

# SPECTRAL ASYMMETRY, ZETA FUNCTIONS, AND THE NONCOMMUTATIVE RESIDUE

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ABSTRACT. In this paper we study the spectral asymmetry of (possibly non-selfadjoint) elliptic  $\Psi$ DO's in terms of the difference of zeta functions coming from different cuttings. Refining previous formulas of Wodzicki in the case of odd class elliptic  $\Psi$ DO's, our main results have several consequence concerning the local independence with respect to the cutting, the regularity at integer points of eta functions and a geometric expression for the spectral asymmetry of Dirac operators which, in particular, yields a new spectral interpretation of the Einstein-Hilbert action in gravity.

## 1. INTRODUCTION

This paper focuses on the spectral asymmetry of elliptic  $\Psi$ DO's. Given a compact Riemannian manifold  $M^n$  and a Hermitian bundle  $\mathcal{E}$  over  $M$ , the spectral asymmetry was first studied by Atiyah-Patodi-Singer [APS1] in the case of a self-adjoint elliptic  $\Psi$ DO  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  in terms of the eta function,

$$(1.1) \quad \eta(P; s) = \text{Trace } P|P|^{-(s+1)}, \quad s \in \mathbb{C}.$$

This function is meromorphic with at worst simple pole singularities and an important result, due to Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]), is its regularity at  $s = 0$ , so that the eta invariant  $\eta(P) := \eta(P, 0)$  is always well defined.

The residues of the eta function at other integer points are also interesting, e.g., they enter in the index formula of Brüning-Seeley [BS] for first order elliptic operators on a manifold with cone-like singularities

In [Wo1]–[Wo4] Wodzicki took a different point of view. Motivated by an observation of Shubin, he looked at the spectral asymmetry of a (possibly nonselfadjoint) elliptic  $\Psi$ DO  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  of order  $m > 0$  in terms of the difference,

$$(1.2) \quad \zeta_\theta(P; s) - \zeta_{\theta'}(P; s) = \text{Trace } P_\theta^{-s} - \text{Trace } P_{\theta'}^{-s}, \quad s \in \mathbb{C},$$

of zeta functions coming from different spectral cuttings  $L_\theta = \{\arg \lambda = \theta\}$  and  $L_{\theta'} = \{\arg \lambda = \theta'\}$  with  $0 \leq \theta < \theta' < 2\pi$ . In particular, he showed that the spectral asymmetry of  $P$  was encoded by the sectorial projection earlier introduced by Burak [Bu2] and given by

$$(1.3) \quad \Pi_{\theta, \theta'}(P) = \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda^{-1} P(P - \lambda)^{-1} d\lambda,$$

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where  $\Gamma_{\theta, \theta'}$  is a contour separating the part of the spectrum of  $P$  contained in the open sector  $\theta < \arg \lambda < \theta'$  from the rest of the spectrum. More precisely, Wodzicki proved the equality of meromorphic functions,

$$(1.4) \quad \zeta_{\theta}(P; s) - \zeta_{\theta'}(P; s) = (1 - e^{-2i\pi s}) \text{Trace } \Pi_{\theta, \theta'}(P) P_{\theta}^{-s}, \quad s \in \mathbb{C}.$$

In particular, at every integer  $k \in \mathbb{Z}$  the function  $\zeta_{\theta}(P; s) - \zeta_{\theta'}(P; s)$  is regular and there we have

$$(1.5) \quad \text{ord} P \cdot \lim_{s \rightarrow k} (\zeta_{\theta}(P; s) - \zeta_{\theta'}(P; s)) = 2i\pi \text{Res } \Pi_{\theta, \theta'}(P) P^{-k},$$

where  $\text{Res}$  denotes the noncommutative residue of Wodzicki ([Wo2], [Wo5]) and Guillemin [Gu1].

Furthermore, Wodzicki proved in [Wo2] that the regular value  $\zeta(P; 0)$  is independent of the choice of the cutting  $L_{\theta}$  and that the noncommutative residue of a  $\Psi$ DO projection is always zero, which generalize the vanishing of the residue at the origin of the eta function a selfadjoint elliptic  $\Psi$ DO.

In this paper, partly motivated by a recent upsurge of interest in the spectral asymmetry of non-selfadjoint elliptic  $\Psi$ DO's ([BK], [Sc]), we prove various results related to the spectral asymmetry of odd class elliptic  $\Psi$ DO's as a consequence of a refinement of the formulas (1.4)–(1.5) for such operators.

Recall that a  $\Psi$ DO of integer order is said to be odd class when the homogeneous components of its symbol are homogeneous with respect to the dilation by  $-1$ . In particular, the odd class  $\Psi$ DO's form an algebra containing all the differential operators and the parametrices of odd class elliptic  $\Psi$ DO's.

Let  $P : C^{\infty}(M, \mathcal{E}) \rightarrow C^{\infty}(M, \mathcal{E})$  be an odd class  $\Psi$ DO of integer order  $m \geq 1$  and let  $L_{\theta} = \{\arg \lambda = \theta\}$  and  $L_{\theta'} = \{\arg \lambda = \theta'\}$  be spectral cuttings for  $P$  and its principal symbol with  $0 \leq \theta < \theta' < 2\pi$ . Our main results are:

(i) If  $\dim M$  is odd and  $\text{ord} P$  is even then  $\zeta_{\theta}(P; s)$  is regular at every integer point and its value there is independent of the spectral cut  $L_{\theta}$  (Theorem 5.1).

(ii) If  $\dim M$  is even,  $\text{ord} P$  is odd and the principal symbol of  $P$  has all its eigenvalue in the open cone  $\{\theta < \arg \lambda < \theta'\} \cup \{\theta + \pi < \arg \lambda < \theta' + \pi\}$ , then for any integer  $k \in \mathbb{Z}$  we have

$$(1.6) \quad \text{ord} P \cdot \lim_{s \rightarrow k} (\zeta_{\theta}(P; s) - \zeta_{\theta'}(P; s)) = i\pi \text{Res } P^{-k}.$$

In particular, at every integer at which they are not singular the functions  $\zeta_{\theta}(P; s)$  and  $\zeta_{\theta'}(P; s)$  take on the same regular value (Theorem 5.2).

These results are deduced from a careful analysis of the symbol of the sectorial projection  $\Pi_{\theta, \theta'}(P)$ , so that the proofs are purely local in nature. It thus follows that the theorems ultimately hold at the level of the *local* zeta functions  $\zeta_{\theta}(P; 0)(x)$  and  $\zeta_{\theta'}(P; 0)(x)$ , that is, the densities whose integrals yield the zeta functions  $\zeta_{\theta}(P; s)$  and  $\zeta_{\theta'}(P; s)$ . In particular, we obtain that if  $P$  is an odd class elliptic  $\Psi$ DO satisfying either the assumptions of (i) or that of (ii), then the regular value  $\zeta_{\theta}(P; 0)(x)$  is independent of the choice of the spectral cutting (Theorem 5.4).

In fact, the independence with respect to the spectral cutting of the regular values at  $s = 0$  of the local zeta functions is not true for general  $\Psi$ DO's (see [Wo1, pp. 130-131]). Therefore, it is interesting to see that this nevertheless can happen for a wide class of elliptic  $\Psi$ DO's.

Next, these results have further applications when  $P$  is selfadjoint. In this case we shall use the subscript  $\uparrow$  (resp.  $\downarrow$ ) to refer to a spectral cut in the upper halfplane  $\Im\lambda > 0$  (resp. lower halfplane  $\Im\lambda < 0$ ).

First, while the above results tell us that there are many integer points at which there is no spectral asymmetry, they also allow us to single out some points at which the spectral asymmetry always occurs. For instance, we always have

$$(1.7) \quad \lim_{s \rightarrow n} \frac{1}{i} (\zeta_{\uparrow}(P; s) - \zeta_{\downarrow}(P; s)) > 0,$$

when  $\dim M$  is even and  $P$  is a first order selfadjoint elliptic odd class  $\Psi$ DO (see Proposition 6.1).

Second, as the eta function  $\eta(P; s)$  can be nicely related to  $\zeta_{\uparrow}(P; s) - \zeta_{\downarrow}(P; s)$  (see [Sh, p. 114] and Section 6), we can make use of the previous results to study  $\eta(P; s)$ . It is a well known result of Branson-Gilkey [BG] that in even dimension the eta function of a Dirac operator is an entire function. We generalize this result by proving that if  $\text{ord}P$  and  $\dim M$  have opposite parities then  $\eta(P; s)$  is regular at every integer point, so that when  $P$  has order 1 and  $\dim M$  is even the function  $\eta(P; s)$  is entire (Theorem 6.3).

The latter result has been independently obtained by Grubb [Gr] using a different approach. Furthermore, it allows us to simplify in odd dimension the aforementioned index formula of Brüning-Seeley [BS] for first order elliptic operators on a manifold with cone-like singularities (see Remark 6.5).

Third, for Dirac operators our results enable us to express the spectral asymmetry of these operators in geometric terms. More precisely, assume that  $M$  has even dimension, that  $\mathcal{E}$  is a  $\mathbb{Z}_2$ -graded Clifford module over  $M$  equipped with a unitary connection  $\nabla^{\mathcal{E}}$ , and let  $\mathcal{D}_{\mathcal{E}} : C^{\infty}(M, \mathcal{E}) \rightarrow C^{\infty}(M, \mathcal{E})$  be the associated Dirac operator. Then in Proposition 7.1 we show that:

- At every integer that is not an even integer between 2 and  $n$  the zeta functions  $\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s)$  and  $\zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)$  are non-singular and take on the same regular value;
- For  $k = 2, 4, \dots, n$  we can express  $\lim_{s \rightarrow k} (\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s))$  as the integral of a universal polynomial in complete tensorial contractions of the covariant derivatives of the curvature  $R^M$  of  $M$  and the twisted curvature  $F^{\mathcal{E}/\mathcal{S}}$  of  $\mathcal{E}$ .

As a consequence we get a new spectral interpretation of the Einstein-Hilbert action  $\mathcal{I} = \int_M r_M(x) \sqrt{g(x)} dx$ , which is an important issue in noncommutative geometry and we get points at which the spectral asymmetry occurs independently of the choice of the Clifford data  $(\mathcal{E}, \nabla^{\mathcal{E}})$  (see Proposition 7.2).

The paper is organized as follows. In Section 2 we recall the general background needed in this paper about complex powers of elliptic operators, the noncommutative residue trace of Wodzicki and Guillemin and the zeta and eta functions of elliptic  $\Psi$ DO's. In Section 3 we gather some of the main facts about the sectorial projection of an elliptic  $\Psi$ DO, but we postpone to the Appendix those concerning its spectral interpretation. In Section 4 we give a detailed review of Wodzicki's results on the spectral asymmetry elliptic  $\Psi$ DO's needed in this paper. In Section 5 we refine the latter formulas for odd class elliptic  $\Psi$ DO's and prove our main results. We then specialize these results to the selfadjoint case in Section 6 and to Dirac operators in Section 7.

