital algebra $\mathcal{A}$ is represented in the (separable) Hilbert space $\mathcal{H}$ and D is an unbounded selfadjoint operator on $\mathcal{H}$ with compact resolvent and which almost commutes with $\mathcal{A}$, i.e. $[D,a]$ is bounded for any element a of $\mathcal{A}$.

In the sequel we assume $\mathcal{A}$ stable by holomorphic calculus, i.e. if $a \in \overline{\mathcal{A}}$ is invertible then $a^{-1} \in \mathcal{A}$; this has the effect that the K-groups of $\mathcal{A}$ and $\bar{\mathcal{A}}$ coincide.

The spectral triple is *even* if $\mathcal{H}$ is endowed with a $\mathbb{Z}_2$-grading $\gamma \in \mathcal{L}(\mathcal{H})$, $\gamma = \gamma^*$, $\gamma^2 = 1$, such that $\gamma D = -D\gamma$ and $\gamma a = a\gamma$ for all $a \in \mathcal{A}$. Otherwise the spectral triple is *odd*.

The datum of D above defines an additive index map $\mathrm{ind}_D : K_* \to \mathbb{Z}$ as follows (see also [Mo, sect. 2]).

In the even case, with respect to the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ given by the $\mathbb{Z}_2$-grading of $\mathcal{H}$ the operator D takes the form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \qquad D_\pm : \mathcal{H}^\mp \to \mathcal{H}^\pm. \tag{3.1}$$

For any selfadjoint idempotent $e \in M_q(\mathcal{A})$ the operator $e(D^+ \otimes 1)e$ from $e(\mathcal{H}^+ \otimes \mathbb{C}^q)$ to $e(\mathcal{H}^- \otimes \mathbb{C}^q)$ is Fredholm and its index only depends on the homotopy class of e. We then define

$$\mathrm{ind}_D[e] = \mathrm{ind}\, eD^+e. \tag{3.2}$$

In the odd case, given an unitary $U \in GL_q(\mathcal{A})$ the operator $[D,U]$ is bounded and so the compression PUP, where $P = \frac{1+F}{2}$ and $F = \mathrm{sign}\, D$, is Fredholm. The index of PUP then depends only on the homotopy class of U and we let

$$\mathrm{ind}_D[U] = \mathrm{ind}\, PUP. \tag{3.3}$$

The index map (3.3) can also be interpreted in terms of spectral flows as follows. Recall that given a family $(D_t)_{0\leq t\leq 1}$ of (unbounded) selfadjoint operators with discrete spectrum such that $D_0 - D_t$ is a C^1-family of bounded operators, the spectral flow $\mathrm{Sf}(D_t)_{0\leq t\leq 1}$ counts the net number of eigenvalues of D_t crossing the origin as t ranges over $[0,1]$ (see [APS]). The spectral flow depends only on the endpoints D_0 and D_1 and we define

$$\mathrm{Sf}(D_0, D_1) = \mathrm{Sf}(D_t)_{0\leq t\leq 1}. \tag{3.4}$$

Here $D - U^*DU = U^*[D,U]$ is bounded and one can prove that

$$\mathrm{Sf}(D, U^*DU) = \mathrm{ind}\, PUP. \tag{3.5}$$

The *cyclic cohomology groups* $HC^*(\mathcal{A})$ of the algebra $\mathcal{A}$ are obtained from the spaces $C^k(\mathcal{A}) = \{(k+1)\text{-linear forms on } \mathcal{A}\}$, $k \in \mathbb{N}$, by restricting the Hochschild coboundary,

$$\begin{aligned} b\psi(a^0, \cdots, a^{k+1}) &= \sum (-1)^j \psi(a^0, \cdots, a^j a^{j+1}, \cdots, a^{k+1}) \\ &\quad + (-1)^{k+1}\psi(a^{k+1}a^0, \cdots, a^k), \qquad a^j \in \mathcal{A}, \end{aligned} \tag{3.6}$$

to cyclic cochains, i.e. those satisfying

$$\psi(a^1, \cdots, a^k, a^0) = (-1)^k \psi(a^0, a^1, \cdots, a^k) \qquad a^j \in \mathcal{A}. \tag{3.7}$$

It can equivalently be described as the second filtration of the (b, B)-bicomplex of (arbitrary) cochains, where $B : C^m(\mathcal{A}) \to C^{m-1}(\mathcal{A})$ is given by

$$B = AB_0, \qquad (A\phi)(a^0, \cdots, a^{m-1}) = \sum (-1)^{(m-1)j} \psi(a^j, \cdots, a^{j-1}), \tag{3.8}$$

$$B_0\psi(a^0, \cdots, a^{m-1}) = \psi(1, a^0, \cdots, a^{m-1}), \qquad a^j \in \mathcal{A}. \tag{3.9}$$

The *periodic cyclic cohomology* is obtained by taking the inductive limit of the groups $HC^k(\mathcal{A})$, $k \geq 0$, with respect to the periodicity operator given by the cup product with the generator of $HC^2(\mathbb{C})$. In terms of the (b, B)-bicomplex this is the cohomology of the short complex

$$C^{\mathrm{ev}}(\mathcal{A}) \overset{b+B}{\leftrightarrows} C^{\mathrm{odd}}(\mathcal{A}), \qquad C^{\mathrm{ev/odd}}(\mathcal{A}) = \bigoplus_{k \text{ even/odd}} C^k(\mathcal{A}), \tag{3.10}$$

whose cohomology groups are denoted $HC^{\mathrm{ev}}(\mathcal{A})$ and $HC^{\mathrm{odd}}(\mathcal{A})$.

There is a pairing between $HC^{\mathrm{ev}}(\mathcal{A})$ and $K_0(\mathcal{A})$ such that for any cocycle $\varphi = (\varphi_{2k})$ in $C^{\mathrm{ev}}(\mathcal{A})$ and for any selfadjoint idempotent e in $M_q(\mathcal{A})$ we have

$$\langle [\varphi], [e] \rangle = \sum_{k \geq 0} (-1)^k \frac{(2k)!}{k!} \varphi_{2k} \# \operatorname{Tr}(e, \cdots, e), \tag{3.11}$$

where $\varphi_{2k} \# \operatorname{Tr}$ is the$(2k+1)$-linear map on $M_q(\mathcal{A}) = M_q(\mathbb{C}) \otimes \mathcal{A}$ given by

$$\varphi_{2k} \# \operatorname{Tr}(\mu^0 \otimes a^0, \cdots, \mu^{2k} \otimes a^{2k}) = \operatorname{Tr}(\mu^0 \dots \mu^{2k}) \varphi_{2k}(a^0, \cdots, a^{2k}), \tag{3.12}$$

for $\mu^j \in M_q(\mathbb{C})$ and $a^j \in \mathcal{A}$.

The pairing between $HC^{\mathrm{odd}}(\mathcal{A})$ and $K_1(\mathcal{A})$ is such that

$$\langle [\varphi], [U] \rangle = \frac{1}{\sqrt{2i\pi}} \sum_{k \geq 0} (-1)^k k! \varphi_{2k+1} \# \operatorname{Tr}(U^{-1}, U, \cdots, U^{-1}, U), \tag{3.13}$$

for any $\varphi = (\varphi_{2k+1})$ in $C^{\mathrm{odd}}(\mathcal{A})$ and any U in $U_q(\mathcal{A})$.

Example 3.1. Let $\mathcal{A}$ be the algebra $C^\infty(M)$ of smooth functions on a compact manifold of dimension n and let $\mathcal{D}_k(M)$ denote the space of k-dimensional de Rham current on M. Any $C \in \mathcal{D}_k(M)$ define a Hochschild cochain on $C^\infty(M)$ by letting

$$\psi_C(f^0, f^1, \dots, f^n) = \langle C, f^0 df^1 \wedge \dots \wedge df^k \rangle \qquad f^j \in C^\infty(M). \tag{3.14}$$

This cochain satisfies $B\psi_C = k\psi_{d^tC}$, where d^t is the de Rham boundary for currents. Thus the map

$$\mathcal{D}_{\mathrm{ev/odd}}(M) \ni C = (C_k) \longrightarrow \varphi_C = (\frac{1}{k!}\psi_{C_k}) \in C^{\mathrm{ev/odd}}(C^\infty(M)) \tag{3.15}$$

induces a morphism from the de Rham's homology group $H^{\mathrm{ev/odd}}(M)$ to the cyclic cohomology group $HC^{\mathrm{ev/odd}}(C^\infty(M))$. This is actually an isomorphism if we restrict ourselves to the cohomology of continuous cyclic cochains [Co].