*Notation.* Throughout all this paper we let  $M$  denote a compact Riemannian manifold of dimension  $n$  and let  $\mathcal{E}$  be a Hermitian vector bundle over  $M$  of rank  $r$ .

## 2. GENERAL BACKGROUND

In this section we recall the main facts about complex powers of elliptic  $\Psi$ DO's, the noncommutative residue trace of Wodzicki and Guillemin and the zeta and eta functions of elliptic  $\Psi$ DO's.

**2.1. Complex powers of elliptic  $\Psi$ DO's.** For  $m \in \mathbb{C}$  we let  $\Psi^m(M, \mathcal{E})$  denote the space of (classical)  $\Psi$ DO's of order  $m$  on  $M$  acting on sections of  $\mathcal{E}$ , i.e., continuous operators  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  such that:

- The distribution kernel of  $P$  is smooth off the diagonal of  $M \times M$ ;
- In any local trivializing chart  $U \subset \mathbb{R}^n$  the operator  $P$  is of the form  $P = p(x, D) + R$ , for some polyhomogeneous symbol  $p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$  of degree  $m$  and some smoothing operator  $R$ , where  $p(x, D)$  denotes the linear operator from  $C_c^\infty(U, \mathbb{C}^r)$  to  $C^\infty(U, \mathbb{C}^r)$  such that

$$(2.1) \quad p(x, D)u(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi \quad \forall u \in C_c^\infty(U, \mathbb{C}^r).$$

Let  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  be an elliptic  $\Psi$ DO of degree  $m > 0$  with principal symbol  $p_m(x, \xi)$  and assume that the ray  $L_\theta = \{\arg \lambda = \theta\}$ ,  $0 \leq \theta < 2\pi$ , is a spectral cutting for  $p_m$ , that is,  $p_m(x, \xi) - \lambda$  is invertible for every  $\lambda \in L_\theta$ . Then there is a conical neighborhood  $\Lambda$  of  $L_\theta$  such that any ray contained in  $\Lambda$  is also a spectral cutting for  $p_m$ . It then follows that  $P$  admits an asymptotic resolvent as a parametrix in a suitable class of  $\Psi$ DO's with parametrized by  $\Lambda$  (see [Se1], [Sh], [GS]). This allows us to show that, for any closed cone  $\Lambda'$  such that  $\overline{\Lambda'} \setminus 0 \subset \Lambda$  and for  $R > 0$  large enough, there exists  $C_{\Lambda'R} > 0$  such that

$$(2.2) \quad \|(P - \lambda)^{-1}\|_{\mathcal{L}(L^2(M, \mathcal{E}))} \leq C_{\Lambda'R} |\lambda|^{-1}, \quad \lambda \in \Lambda', \quad |\lambda| \geq R.$$

Therefore there are infinitely many rays  $L_\theta = \{\arg \lambda = \theta\}$  contained in  $\Lambda$  that are not through an eigenvalue of  $P$  and any such ray is a ray of minimal growth.

On the other hand, (2.2) also implies that the spectrum of  $P$  is not  $\mathbb{C}$ , hence consists of an unbounded set of isolated eigenvalues with finite multiplicities. Thus, we can define the root space and Riesz projection associated to  $\lambda \in \text{Sp } P$  by letting

$$(2.3) \quad E_\lambda(P) = \cup_{j \geq 1} \ker(P - \lambda)^j \quad \text{and} \quad \Pi_\lambda(P) = \frac{-1}{2i\pi} \int_{\Gamma(\lambda)} (P - \mu)^{-1} d\mu,$$

where  $\Gamma(\lambda)$  is a direct-oriented circle about  $\lambda$  with a radius small enough so that apart from  $\lambda$  no other element of  $\text{Sp } P \cup \{0\}$  lies inside  $\Gamma(\lambda)$ .

The family  $\{\Pi_\lambda(P)\}_{\lambda \in \text{Sp } P}$  is a family of disjoint projections, in the sense that we have  $\Pi_\lambda(P)\Pi_\mu(P) = 0$  for  $\lambda \neq \mu$ . Moreover, for every  $\lambda \in \text{Sp } P$  the root space  $E_\lambda(P)$  has finite dimension and  $\Pi_\lambda(P)$  projects onto  $E_\lambda(P)$  and along  $E_\lambda(P)^{\perp}$  (see [RN, Leçon 148], [GK, Sect. I.7]). In addition, since  $P$  is elliptic  $\Pi_\lambda(P)$  is a smoothing operator and  $E_\lambda(P)$  is contained  $C^\infty(M, \mathcal{E})$  (see [Sh, Thm. 8.4]).

Next, assume that the ray  $L_\theta$  is a spectral cutting for both  $p_m$  and  $P$ . Then the family  $(P_\theta^s)_{s \in \mathbb{C}}$  of complex powers of  $P$  associated to  $L_\theta$  can be defined as follows. Thanks to (2.2) we define a bounded operator on  $L^2(M, \mathcal{E})$  by letting

$$(2.4) \quad P_\theta^s = \frac{-1}{2i\pi} \int_{\Gamma_\theta} \lambda_\theta^s (P - \lambda)^{-1} d\lambda, \quad \Re s < 0,$$

$$(2.5) \quad \Gamma_\theta = \{\rho e^{i\theta}; \infty < \rho \leq r\} \cup \{r e^{it}; \theta \geq t \geq \theta - 2\pi\} \cup \{\rho e^{i(\theta - 2\pi)}; r \leq \rho \leq \infty\},$$

where  $r > 0$  is small enough so that there is no nonzero eigenvalue of  $P$  in the disc  $|\lambda| < r$  and  $\lambda_\theta^s = |\lambda|^s e^{is \arg \lambda}$  is defined by means of the continuous determination of the argument on  $\mathbb{C} \setminus L_\theta$  that takes values in  $(\theta - 2\pi, \theta)$ . We then have

$$(2.6) \quad P_\theta^{s_1+s_2} = P_\theta^{s_1} P_\theta^{s_2}, \quad \Re s_j < 0,$$

$$(2.7) \quad P_\theta^{-k} = P^{-k}, \quad k = 1, 2, \dots,$$

where  $P^{-k}$  denote the partial inverse of  $P^k$ , that is, the bounded operator that inverts  $P$  on  $E_0(P^{k*})^\perp = E_0(P^*)^\perp$  and vanishes on  $E_0(P^k) = E_0(P)$ .

On the other hand, the  $\Psi$ DO calculus with parameter allows us to show that  $P_\theta^s$  is a  $\Psi$ DO of order  $ms$  and that the family  $(P_\theta^s)_{\Re s < 0}$  is a holomorphic family of  $\Psi$ DO's in the sense of [Wo2, 7.14] and [Gu2, p. 189] (see [Se1], [Sh], [GS]). Therefore, for any  $s \in \mathbb{C}$  we can define  $P_\theta^s$  as the  $\Psi$ DO such that  $P_\theta^s = P^k P_\theta^{s-k}$ , where  $k$  is any integer  $> \Re s$ .

This gives rise to a holomorphic 1-parameter group of  $\Psi$ DO's such that  $\text{ord} P_\theta^s = ms$  for any  $s \in \mathbb{C}$ . In particular, we have  $P_\theta^0 = PP^{-1} = 1 - \Pi_0(P)$ .

**2.2. Noncommutative residue.** The noncommutative residue trace of Wodzicki ([Wo2], [Wo5]) and Guillemin [Gu1] appears as the residual trace on the algebra  $\Psi^{\mathbb{Z}}(M, \mathcal{E})$  of  $\Psi$ DO's of integer orders induced by the analytic extension of the usual trace to the class  $\Psi^{\mathbb{C}\mathbb{Z}}(M, \mathcal{E})$  of  $\Psi$ DO's of non-integer complex orders. Our exposition essentially follows that of [KV] and [CM].

First, if  $Q$  is in  $\Psi^{\text{int}}(M, \mathcal{E}) = \cup_{\Re m < -n} \Psi^m(M, \mathcal{E})$  then the restriction of its distribution kernel to the diagonal of  $M \times M$  an element  $k_Q(x, x)$  of  $\Gamma(M, |\Lambda|(M) \otimes \text{End } \mathcal{E})$ , the space of smooth  $\text{End } \mathcal{E}$ -valued densities. Therefore, the operator  $Q$  is trace-class and we have  $\text{Trace } Q = \int_M \text{tr}_{\mathcal{E}} k_Q(x, x)$ .

In fact, as shown in [KV] the map  $Q \rightarrow k_Q(x, x)$  has a unique analytic continuation  $Q \rightarrow t_Q(x)$  to the class  $\Psi^{\mathbb{C}\mathbb{Z}}(M, \mathcal{E})$ , where analyticity is meant in the sense that, for every holomorphic family  $(Q_z)_{z \in \Omega}$  with values in  $\Psi^{\mathbb{C}\mathbb{Z}}(M, \mathcal{E})$ , the map  $z \rightarrow t_{Q_z}(x)$  is analytic with values in  $\Gamma(M, |\Lambda|(M) \otimes \text{End } \mathcal{E})$ .

Moreover, if  $Q$  is in  $\Psi^{\mathbb{Z}}(M, \mathcal{E})$  and  $(Q_z)_{z \in \Omega}$  is a holomorphic family of  $\Psi$ DO's defined near  $z = 0$  such that  $Q_0 = Q$  and  $\text{ord} Q_z = z + \text{ord} Q$ , then the map  $z \rightarrow t_{Q_z}(x)$  has at worst a simple pole singularity at  $z = 0$  in such way that in local trivializing coordinates we have

$$(2.8) \quad \text{res}_{z=0} t_{Q_z}(x) = -(2\pi)^{-n} \int_{|\xi|=1} q_{-n}(x, \xi) d^{n-1} \xi,$$

where  $q_{-n}(x, \xi)$  denotes the symbol of degree  $-n$  of  $Q$ . Since  $t_{Q_z}(x)$  is a density we see that we get a well defined  $\text{End } \mathcal{E}$ -valued density on  $M$  by letting

$$(2.9) \quad c_Q(x) = (2\pi)^{-n} \int_{|\xi|=1} q_{-n}(x, \xi) d^{n-1} \xi.$$

We can now define the functionals

$$(2.10) \quad \text{TR } Q = \int_M \text{tr}_{\mathcal{E}} t_Q(x), \quad Q \in \Psi^{\mathbb{C}\mathbb{Z}}(M, \mathcal{E}),$$

$$(2.11) \quad \text{Res } Q = \int_M \text{tr}_{\mathcal{E}} c_Q(x), \quad Q \in \Psi^{\mathbb{Z}}(M, \mathcal{E}).$$

**Theorem 2.1** ([KV]). *1) The functional TR is the unique analytic continuation of the usual trace to  $\Psi^{\mathbb{C}\mathbb{Z}}(M, \mathcal{E})$ .*

2) We have  $\text{TR}[Q_1, Q_2] = 0$  whenever  $\text{ord}Q_1 + \text{ord}Q_2 \notin \mathbb{Z}$ .