Moreover, under the Serre-Swan isomorphism $K_*(C^\infty(M)) \simeq K^{-*}(M)$ we have, in the even case,

$$\langle [\varphi_C], \mathcal{E}\rangle = \langle C, \mathrm{Ch}^*_{\mathrm{ev}}\, \mathcal{E}\rangle \qquad \forall \mathcal{E} \in K^0(M), \tag{3.16}$$

where $\mathrm{Ch}^*_{\mathrm{ev}}$ is the even Chern character in cohomology (*cf.* Theorem 2.1), while in the odd case we have

$$\langle [\varphi_C], [U]\rangle = \frac{1}{\sqrt{2i\pi}} \langle C, \mathrm{Ch}^*_{\mathrm{odd}}[U]\rangle \qquad \forall U \in C^\infty(M, U_N(\mathbb{C})), \tag{3.17}$$

where $\mathrm{Ch}^*_{\mathrm{odd}}[U]$ is the Chern character of $[U] \in K^{-1}(M)$, i.e. the cohomology class of the odd form $\mathrm{Ch}\, U = \sum (-1)^k \frac{k!}{(2k+1)!} \mathrm{Tr}(U^{-1}dU)^{2k+1}$.

The index maps (3.2) and (3.3) can be computed by pairing $K^*(\mathcal{A})$ with a cyclic cohomology class as follows. Suppose first that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is *p-summable*, i.e.

$$\mu_k(D^{-1}) = \mathrm{O}(k^{-1/p}) \qquad \text{as } k \to +\infty, \tag{3.18}$$

where $\mu_k(D^{-1})$ is the $(k+1)$'th characteristic value of the compact operator D^{-1}. Then let $\Psi^0_D(\mathcal{A})$ denote the algebra generated by the $\delta^k(a)$'s, $a \in \mathcal{A}$, where δ is the derivation $\delta(T) = [|D|, T]$ (assuming $\mathcal{A}$ is contained in $\cap_{k\geq 0} \operatorname{dom} \delta^k$).

Definition 3.2. *The dimension spectrum of $(\mathcal{A}, \mathcal{H}, D)$ is the union set of the singularities of all the zeta functions $\zeta_b(z) = \mathrm{Tr}\, b|D|^{-z}$, $b \in \Psi^0_D(\mathcal{A})$.*

Assuming simple and discrete dimension spectrum we define an analogue of the Wodzicki-Guillemin residue ([Wo], [Gu]) on $\Psi^0_D(\mathcal{A})$ by letting

$$⨍ b = \mathrm{Res}_{z=0} \mathrm{Tr}\, b|D|^{-z} \qquad \text{for } b \in \Psi^0_D(\mathcal{A}). \tag{3.19}$$

This functional is a trace on the algebra $\Psi^0_D(\mathcal{A})$ and is local in the sense of noncommutative geometry since it vanishes on any element of $\Psi^0_D(\mathcal{A})$ which is traceable.

Theorem 3.3 ([CM, Thm. II.3])**.** *Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is even, p-summable and has a discrete and simple dimension spectrum. Then:*

1) The following formulas define an even cocycle $\varphi^{ev}_{\mathrm{CM}} = (\varphi_{2k})$ in the (b, B)-complex of the algebra $\mathcal{A}$. For $k = 0$,

$$\varphi_0(a^0) = \textit{finite part of } \mathrm{Tr}\, \gamma a^0 e^{-tD^2} \textit{ as } t \to 0^+, \tag{3.20}$$

while for $k \neq 0$,

$$\varphi_{2k}(a^0, \ldots, a^{2k}) = \sum_\alpha c_{k,\alpha} ⨍ \gamma a^0 [D, a^1]^{[\alpha_1]} \ldots [D, a^{2k}]^{[\alpha_{2k}]} |D|^{-2(|\alpha|+k)}, \tag{3.21}$$

where $\Gamma(|\alpha| + k) c^{-1}_{k,\alpha} = 2(-1)^{|\alpha|} \alpha!(\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2k} + 2k)$ and the symbol $T^{[j]}$ denotes the j'th iterated commutator with D^2.