3) Let  $Q \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$  and let  $(Q_z)_{z \in \Omega}$  be a holomorphic family of  $\Psi$ DO's defined near  $z = 0$  such that  $Q_0 = Q$  and  $\text{ord}Q_z = z + \text{ord}Q$ . Then near  $z = 0$  the function  $\text{TR}Q_z$  has at worst a simple pole singularity such that  $\text{res}_{z=0} \text{TR}Q_z = -\text{Res}Q$ .

The functional  $\text{Res}$  is the noncommutative residue of Wodzicki and Guillemin. From Theorem 2.1 we immediately get:

**Theorem 2.2** ([Wo2], [Gu1], [Wo5]). 1) The noncommutative residue is a linear trace on the algebra  $\Psi^{\mathbb{Z}}(M, \mathcal{E})$  which vanishes on differential operators and on  $\Psi$ DO's of integer order  $\leq -(n+1)$ .

2) We have  $\text{res}_{s=0} \text{TR}QP_{\theta}^{-s} = m \text{Res}Q$  for any  $Q \in \Psi^{\mathbb{Z}}(M, \mathcal{E})$ .

Notice also that by a well-known result of Wodzicki ([Wo4], [Ka, Prop. 5.4]; see also [Gu3]) if  $M$  is connected and has dimension  $\geq 2$  then the noncommutative residue induces the only trace on  $\Psi^{\mathbb{Z}}(M, \mathcal{E})$  up to a multiplicative constant.

**2.3. Zeta and eta functions.** The canonical trace  $\text{TR}$  allows us to define the zeta function of  $P$  as the meromorphic function on  $\mathbb{C}$  given by

$$(2.12) \quad \zeta_{\theta}(P; s) = \text{TR}P_{\theta}^{-s}, \quad s \in \mathbb{C}.$$

Then from Theorem 2.1 we obtain:

**Proposition 2.3.** Let  $\Sigma = \{\frac{n-j}{m}; j = 0, 1, \dots\} \setminus \{0\}$ . Then  $\zeta_{\theta}(P; s)$  is analytic outside  $\Sigma$  and on  $\Sigma$  has at worst simple pole singularities such that

$$(2.13) \quad \text{res}_{s=\sigma} \zeta_{\theta}(P; s) = m \text{Res}P_{\theta}^{-\sigma}, \quad \sigma \in \Sigma.$$

Notice that (2.13) is true for  $\sigma = 0$  as well, but in this case it gives

$$(2.14) \quad \text{res}_{s=0} \zeta_{\theta}(P; s) = \text{Res}P_{\theta}^0 = \text{Res}[1 - \Pi_0(P)] = 0,$$

since  $\Pi_0(P)$  is a smoothing operator. Thus  $\zeta_{\theta}(P; s)$  is always regular at  $s = 0$ .

Finally, assume that  $P$  is selfadjoint. Then the eta function of  $P$  is the meromorphic function given by

$$(2.15) \quad \eta(P; s) = \text{TR}F|P|^{-s}, \quad s \in \mathbb{C},$$

where  $F = P|P|^{-1}$  is the sign operator of  $P$ . Then using Theorem 2.1 we get:

**Proposition 2.4.** Let  $\Sigma = \{\frac{n-j}{m}; j = 0, 1, \dots\}$ . Then  $\eta(P; s)$  is analytic outside  $\Sigma$  and on  $\Sigma$  has at worst simple pole singularities such that

$$(2.16) \quad \text{res}_{s=\sigma} \eta(P; s) = m \text{Res}F|P|^{-\sigma}, \quad \sigma \in \Sigma.$$

Showing the regularity at the origin of  $\eta(P; s)$  is a much more difficult task than for the zeta functions. Indeed, from (2.16) we get

$$(2.17) \quad \text{res}_{s=0} \eta(P; s) = m \text{Res}F = m \int_M \text{tr}_{\mathcal{E}} c_F(x),$$

and examples show that  $c_F(x)$  need not vanish locally (see [Gi1]). Therefore, Atiyah-Patodi-Singer [APS2] and Gilkey ([Gi2], [Gi3]) had to rely on global and  $K$ -theoretic arguments to prove:

**Theorem 2.5.** The function  $\eta(P; s)$  is always regular at  $s = 0$ .

This shows that the eta invariant  $\eta(P) := \eta(P; 0)$  is always well defined. Since its appearance as a boundary correcting term in the index formula of Atiyah-Patodi-Singer [APS1], the eta invariant has found many applications and has been extended to various other settings. We refer to the surveys of Bismut [Bi] and Müller [Mü], and the references therein, for an overview of the main results on the eta invariant.

### 3. THE SECTORIAL PROJECTION OF AN ELLIPTIC $\Psi$ DO

In this section we give a detailed account on the sectorial projection of an elliptic  $\Psi$ DO introduced by Burak [Bu2].

Let  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  be an elliptic  $\Psi$ DO of order  $m > 0$  and assume that  $L_\theta = \{\arg \lambda = \theta\}$  and  $L_{\theta'} = \{\arg \lambda = \theta'\}$  are spectral cuttings for both  $P$  and its principal symbol  $p_m(x, \xi)$  with  $\theta < \theta' \leq \theta + 2\pi$ . In addition, we let  $\Lambda_{\theta, \theta'}$  and  $\Lambda_{\theta', \theta + 2\pi}$  respectively denote the angular sectors  $\theta < \arg \lambda < \theta'$  and  $\theta' < \arg \lambda < \theta + 2\pi$ .

The sectorial projection of  $P$  associated to the angular sector  $\Lambda_{\theta, \theta'}$  is

$$(3.1) \quad \Pi_{\theta, \theta'}(P) = \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda^{-1} P (P - \lambda)^{-1} d\lambda,$$

$$(3.2) \quad \Gamma_{\theta, \theta'} = \{\rho e^{i\theta}; \infty > \rho \geq r\} \cup \{r e^{it}; \theta \leq t \leq \theta'\} \cup \{\rho e^{i\theta'}; r \leq \rho < \infty\},$$

where  $r$  is small enough so that no non-zero eigenvalue of  $P$  lies in the disc  $|\lambda| \leq r$ .

In view of (2.2) the integral (3.1) *a priori* gives rise to an unbounded operator on  $L^2(M, \mathcal{E})$  whose domain contains  $L_m^2(M, \mathcal{E})$ . We actually get a bounded operator thanks to:

**Proposition 3.1.** *1) The operator  $\Pi_{\theta, \theta'}(P)$  is a  $\Psi$ DO of order  $\leq 0$ , hence is bounded on  $L^2(M, \mathcal{E})$ .*

*2) The zero'th order symbol of  $\Pi_{\theta, \theta'}(P)$  is the sectorial projection  $\Pi_{\theta, \theta'}(p_m(x, \xi))$ , i.e., the Riesz projection onto the root space associated to eigenvalues in  $\Lambda_{\theta, \theta'}$ .*

*Proof.* Let  $R_{\theta, \theta'} = \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda^{-1} (P - \lambda)^{-1} d\lambda$ . Then the arguments of [Se1, Thm. 3] can be carried through to prove that  $R_{\theta, \theta'}$  is a  $\Psi$ DO of order  $\leq -1$ . Hence  $\Pi_{\theta, \theta'}(P) = P R_{\theta, \theta'}$  is a  $\Psi$ DO of order  $\leq 0$ .

Next, in some local trivializing coordinates let  $p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$  and  $r(x, \xi) \sim \sum_{j \geq 0} r_{-1-j}(x, \xi)$  respectively denote the symbols of  $P$  and  $R_{\theta, \theta'}$ , so that  $\Pi_{\theta, \theta'}(P)$  has symbol  $\pi(x, \xi) \sim \sum \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x, \xi) \partial_x^\alpha r(x, \xi)$ . Furthermore, let  $q(x, \xi) \sim \sum_{j \geq 0} q_{-m-j}(x, \xi; \lambda)$  be the symbol with parameter of  $(P - \lambda)^{-1}$ . Then by [Se1, Thm. 2] we have

$$(3.3) \quad r(x, \xi) = \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda^{-1} q(x, \xi; \lambda) d\lambda = \frac{-1}{2i\pi} \int_{\Gamma(x, \xi)} \lambda^{-1} q(x, \xi; \lambda) d\lambda,$$

where  $\Gamma(x, \xi)$  is a direct-oriented bounded contour contained in the sector  $\Lambda_{\theta, \theta'}$  which isolates from  $\mathbb{C} \setminus \Lambda_{\theta, \theta'}$  the eigenvalues of  $p_m(x, \xi)$  that lie in  $\Lambda_{\theta, \theta'}$ .

On the other hand, using the equality,

$$(3.4) \quad P(P - \lambda)^{-1} = 1 + \lambda(P - \lambda)^{-1},$$

we see that  $\lambda^{-1}(p\#q)(x, \xi; \lambda) = \lambda^{-1} + q(x, \xi; \lambda)$ . Thus  $\pi(x, \xi)$  is equal to

$$(3.5) \quad p\#r(x, \xi) = \frac{-1}{2i\pi} \int_{\Gamma(x, \xi)} \lambda^{-1} p\#q(x, \xi; \lambda) d\lambda = \frac{-1}{2i\pi} \int_{\Gamma(x, \xi)} q(x, \xi; \lambda) d\lambda.$$

Therefore, for  $j = 0, 1, \dots$  we obtain

$$(3.6) \quad \pi_{-j}(x, \xi) = \frac{-1}{2i\pi} \int_{\Gamma(x, \xi)} q_{-m-j}(x, \xi; \lambda) d\lambda.$$

Hence  $\pi_0(x, \xi) = \frac{-1}{2i\pi} \int_{\Gamma(x, \xi)} (p_m(x, \xi) - \lambda)^{-1} d\lambda = \Pi_{\theta, \theta'}(p_m(x, \xi))$  as desired.  $\square$

Next, the sectorial root spaces  $E_{\theta, \theta'}(P)$  and  $E_{\theta', \theta+2\pi}(P)$  are

$$(3.7) \quad E_{\theta, \theta'}(P) = \dot{+}_{\lambda \in \Lambda_{\theta, \theta'}} E_\lambda(P), \quad E_{\theta', \theta+2\pi}(P) = \dot{+}_{\lambda \in \Lambda_{\theta', \theta+2\pi}} E_\lambda(P),$$

where  $\dot{+}$  denotes the algebraic direct sum and for  $\lambda \notin \text{Sp } P$  we make the convention that  $E_\lambda(P) = \cup_{k \geq 1} \ker(P - \lambda)^k = \{0\}$ . Then we have:

**Proposition 3.2.**  $\Pi_{\theta, \theta'}(P)$  is a projection on  $L^2(M, \mathcal{E})$  which projects onto a subspace containing  $E_{\theta, \theta'}(P)$  and along a subspace containing  $E_0(P) \dot{+} E_{\theta', \theta+2\pi}(P)$ .