2) We have $\mathrm{ind}_D(\mathcal{E}) = \langle [\varphi_{\mathrm{CM}}], \mathcal{E}\rangle$ for any $\mathcal{E} \in K_0(\mathcal{A})$.

Theorem 3.4 ([CM, Thm. II.2])**.** *Assume $(\mathcal{A}, \mathcal{H}, D)$ is p-summable and has a discrete and simple dimension spectrum. Then:*

1) We define an odd cocycle $\varphi^{odd}_{\mathrm{CM}} = (\varphi_{2k+1})$ in the (b, B)-complex of the algebra $\mathcal{A}$ by letting

$$\begin{aligned} &\varphi_{2k+1}(a^0, \ldots, a^{2k+1}) = \\ &\sqrt{2i\pi} \sum_\alpha c_{k,\alpha} ⨍ a^0 [D, a^1]^{[\alpha_1]} \ldots [D, a^{2k+1}]^{[\alpha_{2k+1}]} |D|^{-2(|\alpha|+k)-1)}, \end{aligned} \tag{3.22}$$

where $\Gamma(|\alpha| + k + \frac{1}{2}) c^{-1}_{k,\alpha} = (-1)^{|\alpha|} \alpha!(\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2k} + 2k + 1)$.

2) We have $\mathrm{ind}_D(U) = \langle [\varphi_{\mathrm{CM}}], U\rangle$ for any $U \in K_1(\mathcal{A})$.

Example 3.5. Let M be a compact manifold of dimension n and let D be a pseudodifferential operator of order 1 on M acting on the sections of a vector bundle $\mathcal{E}$ over M such that D is elliptic and selfadjoint. Then the triple

$$(C^\infty(M), L^2(M,\mathcal{E}), D) \tag{3.23}$$

is an n-summable spectral triple, which is even when $\mathcal{E}$ is equipped with a $\mathbb{Z}_2$-grading anticommuting with D. In any case the algebra $\Psi_D^0(C^\infty(M))$ is contained in the algebra of ΨDO's with order ≤ 0. So by the very construction of the Wodzicki-Guillemin residue ([Wo], [Gu]) this spectral triple has a simple and discrete dimension spectrum contained in $\{k \in \mathbb{Z};\ k \leq n\}$.

In fact, given $P \in \Psi^m(M,\mathcal{E})$, $m \in \mathbb{Z}$, the function $z \to \operatorname{Trace} P|D|^{-z}$ has a meromorphic continuation on $\mathbb{C}$ with at worst simple poles at integers k, $k \leq m+n$. At $z=0$ the residue coincides with the Wodzicki-Guillemin residue $⨍P$ of P, i.e.

$$⨍ P = \operatorname{res}_{z=0} \operatorname{Trace} P|D|^{-z} = \int_M \operatorname{tr}_{\mathcal{E}} c_P(x), \tag{3.24}$$

where $c_P(x)$ is an $\operatorname{END}\mathcal{E}$-valued density on M. Hence the formulas for the CM-cocycle φ_{CM} hold using the Wodzicki-Guillemin residue as residual trace.

4. The CM cocycle of a Dirac spectral triple (even case)

Let (M^n,g) be a compact Riemannian spin manifold of even dimension and let $\not{D}_M$ denote the Dirac operator acting its spin bundle $\not{S}$. Then the spectral triple $(C^\infty(M), L^2(M,\not{S}), \not{D}_M)$ is even and has a discrete and simple dimension spectrum. In this section we shall compute the associated even CM cocycle and explain how this allows us to recover the index formula of Atiyah and Singer.

Theorem 4.1. *The components of the even CM cyclic cocycle $\varphi_{\mathrm{CM}}^{ev} = (\varphi_{2k})$ associated to the spectral triple $(C^\infty(M), L^2(M,\not{S}), \not{D}_M)$ are given by*

$$\varphi_{2k}(f^0,\ldots,f^{2k}) = \frac{1}{(2k)!}\int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R_M)^{(n-2k)}, \tag{4.1}$$

for $f^0, f^1, \ldots, f^n$ in $C^\infty(M)$.

Proof. First, it follows from Theorem 2.2 that

$$\varphi_0(f^0) = \lim_{t\to 0^+} \operatorname{Str} f^0 e^{-t\not{D}_M^2} = \int_M f^0 \hat{A}(R_M)^{(n)}. \tag{4.2}$$

Second, let α be a $2k$-fold index, $k \geq 1$, and define

$$\not{P}_\alpha = f^0[\not{D}_M, f^1]^{[\alpha_1]} \cdots [\not{D}_M, f^{2k}]^{[\alpha_{2k}]} = f^0 c(df^1)^{[\alpha_1]} \cdots c(df^{2k})^{[\alpha_{2k}]}. \tag{4.3}$$

Then in order to use Formula (3.21) for $\varphi_{2k}(f^0,\ldots,f^{2k})$ we need to compute

$$⨍ \gamma \not{P}_\alpha |\not{D}_M|^{-2(k+|\alpha|)} = \operatorname{res}_{z=0} \operatorname{Str} \not{P}_\alpha |\not{D}_M|^{-2(k+|\alpha|)-z}. \tag{4.4}$$

The main step is to prove the lemma below.

Lemma 4.2. *For $t>0$ let $k_{\alpha,t}(x,y)$ be the kernel of $\not{P}_\alpha e^{-t\not{D}_M^2}$. Then as $t \to 0^+$ we have the following asymptotics in $C^\infty(M, |\Lambda|(M))$:*

- $\operatorname{Str}_x k_{\alpha,t}(x,x) = \mathrm{O}(t^{-(k+|\alpha|)+1})$ *if $\alpha \neq 0$;*
- $\operatorname{Str}_x k_{0,t}(x,x) = \frac{t^{-k}}{(2i\pi)^{\frac{n}{2}}} f^0 df^1 \wedge \ldots \wedge df^{2k} \wedge \hat{A}(R_M)^{(n-2k)} + \mathrm{O}(t^{-k+1})$.

Proof. In synchronous normal coordinates $c(df^j)$ and $\not{D}^2$ have respectively Getzler orders 1 and 2 and model operators $df^j(0)$ and $H_R = -\sum(\partial_i - R^M_{ij}(0)x^j)^2$. Therefore, by Lemma 2.8 the operator $\not{P}_\alpha$ has Getzler order $\leq 2(k+|\alpha|)$ and we have

$$\not{P}_\alpha = c[f^0(0)df^1(0)^{[\alpha_1]} \wedge \cdots \wedge df^{2k}(0)^{[\alpha_{2k}]}] + \mathrm{O}_G(2(k+|\alpha|)-1), \tag{4.5}$$

where $T^{[j]}$ denotes the j'th iterated commutator of T with H_R. Remark that $[H_R, df^j(0)] = 0$, so if $\alpha \neq 0$ then $\not{P}_\alpha Q$ has Getzler order $\leq 2(k+|\alpha|)-1$. Moreover as the model operator of P_0 is $\not{P}_{0(2k)} = f^0(0)df^1(0)\wedge\ldots\wedge df^{2k}(0)$ we get $(\not{P}_{0(2k)}G_R)(0,1) = f^0(0)df^1(0)\wedge\ldots\wedge df^{2k}(0)\wedge\hat{A}(R^M(0))$. The result then follows by applying Proposition 2.12. $\square$

Now, by the Mellin formula we have $\not{D}_M^{-2s} = \Gamma(s)^{-1}\int_0^\infty t^{s-1}e^{-t\not{D}_M^2}dt$ for $\Re s > 1$, so the function $\operatorname{Str}\not{P}_\alpha|\not{D}_M|^{-(2+|\alpha|)-2z}$ coincides with

$$\Gamma(k+|\alpha|+z)^{-1}\int_0^1 t^{k+|\alpha|+z}\operatorname{Str}(\not{P}_\alpha e^{-t\not{D}_M^2})\frac{dt}{t}, \tag{4.6}$$

up to a holomorphic function on the halfplane $\Re z > -1$. Therefore, it follows from Lemma 4.2 that if $\alpha \neq 0$ then $\operatorname{Str}\not{P}_\alpha|\not{D}_M|^{-(2+|\alpha|)-2z}$ has an analytic continuation on the halfplane $\Re z > -1$, while $\operatorname{Str}\not{P}_0|\not{D}_M|^{-2-2z}$ is equal to

$$\frac{(2i\pi)^{-\frac{n}{2}}}{z\Gamma(z+k)}\int_M f^0df^1\wedge\ldots\wedge df^{2k}\wedge\hat{A}(R_M)^{(n-2k)}, \tag{4.7}$$