*Proof.* Let  $L_{\theta_1}$  and  $L_{\theta_2}$  be rays with  $\theta_1 < \theta < \theta' < \theta'_1 < \theta + 2\pi$  and such that no eigenvalues of  $P$  and  $p_m$  lie in the angular sectors  $\theta_1 < \arg \lambda < \theta$  and  $\theta' < \arg \lambda < \theta'_1$ . This allows us to replace in the formula (3.1) for  $\Pi_{\theta, \theta'}(P)$  the integration over  $\Gamma_{\theta, \theta'}$  by that over a contour  $\Gamma_{\theta_1, \theta'_1}$  defined as in (3.2) using  $\theta_1$  and  $\theta'_1$  and a radius  $r_1$  smaller than that of  $\Gamma_{\theta, \theta'}$ . Then we have

$$(3.8) \quad \Pi_{\theta, \theta'}(P)^2 = \frac{-1}{4\pi^2} \int_{\Gamma_{\theta, \theta'}} \int_{\Gamma_{\theta_1, \theta'_1}} \lambda^{-1} \mu^{-1} P^2 (P - \lambda)^{-1} (P - \mu)^{-1} d\lambda d\mu.$$

Therefore, by using the identity,

$$(3.9) \quad (P - \lambda)^{-1} (P - \mu)^{-1} = (\lambda - \mu)^{-1} [(P - \lambda)^{-1} - (P - \mu)^{-1}],$$

we deduce that  $-4\pi^2 \Pi_{\theta, \theta'}(P)^2$  is equal to

$$(3.10) \quad \int_{\Gamma_{\theta, \theta'}} \frac{P^2}{\lambda(P - \lambda)} \left( \int_{\Gamma_{\theta_1, \theta'_1}} \frac{\mu^{-1} d\mu}{\mu - \lambda} \right) d\lambda + \int_{\Gamma_{\theta_1, \theta'_1}} \frac{P^2}{\mu(P - \mu)} \left( \int_{\Gamma_{\theta, \theta'}} \frac{\lambda^{-1} d\lambda}{\lambda - \mu} \right) d\mu,$$

from which we see that  $\Pi_{\theta, \theta'}(P)^2 = \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda^{-2} P^2 (P - \lambda)^{-1} d\lambda$ . Combining this with (3.4) then gives

$$(3.11) \quad \Pi_{\theta, \theta'}(P)^2 = \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda^{-2} P d\lambda + \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \frac{P}{\lambda(P - \lambda)} d\lambda = \Pi_{\theta, \theta'}(P).$$

Hence  $\Pi_{\theta, \theta'}(P)$  is a projection.

Next, let  $\lambda_0 \in \text{Sp } P$ . We may assume that the contour  $\Gamma_{(\lambda_0)}$  does not intersect  $\Gamma_{\theta, \theta'}$ . Then thanks to (3.9) we see that  $4\pi^2 \Pi_{\theta, \theta'}(P) \Pi_{\lambda_0}(P)$  is equal to

$$(3.12) \quad \int_{\Gamma_{\theta, \theta'}} \int_{\Gamma_{(\lambda_0)}} \frac{P}{\lambda(P - \lambda)(P - \mu)} d\lambda d\mu = \int_{\Gamma_{(\lambda_0)}} \frac{P}{P - \mu} \left( \int_{\Gamma_{\theta, \theta'}} \frac{d\lambda}{\lambda(\lambda - \mu)} \right) d\mu.$$

Therefore, if  $\lambda_0$  lies outside  $\Lambda_{\theta, \theta'}$  then  $\Pi_{\theta, \theta'}(P) \Pi_{\lambda_0}(P)$  is zero, while when  $\lambda_0$  lies inside  $\Lambda_{\theta, \theta'}$  using (3.4) we see that  $\Pi_{\theta, \theta'}(P) \Pi_{\lambda_0}(P)$  is equal to

$$(3.13) \quad \frac{-1}{2i\pi} \int_{\Gamma_{(\lambda_0)}} \frac{P}{\mu(P - \mu)} d\mu = \frac{-1}{2i\pi} \int_{\Gamma_{(\lambda_0)}} \frac{d\mu}{\mu} + \frac{-1}{2i\pi} \int_{\Gamma_{(\lambda_0)}} \frac{d\mu}{P - \mu} = \Pi_{\lambda_0}(P).$$

Since  $\Pi_{\lambda_0}(P)$  has range  $E_{\lambda_0}(P)$  it then follows that the range of  $\Pi_{\theta, \theta'}(P)$  contains  $E_{\theta, \theta'}(P)$  and its kernel contains  $E_0(P) \dot{+} E_{\theta', \theta+2\pi}(P)$ . Hence the result.  $\square$

*Remark 3.3.* Since Propositions 3.1 and 3.2 tell us that  $\Pi_{\theta, \theta'}(P)$  is a (bounded)  $\Psi$ DO projection, we see that  $\Pi_{\theta, \theta'}(P)$  has either order 0 or is smoothing.

**Proposition 3.4.** *Let  $L_{\theta_1}$  and  $L_{\theta'_1}$  be spectral cuttings for  $P$  and its principal symbol in such way that  $\theta' \leq \theta_1 < \theta'_1 < \theta + 2\pi$ . Then:*

- 1) *The projections  $\Pi_{\theta, \theta'}(P)$  and  $\Pi_{\theta_1, \theta'_1}(P)$  are disjoint.*
- 2) *We have  $\Pi_{\theta, \theta'}(P) + \Pi_{\theta', \theta'_1}(P) = \Pi_{\theta, \theta'_1}(P)$  and  $\Pi_{\theta, \theta'}(P) + \Pi_{\theta', \theta + 2\pi}(P) = 1 - \Pi_0(P)$ .*

*Proof.* First, using (3.9) we see that  $4\pi^2\Pi_{\theta, \theta'}(P)\Pi_{\theta_1, \theta'_1}(P)$  and  $4\pi^2\Pi_{\theta_1, \theta'_1}(P)\Pi_{\theta, \theta'}(P)$  are both equal to

$$(3.14) \quad \int_{\Gamma_{\theta, \theta'}} \int_{\Gamma_{\theta_1, \theta'_1}} \frac{P^2}{\lambda\mu(P-\lambda)(P-\mu)} d\lambda d\mu = \int_{\Gamma_{\theta, \theta'}} \frac{P^2}{\lambda(P-\lambda)} \left( \int_{\Gamma_{\theta_1, \theta'_1}} \frac{\mu^{-1}d\mu}{\mu-\lambda} \right) d\lambda + \int_{\Gamma_{\theta_1, \theta'_1}} \frac{P^2}{\mu(P-\mu)} \left( \int_{\Gamma_{\theta, \theta'}} \frac{\lambda^{-1}d\lambda}{\lambda-\mu} \right) d\mu = 0.$$

Hence  $\Pi_{\theta, \theta'}(P)$  and  $\Pi_{\theta_1, \theta'_1}(P)$  are disjoint projections.

Next, the operator  $\Pi_{\theta, \theta'}(P) + \Pi_{\theta', \theta'_1}(P)$  is equal to

$$(3.15) \quad \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'} \cup \Gamma_{\theta', \theta'_1}} \frac{P}{\lambda(P-\lambda)} d\lambda = \frac{1}{2i\pi} \int_{\Gamma_{\theta, \theta'_1}} \frac{P}{\lambda(P-\lambda)} d\lambda = \Pi_{\theta, \theta'_1}(P),$$

since integrating  $\lambda^{-1}P(P-\lambda)^{-1}$  along  $\Gamma_{\theta, \theta'} \cup \Gamma_{\theta', \theta'_1}$  is the same as integrating it along  $\Gamma_{\theta, \theta'_1}$ . In the special case  $\theta'_1 = \theta + 2\pi$  the integration along  $\Gamma_{\theta, \theta + 2\pi}$  reduces to that along the small circle  $|\lambda| = r$  with clockwise orientation. Therefore, using (3.4) we see that  $\Pi_{\theta, \theta'}(P) + \Pi_{\theta', \theta + 2\pi}(P)$  is equal to

$$(3.16) \quad \frac{1}{2i\pi} \int_{|\lambda|=r} \frac{P}{\lambda(P-\lambda)} d\lambda = \frac{1}{2i\pi} \int_{|\lambda|=r} \frac{d\lambda}{\lambda} + \frac{1}{2i\pi} \int_{|\lambda|=r} \frac{d\lambda}{P-\lambda} = 1 - \Pi_0(P).$$

The proof is thus complete.  $\square$

In general, the closures of  $E_{\theta, \theta'}(P)$  and  $E_0(P) \dot{+} E_{\theta', \theta + 2\pi}(P)$  don't yield the whole range and the whole kernel of  $\Pi_{\theta, \theta'}(P)$  but, as we explain in Appendix, there are special cases where they actually do:

(i) When the principal symbol of  $P$  has no eigenvalues within the angular sector  $\theta < \arg \lambda < \theta'$ , which is equivalent to  $\Pi_{\theta, \theta'}(P)$  being a smoothing operator (see Proposition A.3);

(ii) When  $P$  is normal, i.e., commutes with its adjoint, and in particular when (see Proposition A.5);

(iii) When  $P$  has a complete system of root vectors, that is, the subspace spanned by its root vector is dense (see Proposition A.7).

In the non-normal case it is a difficult issue to determine whether a general closed unbounded operator on a Hilbert space admits a complete system of root vectors. Thanks to a criterion due to Dunford-Schwartz [DS] it can be shown that  $P$  has a complete system of root vectors when its principal symbol admits spectral cuttings dividing the complex planes into angular sectors of apertures  $< \frac{2n\pi}{m}$  (see [Ag], [Bu1], [Agr]). Therefore, in this case  $\Pi_{\theta, \theta'}(P)$  is the projection onto the closure of  $E_{\theta, \theta'}(P)$  and along the closure of  $E_{\theta', \theta + 2\pi}(P)$ .