modulo a holomorphic function on the halfplane $\Re z > -1$. Thus in the formula (3.21) for $\varphi_{2k}(f^0,\ldots,f^{2k})$ all the residues corresponding to $\alpha \neq 0$ are zero, while for $\alpha = 0$ we get

$$-\!\!\!\!\!\!\int \gamma\not{P}_0|\not{D}_M|^{-2k} = \frac{2(2i\pi)^{-\frac{n}{2}}}{(k-1)!}\int_M f^0df^1\wedge\ldots\wedge df^{2k}\wedge\hat{A}(R_M)^{(n-2k)}. \tag{4.8}$$

This gives $\varphi_{2k}(f^0,\ldots,f^{2k}) = \frac{(2i\pi)^{-\frac{n}{2}}}{(2k)!}\int_M f^0df^1\wedge\ldots\wedge df^{2k}\wedge\hat{A}(R_M)^{(n-2k)}$ since $c_{k,0} = \frac{1}{2}\frac{\Gamma(k)}{(2k)!}$. $\square$

We can now recover the local index formula of Atiyah and Singer. Let $\mathcal{E}$ be a Hermitian vector bundle over M together with a unitary connection with curvature $F^{\mathcal{E}}$ and let $\not{D}_{\mathcal{E}}$ denote the associated twisted Dirac operator. The starting point is that under the Serre-Swan isomorphism $K^0(M) \simeq K_0(C^\infty(M))$ we have $\operatorname{ind}\not{D}_{\mathcal{E}}^+ = \operatorname{ind}_{\not{D}_M}[\mathcal{E}]$ (e.g. [Mo, Sect. 2]). Therefore, from Theorem 3.3 we obtain

$$\operatorname{ind}\not{D}_{\mathcal{E}}^+ = \langle[\varphi_{CM}],[\mathcal{E}]\rangle. \tag{4.9}$$

On the other hand, Formula (4.1) shows that φ_{CM} is the image under the map (3.15) of the even de Rham current that is the Poincaré dual of $\hat{A}(R_M)$. Thus using (3.16) we get

$$\operatorname{ind}\not{D}_{\mathcal{E}}^+ = (2i\pi)^{-\frac{n}{2}}\int_M[\hat{A}(R^M)\wedge\operatorname{Ch}F^{\mathcal{E}}]^{(n)}, \tag{4.10}$$

which is precisely the index formula of Atiyah and Singer.

5. The CM cocycle of a Dirac spectral triple (odd case)

In this section we compute the CM cocycle corresponding to a Dirac operator on an odd dimensional spin manifold. As consequence we can recapture the spectral flow formula of Atiyah-Patodi-Singer [APS].

Let (M^n, g) be a compact Riemannian spin manifold of odd dimension and let $\not{S}$ be a spin bundle for M, so that each fiber $\not{S}_x$ is an irreducible representation space for $\mathrm{Cl}_x(M)$. The Dirac operator $\not{D}_M$ acting on the sections of $\not{S}$ is given by the composition,

$$C^\infty(M,\not{S}) \xrightarrow{\nabla^{\not{S}}} C^\infty(M, T^*M\otimes\not{S}) \xrightarrow{c\otimes 1} C^\infty(M,\not{S}), \tag{5.1}$$

where c denotes the action of ΛT^*M on $\not{S}$ by Clifford representation. This gives rise to an odd spectral triple $(C^\infty(M), L^2(M,\not{S}), \not{D}_M)$ with simple and discrete dimension spectrum.

Theorem 5.1. *The components of the odd CM cocycle* $\varphi_{\text{CM}}^{odd} = (\varphi_{2k+1})$ *associated to the spectral triple* $(C^\infty(M), L^2(M,\not{S}), \not{D}_M)$ *are given by*

$$\varphi_{2k+1}(f^0,\ldots,f^{2k+1}) = \sqrt{2i\pi}\frac{(2i\pi)^{-[\frac{n}{2}]+1}}{(2k+1)!}\int_M f^0 df^1\wedge\ldots\wedge df^{2k+1}\wedge\hat{A}(R_M)^{(n-2k-1)}. \tag{5.2}$$

for $f^0,\ldots,f^{(n)}$ *in* $C^\infty(M)$.