In fact, if we content ourselves by determining the range of  $\Pi_{\theta, \theta'}(P)$  then it can be shown that the range agrees with the closure  $E_{\theta, \theta'}(P)$  when we only require

the principal symbol of  $P$  to admit spectral cuttings dividing the angular sector  $\theta < \arg \lambda < \theta'$  into angular sectors of apertures  $< \frac{2n\pi}{m}$  (see Proposition A.9).

Detailed proofs of the above statements are given in Appendix.

#### 4. ZETA FUNCTIONS AND SPECTRAL ASYMMETRY

In this section, we give a detailed review of the spectral asymmetry formulas of Wodzicki ([Wo2]–[Wo4]) for elliptic  $\Psi$ DO's.

Let  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  be an elliptic  $\Psi$ DO of order  $m > 0$ . Let us first assume that  $P$  is selfadjoint. Then we have:

$$(4.1) \quad P_\uparrow^s = \Pi_+(P)|P|^s + e^{-i\pi s}\Pi_-(P)|P|^s, \quad P_\downarrow^s = \Pi_+(P)|P|^s + e^{i\pi s}\Pi_-(P)|P|^s,$$

where  $\Pi_+(P)$  (resp.  $\Pi_-(P)$ ) denotes the orthogonal projections onto the positive (resp. negative) eigenspace of  $P$ . Hence we have

$$(4.2) \quad P_\uparrow^s - P_\downarrow^s = (e^{-i\pi s} - e^{i\pi s})\Pi_-(P)|P|^s = (1 - e^{2i\pi s})\Pi_-(P)P_\uparrow^s.$$

Therefore, in the selfadjoint case, the spectral asymmetry of  $P$  is encoded by  $\Pi_-(P)$ .

Suppose now that  $P$  is not selfadjoint and let  $L_\theta = \{\arg \lambda = \theta\}$  and  $L_{\theta'} = \{\arg \lambda = \theta'\}$  be spectral cuttings for both  $P$  and its principal symbol  $p_m(x, \xi)$  with  $0 \leq \theta < \theta' < 2\pi$ . As observed by Wodzicki ([Wo3], [Wo4]) in this context a substitute to the projection  $\Pi_-(P)$  is provided by the sectorial projection  $\Pi_{\theta, \theta'}(P)$  in (3.1). This stems from:

**Proposition 4.1** ([Wo3], [Wo4]). *For any  $s \in \mathbb{C}$  we have*

$$(4.3) \quad P_\theta^s - P_{\theta'}^s = (1 - e^{2i\pi s})\Pi_{\theta, \theta'}(P)P_\theta^s.$$

*Proof.* Since in the integral (2.4) defining  $P_\theta^s$  the value of the argument has shifted of  $-2\pi$  once  $\lambda$  has turned around the circle we have

$$(4.4) \quad P_\theta^s = \frac{e^{2i\pi s} - 1}{2i\pi} \int_\infty^r \frac{x^s e^{is(\theta-2\pi)}}{P - xe^{i\theta}} d(xe^{i\theta}) + \int_\theta^{\theta-2\pi} \frac{r^s e^{ist}}{P - re^{it}} d(re^{it}).$$

Similarly, we have

$$(4.5) \quad P_{\theta'}^s = \frac{-e^{2i\pi s} + 1}{2i\pi} \int_r^\infty \frac{x^s e^{is(\theta'-2\pi)}}{P - xe^{i\theta'}} d(xe^{i\theta'}) + \int_{\theta'}^{\theta'-2\pi} \frac{r^s e^{ist}}{P - re^{it}} d(re^{it}).$$

Observe that  $\int_\theta^{\theta-2\pi} \frac{r^s e^{ist}}{P - re^{it}} d(re^{it}) - \int_{\theta'}^{\theta'-2\pi} \frac{r^s e^{ist}}{P - re^{it}} d(re^{it})$  is equal to

$$(4.6) \quad \int_{\theta'-2\pi}^{\theta-2\pi} \frac{r^s e^{ist}}{P - re^{it}} d(re^{it}) - \int_{\theta'}^\theta \frac{r^s e^{ist}}{P - re^{it}} d(re^{it}) \\ = (1 - e^{2i\pi s}) \int_{\theta'-2\pi}^{\theta-2\pi} \frac{r^s e^{ist}}{P - re^{it}} d(re^{it}).$$

Therefore, the operator  $P_\theta^s - P_{\theta'}^s$  agrees with

$$(4.7) \quad \frac{e^{2i\pi s} - 1}{2i\pi} \left( \int_\infty^r \frac{x^s e^{is(\theta-2\pi)}}{P - xe^{i\theta}} d(xe^{i\theta}) + \int_{\theta-2\pi}^{\theta'-2\pi} \frac{r^s e^{ist}}{P - re^{it}} d(re^{it}) \right. \\ \left. + \int_r^\infty \frac{x^s e^{is(\theta'-2\pi)}}{P - xe^{i\theta'}} d(xe^{i\theta'}) \right).$$

In view of the definition (3.2) of the contour  $\Gamma_{\theta, \theta'}$  this gives

$$(4.8) \quad P_{\theta}^s - P_{\theta'}^s = \frac{e^{2i\pi s} - 1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda^s (P - \lambda)^{-1} d\lambda.$$

Next, let  $\theta_1 \in (\theta' - 2\pi, \theta)$  be such that no eigenvalues of  $P$  lie in the sector  $\theta_1 \leq \arg \lambda \leq \theta$ . Then in the formula (2.4) for  $P_{\theta}^s$  we may replace the integration over  $\Gamma_{\theta}$  by that over a contour  $\Gamma_{\theta_1}$  defined by (2.5) using  $\theta_1$  and a radius  $r$  smaller than that of  $\Gamma_{\theta, \theta'}$  in (3.2). Thus,

$$(4.9) \quad \Pi_{\theta, \theta'}(P)P_{\theta}^s = \frac{1}{4\pi^2} \int_{\Gamma_{\theta, \theta'}} \int_{\Gamma_{\theta_1}} \lambda^{-1} \mu^s P(P - \lambda)^{-1} (P - \mu)^{-1} d\lambda d\mu.$$

Using (3.9) we see that  $\Pi_{\theta, \theta'}(P)P_{\theta}^s$  is equal to

$$(4.10) \quad \frac{1}{4\pi^2} \int_{\Gamma_{\theta, \theta'}} \frac{P}{\lambda(P - \lambda)} \left( \int_{\Gamma_{\theta_1}} \frac{\mu^s d\mu}{\mu - \lambda} \right) d\lambda + \frac{1}{4\pi^2} \int_{\Gamma_{\theta_1}} \frac{\mu^s P}{(P - \mu)} \left( \int_{\Gamma_{\theta, \theta'}} \frac{\lambda^{-1} d\lambda}{\lambda - \mu} \right) d\mu \\ = \frac{-1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \frac{\lambda_{\theta}^{s-1} P}{P - \lambda} d\lambda.$$

Combining this with (3.4) we obtain:

$$(4.11) \quad \Pi_{\theta, \theta'}(P)P_{\theta}^s = \frac{-1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \lambda_{\theta}^{s-1} d\lambda + \frac{-1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \frac{\lambda_{\theta}^s}{P - \lambda} d\lambda = \frac{-1}{2i\pi} \int_{\Gamma_{\theta, \theta'}} \frac{\lambda_{\theta}^s}{P - \lambda} d\lambda.$$

Comparing this to (4.8) then gives

$$(4.12) \quad P_{\theta}^s - P_{\theta'}^s = (1 - e^{2i\pi s}) \Pi_{\theta, \theta'}(P)P_{\theta}^s.$$

This proves Proposition 4.1 for  $\Re s < 0$ . Since both sides of (4.12) involve holomorphic families of  $\Psi$ DO's the general case follows by analytic continuation.  $\square$

Next, as the two sides of (4.3) are given by holomorphic families of  $\Psi$ DO's, from Theorem 2.1 we immediately get:

**Theorem 4.2** ([Wo3], [Wo4]). *We have the equality of meromorphic functions,*

$$(4.13) \quad \zeta_{\theta}(P; s) - \zeta_{\theta'}(P; s) = (1 - e^{-2i\pi s}) \text{TR} \Pi_{\theta, \theta'}(P)P_{\theta}^{-s}, \quad s \in \mathbb{C}.$$

*In particular, at any integer  $k \in \mathbb{Z}$  the function  $\zeta_{\theta}(P; s) - \zeta_{\theta'}(P; s)$  is regular and there we have*

$$(4.14) \quad \text{ord} P \lim_{s \rightarrow k} (\zeta_{\theta}(P; s) - \zeta_{\theta'}(P; s)) = 2i\pi \text{Res} \Pi_{\theta, \theta'}(P)P^{-k}.$$

As a consequence of (4.14) we see that if at some integer  $k$  we have  $\text{Res} P^{-k} = 0$ , so that  $\zeta_{\theta}(P; s)$  and  $\zeta_{\theta'}(P; s)$  are regular at  $s = k$ , then we have:

$$(4.15) \quad \zeta_{\theta}(P; k) = \zeta_{\theta'}(P; k) \iff \text{Res} \Pi_{\theta, \theta'}(P)P^{-k} = 0.$$

Furthermore, Wodzicki also proved the remarkable result below.

**Theorem 4.3** ([Wo2, 1.24]). *Let  $P : C^{\infty}(M, \mathcal{E}) \rightarrow C^{\infty}(M, \mathcal{E})$  be an elliptic  $\Psi$ DO of order  $m > 0$  and let  $L_{\theta} = \{\arg \lambda = \theta\}$  be a spectral cutting for  $P$  and its principal symbol. Then the regular value  $\zeta_{\theta}(P; 0)$  is independent of  $\theta$ .*

*Remark 4.4.* As with the vanishing of the residue at the origin of the eta function of a selfadjoint elliptic  $\Psi$ DO Theorem 4.3 is not a local result, since it is not true that in general the regular value at  $s = 0$  of the local zeta function  $t_{P_{\theta}^{-s}}(x)$  is independent of the spectral cutting (see [Wo1, pp. 130-131]).

*Remark 4.5.* The proof of Theorem 4.3 in [Wo2] is quite difficult because it relies on a very involved characterization of local invariants of spectral asymmetry. Notice that from (4.14) we get

$$(4.16) \quad \text{ord}P.(\zeta_\theta(P;0) - \zeta_{\theta'}(P;0)) = 2i\pi \text{Res} \Pi_{\theta,\theta'}(P),$$

so that  $\zeta_\theta(P;0) - \zeta_{\theta'}(P;0)$  is a constant multiple of the noncommutative residue of a  $\Psi$ DO projection. In fact, Wodzicki [Wo2, 7.12] used Theorem 4.3 to prove that the noncommutative residue of a  $\Psi$ DO projection is always zero. However, it follows from an observation of Brüning-Lesch [BL, Lem. 2.7] that the latter result can be deduced in a rather elementary way from the vanishing of the residue at the origin of the eta function of a selfadjoint elliptic  $\Psi$ DO. Therefore, combining this with (4.16) allows us to prove Theorem 4.3 without any appeal to Wodzicki's characterization of local invariants of spectral asymmetry.

## 5. SPECTRAL ASYMMETRY OF ODD CLASS ELLIPTIC $\Psi$ DO'S

In this section we study the spectral asymmetry of odd class elliptic  $\Psi$ DO's. Recall that according to [KV] a  $\Psi$ DO  $Q$  of integer order  $m$  is an odd class  $\Psi$ DO when, in local trivializing coordinates, its symbol  $q(x, \xi) \sim \sum_{j \geq 0} q_{m-j}(x, \xi)$  is polyhomogeneous with respect to the dilation by  $-1$ , i.e., for  $j = 0, 1, \dots$  we have

$$(5.1) \quad q_{m-j}(x, -\xi) = (-1)^{m-j} q_{m-j}(x, \xi).$$

This gives rise to a subalgebra of  $\Psi^{\mathbb{Z}}(M, \mathcal{E})$  which contains all the differential operators and the parametrices of elliptic odd class  $\Psi$ DO's.

Moreover, the condition  $q_{-n}(x, -\xi) = (-1)^n q_{-n}(x, \xi)$  implies that, when the dimension of  $M$  is odd, the noncommutative residue of an odd class  $\Psi$ DO vanishes locally, i.e., the density  $c_Q(x)$  given by (2.9) vanishes.

**Theorem 5.1.** *Suppose that  $\dim M$  is odd and that  $P$  is an odd class  $\Psi$ DO of even integer order  $m \geq 2$ . Then  $\zeta_\theta(P; s)$  is regular at every integer point and its values there are independent of the cutting.*

*Proof.* In some local trivializing coordinates let  $p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$  denote the symbol of  $P$  and let  $q(x, \xi, \lambda) \sim \sum_{j \geq 0} q_{-m-j}(x, \xi, \lambda)$  be the symbol with parameter of  $(P - \lambda)^{-1}$  as in [Se1], so that  $q_{-m-j}(x, t\xi, t^m \lambda) = t^{-m-j} q_{-m-j}(x, \xi, \lambda)$  for  $t \neq 0$  and  $\sim$  is taken in the sense of symbols with parameter of [Se1, p. 295]. Then by (3.6) the symbol  $\pi(x, \xi) \sim \sum_{j \geq 0} \pi_{-j}(x, \xi)$  is given by

$$(5.2) \quad \pi_{-j}(x, \xi) = \frac{-1}{2i\pi} \int_{\Gamma(x, \xi)} q_{-m-j}(x, \xi; \lambda) d\lambda,$$

where  $\Gamma(x, \xi)$  is a direct-oriented bounded contour contained in the angular sector  $\Lambda_{\theta, \theta'} = \{\theta < \arg \lambda < \theta'\}$  which isolates from  $\mathbb{C} \setminus \Lambda_{\theta, \theta'}$  the eigenvalues of  $p_m(x, \xi)$  that lie in  $\Lambda_{\theta, \theta'}$ .

At the level of symbols the equality  $(P - \lambda)(P - \lambda)^{-1} = 1$  gives

$$(5.3) \quad 1 = p\#(q - \lambda) \sim (p(x, \xi) - \lambda)q(x, \xi, \lambda) + \sum_{\alpha \neq 0} \frac{1}{\alpha!} \partial_\xi^\alpha p(x, \xi) D_x^\alpha q(x, \xi, \lambda).$$

From this we get

$$(5.4) \quad q_{-m}(x, \xi, \lambda) = (p_m(x, \xi) - \lambda)^{-1},$$

and for  $j = 1, 2, \dots$  we see that  $q_{-m-j}(x, \xi, \lambda)$  is equal to

$$(5.5) \quad -(p_m(x, \xi) - \lambda)^{-1} \sum_{\substack{|\alpha|+k+l=j, \\ l \neq j}} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-k}(x, \xi) D_x^\alpha q_{-m-l}(x, \xi, \lambda).$$

Since the symbol  $p(x, \xi)$  satisfies (5.1), it follows from (5.4) and (5.5) that for  $j = 0, 1, \dots$  we have

$$(5.6) \quad q_{-m-j}(x, -\xi, (-1)^m \lambda) = (-1)^{-m-j} q_{-m-j}(x, \xi, \lambda).$$

Now, assume  $n$  is odd and  $m$  is even. As alluded to above the noncommutative residue of an odd class  $\Psi$ DO is zero in odd dimension. Since the odd class  $\Psi$ DO's form an algebra containing all the parametrices of odd class elliptic  $\Psi$ DO's it follows that for any integer  $k$  the operator  $P^{-k}$  is an odd class  $\Psi$ DO and its noncommutative residue is zero. Therefore, the zeta functions  $\zeta_\theta(P; s)$  and  $\zeta_{\theta'}(P; s)$  are regular at all integer points.

On the other hand, since  $m$  is even thanks to (5.6) we see that

$$(5.7) \quad \begin{aligned} \pi_{-j}(x, -\xi) &= \frac{-1}{2i\pi} \int_{\Gamma(x, -\xi)} q_{-m-j}(x, -\xi, \lambda) d\lambda = \\ &= \frac{(-1)^{m-j+1}}{2i\pi} \int_{\Gamma(x, \xi)} q_{-m-j}(x, \xi, \lambda) d\lambda = (-1)^{-j} \pi_{-j}(x, \xi). \end{aligned}$$

Hence  $\Pi_{\theta, \theta'}(P)$  is an odd class  $\Psi$ DO. Therefore, for any  $k \in \mathbb{Z}$  the operator  $\Pi_{\theta, \theta'}(P)P^{-k}$  is an odd class  $\Psi$ DO as well, and so  $\text{Res } \Pi_{\theta, \theta'}(P)P^{-k} = 0$ . It then follows from Theorem 4.2 that  $\zeta_\theta(P; k) = \zeta_{\theta'}(P; k)$ .  $\square$

**Theorem 5.2.** *Assume  $\dim M$  is even,  $P$  is an odd class  $\Psi$ DO of odd integer order  $m \geq 1$  such all the eigenvalues of its principal symbol lie in the open cone  $\{\theta < \arg \lambda < \theta'\} \cup \{\theta + \pi < \arg \lambda < \theta' + \pi\}$ . Then:*

1) *For any integer  $k \in \mathbb{Z}$  we have*

$$(5.8) \quad \text{ord } P \cdot \lim_{s \rightarrow k} (\zeta_\theta(P; s) - \zeta_{\theta'}(P; s)) = i\pi \text{Res } P^{-k}.$$

2) *At every integer at which they are not singular the functions  $\zeta_\theta(P; s)$  and  $\zeta_{\theta'}(P; s)$  take on the same regular value.*

*Proof.* Since all the eigenvalues of  $p_m(x, \xi)$  are contained in the cone  $\mathcal{C}_{\theta, \theta'} := \{\theta < \arg \lambda < \theta'\} \cup \{\theta + \pi < \arg \lambda < \theta' + \pi\}$ . Then  $P$  has at most finitely many eigenvalues in  $\mathcal{C}_{\theta, \theta'}$  and by Proposition A.3 the sectorial projections  $\Pi_{\theta', \theta + \pi}(P)$  and  $\Pi_{\theta + \pi, \theta + 2\pi}(P)$  are smoothing operators.

On the other hand, by Proposition 3.4 we have

$$(5.9) \quad \Pi_{\theta, \theta'}(P) + \Pi_{\theta', \theta + \pi}(P) + \Pi_{\theta + \pi, \theta' + \pi}(P) + \Pi_{\theta' + \pi, \theta + 2\pi}(P) = 1 - \Pi_0(P).$$

Since  $\Pi_{\theta', \theta + \pi}(P)$  and  $\Pi_{\theta' + \pi, \theta + 2\pi}(P)$ , as well as  $\Pi_0(P)$ , are smoothing operators it follows that

$$(5.10) \quad \Pi_{\theta, \theta'}(P) + \Pi_{\theta + \pi, \theta' + \pi}(P) = 1 \quad \text{mod } \Psi^{-\infty}(M, \mathcal{E}).$$

Combining this with (5.2) we see that at the level of symbols we get

$$(5.11) \quad \frac{-1}{2i\pi} \int_{\Gamma(x,\xi)} q_{-m}(x, \xi, \lambda) d\lambda + \frac{-1}{2i\pi} \int_{-\Gamma(x,\xi)} q_{-m}(x, \xi, \lambda) d\lambda = 1,$$

$$(5.12) \quad \frac{-1}{2i\pi} \int_{\Gamma(x,\xi)} q_{-m-j}(x, \xi, \lambda) d\lambda + \frac{-1}{2i\pi} \int_{-\Gamma(x,\xi)} q_{-m-j}(x, \xi, \lambda) d\lambda = 0, \quad j \geq 1.$$

Next, observe that the formula (5.6) in the proof of Theorem 5.1 is actually true independently of the parities of  $m$  and  $n$ . Therefore, we may combine it with (5.11) to get

$$(5.13) \quad \begin{aligned} \pi_0(x, -\xi) - 1 &= \frac{1}{2i\pi} \int_{-\Gamma(x,\xi)} q_{-m}(x, -\xi, \lambda) d\lambda \\ &= \frac{-1}{2i\pi} \int_{\Gamma(x,\xi)} q_{-m}(x, -\xi, -\lambda) d\lambda = (-1)^m \pi_0(x, \xi) = -\pi_0(x, \xi). \end{aligned}$$

Similarly, using (5.6) and (5.12) for  $j = 1, 2, \dots$  we get

$$(5.14) \quad \begin{aligned} \pi_{-j}(x, -\xi) &= \frac{-1}{2i\pi} \int_{-\Gamma(x,\xi)} q_{-m-j}(x, -\xi, -\lambda) d\lambda \\ &= (-1)^{m-j} \pi_{-j}(x, \xi) = (-1)^{j+1} \pi_{-j}(x, \xi). \end{aligned}$$

Now, let  $k \in \mathbb{Z}$  and let  $p^{(k)} \sim \sum_{j \geq 0} p_{-km-j}^{(k)}$  denote the symbol of  $P^{-k}$ . Then the symbol  $r_{-n}^{(k)}$  of degree  $-n$  of  $R^{(k)} = \Pi_{\theta, \theta'}(P)P^{-k}$  is given by

$$(5.15) \quad r_{-n}^{(k)}(x, -\xi) = \sum_{|\alpha|+j+l=n-km} \frac{1}{\alpha!} \partial_\xi^\alpha \pi_{-j}(x, \xi) D_x^\alpha p_{-km-l}^{(k)}(x, \xi).$$