Before tackling the proof of Theorem 5.1 let us explain the similarities with the even case. Since the dimension of M is odd there is not anymore an isomorphism between $\mathrm{Cl}(M)$ and $\mathrm{End}\,\not{S}$ and so we need to distinguish between them. In fact if $e_1,\ldots,e_n$ is an orthonormal frame for T_xM then $c(e^1)\cdots c(e^n)$ acts like $(-i)^{[\frac{n}{2}]+1}$ on $\not{S}_x$ (*cf.* [Ge1], [BF]). Therefore, if we look at $c(e^{i_1})\cdots c(e^{i_k})$, $i_1<\ldots<i_k$, as an endomorphism of $\not{S}_x$ then we have

$$\operatorname{tr}_{\not{S}_x} c(e^{i_1})\cdots c(e^{i_k}) = \begin{cases} 0 & \text{if } 0<k<n,\\ (-i)^{[\frac{n}{2}]+1}2^{[\frac{n}{2}]} & \text{if } k=n.\end{cases} \tag{5.3}$$

Therefore, provided only an odd number of Clifford variables are involved the trace behaves as the supertrace in even dimension.

Bearing this in mind let $Q\in\Psi_{\mathrm{v}}^*(M\times\mathbb{R},\not{S})$. In synchronous normal coordinates Q is given near the origin by a Volterra ΨDO operator $\tilde{Q}$ which acts on the trivial bundle with fiber $\not{S}_n$, i.e. is with coefficients in $\mathrm{End}\,\not{S}_n$. Thus, via the Clifford representation $\tilde{Q}$ comes from a Volterra ΨDO ${}^{\mathrm{Cl}}Q$ on $\mathbb{R}^n\times\mathbb{R}$ with coefficients in $\mathrm{Cl}^{\mathrm{odd}}(\mathbb{R}^n)$. Then using (5.3) we get

$$\operatorname{Tr} K_Q(0,0,t) = -i(-2i)^{[\frac{n}{2}]}\sigma[K_{{}^{\mathrm{Cl}}\not{P}{}^{\mathrm{Cl}}Q_1}(0,0,t)]^{(n)} + \mathrm{O}(t^\infty). \tag{5.4}$$

On the other hand, in the the proof of Theorem 2.2 we identified $\mathrm{End}\,\not{S}_n$ and $\mathrm{Cl}(\mathbb{R}^n)$. Thus definition 2.3 and all the lemmas 2.7-2.9 hold *verbatim* for Volterra ΨDO's with coefficients in $\mathrm{Cl}(\mathbb{R}^n)$, and this independently of the parity of n. For instance, if we let m be the Getzler order of ${}^{\mathrm{Cl}}Q$ then from Lemma 2.7 we get:

- $\sigma[K_{{}^{\mathrm{Cl}}Q}(0,0,t)]^{(j)} = \mathrm{O}(t^{\frac{j-m-n-1}{2}})$ if $m-j$ is odd,
- $\sigma[K_{{}^{\mathrm{Cl}}Q}(0,0,t)]^{(j)} = t^{\frac{j-m-n}{2}-1}K_{{}^{\mathrm{Cl}}Q_{(m)}}(0,0,1)^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}})$ otherwise.

Note this is consistent with (2.17) because as n is odd by the proof of Lemma 2.7 whenever m is even we have

$$K_{{}^{\mathrm{Cl}}Q_{(m)}}(0,0,1) = \sigma[\check{q}_{-2-n}(0,0,1)]^{(n)} = 0. \tag{5.5}$$

Definition 5.2. *We say that* $Q\in\Psi_{\mathrm{v}}^*(M\times\mathbb{R},\not{S}\otimes\mathcal{E})$ *has Getzler order* m *if in synchronous normal coordinates centered at any* $x_0\in M$ *the operator* ${}^{\mathrm{Cl}}Q$ *defined as above has Getzler order* m. *Moreover we let* $Q_{(m)} = {}^{\mathrm{Cl}}Q_{(m)}$ *be the model operator of* Q.

Along similar lines as that of the proof of Proposition 2.12 we obtain:

Proposition 5.3. *Let* $\not{P}$ *be a differential operator on* M *acting on* $\not{S}$ *with Getzler order* m *and let* $h_t(x,y)$ *denote the kernel of* $\not{P}e^{-t\not{D}_{\mathcal{E}}^2}$. *Then as* $t\to 0^+$ *we have an asymptotics in* $C^\infty(M,|\Lambda|(M))$ *of the form:*

- $\operatorname{Tr}_x h_t(x,x) = \mathrm{O}(t^{\frac{-m+1}{2}})$ *if* m *is even;*
- $\operatorname{Tr}_x h_t(x,x) = t^{\frac{-m}{2}}B_0(\not{D}_{\mathcal{E}}^2,\not{P})(x) + \mathrm{O}(t^{\frac{-m}{2}+1})$ *if* m *is odd, where in synchronous normal coordinates centered at* x_0 *and with* $\not{P}_{(m)}$ *denoting the model operator of* $\not{P}$ *we have* $B_0(\not{D}_{\mathcal{E}}^2,\not{P})(0) = (-i)^{[\frac{n}{2}]+1}2^{[\frac{n}{2}]}[(\not{P}_{(m)}G_R)(0,1)]^{(n)}$.

Proof of Theorem 5.1. Let $\not{P}_\alpha = f^0[\not{D}_M, f^1]^{[\alpha_1]} \cdots [\not{D}_M, f^{2k+1}]^{[\alpha_{2k+1}]}$ where α is a $(2k+1)$-fold index. Then applying Proposition 5.3 and arguing as in the proof of Lemma 4.2 shows that as $t \to 0^+$ we have:

- $\operatorname{Tr} \not{P}_\alpha e^{-t\not{D}_M^2} = \mathrm{O}(t^{-(k+|\alpha|)+\frac{1}{2}})$ if $\alpha \neq 0$,
- $\operatorname{Tr} \not{P}_0 e^{-t\not{D}_M^2} = \frac{t^{-k-\frac{1}{2}}}{(2i\pi)^{[\frac{n}{2}]}} \int_M f^0 df^1 \wedge \ldots \wedge df^{2k+1} \wedge \hat{A}(R_M)^{(n-2k-1)} + \mathrm{O}(t^{-k+\frac{1}{2}})$.

Then as in the proof of Theorem 4.1 we deduce that in the formula (3.22) for $\varphi_{2k+1}(f^0, \ldots, f^{2k+1})$ only $⨍ \not{P}_0 |\not{D}_M|^{-(2k+1)}$ is nonzero and equal to

$$\frac{2}{\Gamma(k+\frac{1}{2})} \frac{(2i\pi)^{-[\frac{n}{2}]}}{2i\sqrt{\pi}} \int_M f^0 df^1 \wedge \ldots \wedge df^{2k+1} \wedge \hat{A}(R_M)^{(n-2k-1)}. \tag{5.6}$$

Hence $\varphi_{2k+1}(f^0, \ldots, f^{2k+1}) = \sqrt{2i\pi} \frac{(2i\pi)^{-[\frac{n}{2}]+1}}{(2k+1)!} \int_M f^0 df^1 \wedge \ldots \wedge df^{2k+1} \wedge \hat{A}(R_M)^{(n-2k-1)}$. ☐

As a consequence of Theorem 5.1 we can recover the spectral flow formula of Atiyah-Patodi-Singer [APS] in the case of a Dirac operator (see also [Ge3]).

Theorem 5.4 ([APS, p. 95]). *For any $U \in C^\infty(M, U(N))$ we have*

$$\operatorname{Sf}(\not{D}_M, U^*\not{D}_M U) = (2i\pi)^{-[\frac{n}{2}]-1} \int_M [\hat{A}(R^M) \wedge \operatorname{Ch}(U)]^{(n)}. \tag{5.7}$$

Proof. Thanks to (3.5) and Theorem 3.4 we have

$$\operatorname{Sf}(\not{D}_M, U^*\not{D}_M U) = \operatorname{ind}_{\not{D}_M}[U] = \langle [\varphi_{CM}], [U] \rangle. \tag{5.8}$$

Moreover, Formula (5.2) shows that φ_{CM} is the image under the map (3.15) of the odd de Rham current that is the Poincaré dual of $\hat{A}(R_M)$. Formula (5.7) then follows by using (3.17). ☐

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Department of Mathematics, Ohio State University, Columbus, USA.
E-mail address: `ponge@math.ohio-state.edu`