Since  $P^{-k}$  is an odd class  $\Psi$ DO, using (5.11) and (5.12) we obtain:

$$(5.16) \quad \begin{aligned} r_{-n}^{(k)}(x, -\xi) &= \sum_{j+l+|\alpha|=n-km} \frac{1}{\alpha!} (\partial_\xi^\alpha \pi_{-j})(x, -\xi) (D_x^\alpha p_{-km-l}^{(k)})(x, -\xi) \\ &= \sum_{l+|\alpha|=n-km} \frac{(-1)^{|\alpha|-km-l}}{\alpha!} \partial_\xi^\alpha [1 - \pi_0(x, \xi)] D_x^\alpha p_{-km-l}^{(k)}(x, \xi) \\ &\quad - \sum_{j+l+|\alpha|=n-km} \frac{(-1)^{j+|\alpha|-km-l}}{\alpha!} \partial_\xi^\alpha \pi_{-j}(x, \xi) D_x^\alpha p_{-km-l}^{(k)}(x, \xi) \\ &= (-1)^n p_{-n}^{(k)}(x, \xi) - (-1)^n \sum_{|\alpha|+j+l=n-km} \frac{1}{\alpha!} (\partial_\xi^\alpha \pi_{-j})(x, \xi) (D_x^\alpha p_{-km-l}^{(k)})(x, \xi). \end{aligned}$$

Combining this with (5.15) and the fact that  $n$  is even we get

$$(5.17) \quad r_{-n}^{(k)}(x, \xi) + r_{-n}^{(k)}(x, -\xi) = p_{-n}^{(k)}(x, \xi).$$

Moreover, we have

$$(5.18) \quad \int_{|\xi|=1} r_{-n}^{(k)}(x, -\xi) d^{n-1}\xi = (-1)^n \int_{|\xi|=1} r_{-n}^{(k)}(x, \xi) d^{n-1}\xi = (2\pi)^{-n} c_{R^{(k)}}(x),$$

where  $c_{R^{(k)}}(x)$  is the residual density (2.9). Thus (5.17) yields  $2c_{R^{(k)}}(x) = c_{P^{-k}}(x)$ , from which we get  $\text{Res } \Pi_{\theta, \theta'}(P)P^{-k} = \text{Res } R^{(k)} = \frac{1}{2} \text{Res } P^{-k}$ . Combining this with

Theorem 4.2 then gives

$$(5.19) \quad \text{ord}P. \lim_{s \rightarrow k} (\zeta_\theta(P; s) - \zeta_{\theta'}(P; s)) = 2i\pi \text{Res } P^{-k}.$$

Finally, by Proposition 2.3 the functions  $\zeta_\theta(P; s)$  and  $\zeta_{\theta'}(P; s)$  are regular at  $k \in \mathbb{Z}$  iff  $\text{Res } P^{-k} = 0$ . As  $\text{ord}P. \lim_{s \rightarrow k} (\zeta_\theta(P; s) - \zeta_{\theta'}(P; s)) = 2i\pi \text{Res } P^{-k}$  it follows that whenever  $\zeta_\theta(P; s)$  and  $\zeta_{\theta'}(P; s)$  are regular at an integer their regular values there coincide. In particular, as they are always regular at the origin we have  $\zeta_\theta(P; 0) = \zeta_{\theta'}(P; 0)$ .  $\square$

*Remark 5.3.* As the noncommutative residue of a differential operator is always zero, we see that if in Theorems 5.1 and 5.2 we further assume that  $P$  is a differential operator, then at every integer not between 1 and  $\frac{n}{m}$  the functions  $\zeta_\theta(P; s)$  and  $\zeta_{\theta'}(P; s)$  are non-singular and share the same regular value.

Finally, the proofs of Theorems 5.1 and 5.2 are based on the analysis of the symbol of  $\Pi_{\theta, \theta'}(P)$ , so the theorems ultimately hold at level the local zeta functions  $\zeta_\theta(P; s)(x) := \text{tr}_\mathcal{E} t_{P^{-s}}(x)$ . In particular, for the regular value at  $s = 0$  we get:

**Theorem 5.4.** *If  $P$  satisfies either the assumptions of Theorem 5.1 or that of Theorem 5.2, then  $\zeta_\theta(P; 0)(x)$  is independent of the cutting.*

This shows that the independence of  $\zeta_\theta(P; 0)(x)$  with respect to the cutting, while not true in general (see [Wo1, pp. 130-131]), nevertheless occurs for a large class of elliptic  $\Psi$ DO's.

## 6. SPECTRAL ASYMMETRY OF SELFADJOINT ODD CLASS ELLIPTIC $\Psi$ DO'S

In this section we specialize the results from the previous sections to selfadjoint odd class elliptic  $\Psi$ DO's and use them to study the eta function of such operators.

Let  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  be a selfadjoint odd class elliptic  $\Psi$ DO of integer order  $m \geq 1$ . Since the principal symbol  $p_m(x, \xi)$  of  $P$  is selfadjoint, the assumption in Theorem 5.2 on the location of the eigenvalues of  $p_m$  is always satisfied if we take  $0 < \theta < \pi < \theta' < 2\pi$ .

Now, Theorems 5.1 and 5.2 tell us that if  $\dim M$  and  $\text{ord}P$  have opposite parities then there are many integer points at which the zeta functions  $\zeta_\uparrow(P; s)$  and  $\zeta_\downarrow(P; s)$  are not asymmetric. However, they also allow us to single out points at which the asymmetry of zeta functions always occurs. For instance, we have:

**Proposition 6.1.** *If  $\dim M$  is even and  $P$  is an odd class selfadjoint elliptic  $\Psi$ DO of order 1, then we always have  $\lim_{s \rightarrow n} \frac{1}{i}(\zeta_\uparrow(P; s) - \zeta_\downarrow(P; s)) > 0$ .*

*Proof.* By Theorem 5.2 we have  $\lim_{s \rightarrow n} \frac{1}{i}(\zeta_\uparrow(P; s) - \zeta_\downarrow(P; s)) = \pi \text{Res } P^{-n}$ . Moreover, since  $P^{-n}$  has order  $-n$  its symbol of degree  $-n$  is its principal symbol  $p_m(x, \xi)^{-n}$ , so we have  $\text{Res } P^{-n} = (2\pi)^{-n} \int_{S^*M} \text{tr } p_m(x, \xi)^{-n} dx d\xi$ , where  $S^*M$  denotes the cosphere bundle of  $M$  with its induced metric.

On the other hand, as  $p_m(x, \xi)$  is selfadjoint and  $n$  is even we have  $\text{tr } p_m(x, \xi)^{-n} = \text{tr}[p_m(x, \xi)^{-\frac{n}{2}} * p_m(x, \xi)^{-\frac{n}{2}}] > 0$ . Hence  $\text{Res } P^{-n}$  and  $\lim_{s \rightarrow n} \frac{1}{i}(\zeta_\uparrow(P; s) - \zeta_\downarrow(P; s))$  are positive numbers.  $\square$

Next, as observed by Shubin [Sh, p. 114] (see also [Wo1, p. 116]), we can relate  $\zeta_\uparrow(P; s) - \zeta_\downarrow(P; s)$  to the eta function  $\eta(P; s)$  as follows. Let  $F = \Pi_+(P) - \Pi_-(P)$  be the sign operator of  $P$ . Then using (4.1) we get:

$$(6.1) \quad P_\uparrow^s - F|P|^s = (1 + e^{-i\pi s})\Pi_-(P)|P|^s.$$

Combining this with (4.2) and the fact that  $(1 - e^{i\pi s})(e^{-i\pi s} + 1) = e^{-i\pi s} - e^{i\pi s}$  we obtain

$$(6.2) \quad P_{\uparrow}^s - P_{\downarrow}^s = (e^{-i\pi s} - e^{i\pi s})(1 + e^{-i\pi s})^{-1}(P_{\uparrow}^s - F|P|^s) = (1 - e^{i\pi s})(P_{\uparrow}^s - F|P|^s).$$

Since  $\eta(P; s) = \text{TR } F|P|^{-s}$  we get:

**Proposition 6.2.** 1) *We have the equality of meromorphic functions,*

$$(6.3) \quad \zeta_{\uparrow}(P; s) - \zeta_{\downarrow}(P; s) = (1 - e^{-i\pi s})\zeta_{\uparrow}(P; s) - (1 - e^{-i\pi s})\eta(P; s), \quad s \in \mathbb{C}.$$

*In particular, for any  $k \in \mathbb{Z}$  we have*

$$(6.4) \quad \text{ord}P \cdot \lim_{s \rightarrow k} (\zeta_{\uparrow}(P; s) - \zeta_{\downarrow}(P; s)) = i\pi \text{Res } P^{-k} - i\pi \text{ord}P \cdot \text{res}_{s=k} \eta(P; s).$$

2) *Let  $k \in \mathbb{Z}$  and suppose that  $\text{Res } P^{-k} = 0$ , so that  $\zeta_{\uparrow}(s)$  and  $\zeta_{\downarrow}(s)$  are both regular at  $s = k$ . Then we have:*

$$(6.5) \quad \zeta_{\uparrow}(P; k) = \zeta_{\downarrow}(P; k) \iff \eta(P; s) \text{ is regular at } s = k.$$

Now, by a well known result of Branson-Gilkey [BG] in even dimension the eta function of a geometric Dirac operator is an entire function. In fact, the latter is a special case of the more general result below.

**Theorem 6.3.** 1) *If  $\dim M$  and  $\text{ord}P$  have opposite parities then  $\eta(P; s)$  is regular at every integer point.*

2) *If  $P$  has order 1 and  $\dim M$  is even then  $\eta(P; s)$  is an entire function.*

*Proof.* Let  $k \in \mathbb{Z}$ . Since  $\dim M$  and  $\text{ord}P$  have opposite parities Theorem 5.1 and Theorem 5.2 tell us that  $i\pi \text{Res } P^{-k}$  and  $\text{ord}P \cdot \lim_{s \rightarrow k} (\zeta_{\uparrow}(P; s) - \zeta_{\downarrow}(P; s))$  in (6.4) either are both equal to zero (when  $\dim M$  is odd and  $\text{ord}P$  is even) or are equal to each other (when  $\dim M$  is even and  $\text{ord}P$  is odd). In any case (6.4) shows that  $\eta(P; s)$  is regular at  $s = k$ .

On the other hand, when  $P$  has order 1 Proposition 2.4 implies that  $\eta(P; s)$  is holomorphic on  $\mathbb{C} \setminus \mathbb{Z}$ . Thus, when  $\dim M$  is even and  $P$  has order 1 the function  $\eta(P; s)$  is entire.  $\square$

*Remark 6.4.* Theorem 6.3 has been obtained independently by Grubb [Gr] using a different approach.

*Remark 6.5.* Theorem 6.3 allows us to simplify in the odd dimensional case the index formula of Brüning-Seeley [BS, Thm. 4.1] for a first order elliptic differential operator on a manifold  $M$  with cone-like singularities. The contribution of the singularities to this formula involves the residues at integer points of some first order selfadjoint elliptic differential operators on manifolds of dimension  $\dim M - 1$ . Thus when  $\dim M$  is odd Theorem 6.3 insures us that all these residues are zero, hence disappear from the formula.

## 7. SPECTRAL ASYMMETRY OF DIRAC OPERATORS

In this section we make use of the results of the previous sections to express in geometric terms the spectral asymmetry of Dirac operators.

Throughout all the section we assume that  $\dim M$  is even and that  $\mathcal{E}$  is endowed with a Clifford module structure, that is, a  $\mathbb{Z}_2$ -grading  $\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$  and an action

of the Clifford bundle  $\text{Cl}(M)$  on  $\mathcal{E}$  anticommuting with each other. Given a unitary Clifford connection  $\nabla^{\mathcal{E}}$  we get a Dirac operator  $\mathcal{D}_{\mathcal{E}}$  as the composition,

$$(7.1) \quad \mathcal{D}_{\mathcal{E}} : C^{\infty}(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} C^{\infty}(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} C^{\infty}(M, \mathcal{E}),$$

where  $c$  denotes the Clifford action of  $T^*M$  on  $\mathcal{E}$  (see [BGV, Sect. 3.3]).

This setting covers many geometric examples, e.g., the Dirac operator on a spin Riemannian manifold with coefficients in a Hermitian vector bundle, the Gauss-Bonnet and signature operators on an oriented Riemannian manifold, or even the  $\bar{\partial} + \bar{\partial}^*$ -operator on a Kaehler manifold.

The main result of this section is the following.

**Theorem 7.1.** 1) The function  $\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)$  is entire.

2) At every odd integer and at every even integer not between 2 and  $n$  the functions  $\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s)$  and  $\zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)$  are regular and have the same regular value.

3) For  $k = 2, 4, \dots, n$  we have

$$(7.2) \quad \lim_{s \rightarrow k} (\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)) = i\pi \int_M A_k(R^M, F^{\mathcal{E}/\mathcal{S}})(x) \sqrt{g(x)} d^n x,$$

where  $A_k(R^M, F^{\mathcal{E}/\mathcal{S}})(x)$  is a universal polynomial in complete tensorial contractions of the covariant derivatives of the Riemannian curvature  $R^M$  of  $M$  and of the twisted curvature  $F^{\mathcal{E}/\mathcal{S}}$  of  $\mathcal{E}$  as defined in [BGV, Prop. 3.43]. In particular,

$$(7.3) \quad \lim_{s \rightarrow n} (\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)) = 2i\pi (4\pi)^{-n/2} \Gamma\left(\frac{n}{2}\right)^{-1} \text{rk } \mathcal{E} \cdot \text{vol } M,$$

$$(7.4) \quad \lim_{s \rightarrow n-2} (\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)) = -i\pi c_n \text{rk } \mathcal{E} \int_M r_M(x) \sqrt{g(x)} d^n x,$$

where  $c_n = \frac{1}{12}(n-2)(4\pi)^{-n/2} \Gamma\left(\frac{n}{2}\right)^{-1}$  and  $r_M$  denotes the scalar curvature of  $M$ .

*Proof.* First, as  $\mathcal{D}_{\mathcal{E}}$  is a first order differential operator Proposition 2.3 tells us that the function  $\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)$  can have poles only at integer points and by Theorem 4.2 the function is regular at these points. Thus  $\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)$  is an entire function.

Second, since  $n$  is even it follows from Theorem 5.2 and Remark 5.3 that at every integer  $k$  not between 1 and  $n$  the functions  $\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s)$  and  $\zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)$  are regular and have the same regular value.

Next, by construction  $\mathcal{D}_{\mathcal{E}}$  anticommutes with the  $\mathbb{Z}_2$ -grading of  $\mathcal{E}$ , so when  $k$  is odd  $\mathcal{D}_{\mathcal{E}}^{-k}$  also anticommutes with the  $\mathbb{Z}_2$ -grading. At the level of the residual density  $c_{\mathcal{D}_{\mathcal{E}}^{-k}}(x)$  this implies that it take values in endomorphisms of  $\mathcal{E}$  intertwining  $\mathcal{E}^+$  and  $\mathcal{E}^-$ , so that we have  $\text{tr}_{\mathcal{E}} c_{\mathcal{D}_{\mathcal{E}}^{-k}}(x) = 0$  and  $\text{Res } P^{-k}$  vanishes. Thus, at  $s = k$  the functions  $\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s)$  and  $\zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)$  are regular and so have same regular value by Theorem 5.2.

Now, let us assume that  $k = 2l$  for some integer  $l$  between 0 and  $\frac{n}{2}$ . Thanks to (5.8) we have

$$(7.5) \quad \lim_{s \rightarrow k} (\zeta_{\uparrow}(\mathcal{D}_{\mathcal{E}}; s) - \zeta_{\downarrow}(\mathcal{D}_{\mathcal{E}}; s)) = i\pi \int_M \text{tr}_{\mathcal{E}} c_{(\mathcal{D}_{\mathcal{E}}^2)^{-l}}(x).$$

As it is well-known (see [Wo5, 3.23]) the densities  $c_{(\mathcal{D}_\mathcal{E}^2)^{-l}}(x)$ ,  $l = 1, \dots, \frac{n}{2}$ , are related to the coefficient of the small time heat-kernel asymptotics,

$$(7.6) \quad k_t(x, x) \sim t^{-\frac{n}{2}} \sum_{j \geq 0} t^j a_j(\mathcal{D}_\mathcal{E}^2)(x) \quad \text{as } t \rightarrow 0^+,$$

where  $k_t(x, y)$ ,  $t > 0$ , denotes the heat kernel of  $\mathcal{D}_\mathcal{E}^2$ . More precisely, we have

$$(7.7) \quad c_{(\mathcal{D}_\mathcal{E}^2)^{-l}}(x) = \frac{2}{(l-1)!} a_{\frac{n}{2}-l}(\mathcal{D}_\mathcal{E}^2)(x).$$

On the other hand, the operator  $\mathcal{D}_\mathcal{E}^2$  is a Laplace type operator, since by the Lichnerowicz's formula we have  $\mathcal{D}_\mathcal{E}^2 = (\nabla^\mathcal{E})^* \nabla^\mathcal{E} + c(F^\mathcal{E}/\mathcal{F}) + \frac{1}{4}r_M$  (see [BGV, Thm. 3.52]). Therefore, by [Gi3, pp. 334-336] each density  $a_j(\mathcal{D}_\mathcal{E}^2)(x)$ 's is of the form  $\tilde{A}_j(R^M, F^\mathcal{E}/\mathcal{F})\sqrt{g(x)}dx$ , for some universal polynomial  $\tilde{A}_j(R^M, F^\mathcal{E}/\mathcal{F})$  in complete tensorial contractions of the curvatures  $R^M$  and  $F^\mathcal{E}/\mathcal{F}$ . In particular, we have

$$(7.8) \quad \tilde{A}_0(R^M, F^\mathcal{E}/\mathcal{F}) = (4\pi)^{-n/2} \text{id}_\mathcal{E},$$

$$(7.9) \quad \tilde{A}_1(R^M, F^\mathcal{E}/\mathcal{F}) = \frac{-(4\pi)^{-n/2}}{12} (r_M \text{id}_\mathcal{E} + 2c(F^\mathcal{E}/\mathcal{F})).$$

Combining this with (7.5) and (7.7) and the fact that  $\text{Tr}_\mathcal{E} c(F^\mathcal{E}/\mathcal{F}) = 0$  then gives the formulas (7.2)–(7.4).  $\square$

As an immediate consequence of (7.3) we get:

**Proposition 7.2.** *1) The value of  $\lim_{s \rightarrow n-2} (\zeta_\uparrow(\mathcal{D}_\mathcal{E}; s) - \zeta_\downarrow(\mathcal{D}_\mathcal{E}; s))$  is independent of the Clifford data  $(\mathcal{E}, \nabla^\mathcal{E})$ .*

*2) If  $\int_M r_M \sqrt{g(x)} dx \neq 0$  then we have  $\lim_{s \rightarrow n-2} (\zeta_\uparrow(\mathcal{D}_\mathcal{E}; s) - \zeta_\downarrow(\mathcal{D}_\mathcal{E}; s)) \neq 0$  for any Clifford data  $(\mathcal{E}, \nabla^\mathcal{E})$ .*

Finally, the integral  $\int_M r_M(x) \sqrt{g(x)} dx$  is the Einstein-Hilbert action of the metric  $g$ , which gives the contribution of gravity forces to the action functional in general relativity. Therefore, it is an important issue in noncommutative geometry and mathematical physics to give an operator theoretic formulation of this action. The first one by given by Connes [Co2] in terms of  $\text{Res}_M \mathcal{D}_M^{-n+2}$  (see also [KW], [Kas]), but we see here that thanks to (7.3) we get another spectral interpretation of the Einstein-Hilbert action.

## APPENDIX

In this appendix we gather the main results regarding the spectral interpretation of the sectorial projection of an elliptic  $\Psi$ DO.

Let  $P : C^\infty(M, \mathcal{E}) \rightarrow C^\infty(M, \mathcal{E})$  be an elliptic  $\Psi$ DO of order  $m > 0$  and assume that  $L_\theta = \{\arg \lambda = \theta\}$  and  $L_{\theta'} = \{\arg \lambda = \theta'\}$  are spectral cuttings for both  $P$  and its principal symbol  $p_m(x, \xi)$  with  $\theta < \theta' \leq \theta + 2\pi$ . We let  $\Pi_{\theta, \theta'}(P)$  be the corresponding sectorial projection as defined in (3.1) and we shall use in the sequel the notation introduced in Section 3.

As alluded to in Section 3 we cannot say in general whether  $\Pi_{\theta, \theta'}(P)$  is the projection onto the closure of  $E_{\theta, \theta'}(P)$  and along that of  $E_0(P) \dot{+} E_{\theta', \theta+2\pi}(P)$ , but there are some important cases for which we can. First, we have:

**Proposition A.3.** *The following are equivalent:*

- (i) *For any  $(x, \xi) \in T^*M \setminus 0$  there are no eigenvalues of  $p_m(x, \xi)$  within  $\overline{\Lambda_{\theta, \theta'}}$ .*
- (ii) *The sectorial projection  $\Pi_{\theta, \theta'}(P)$  is a smoothing operator.*

Moreover, if (i) and (ii) hold then  $\text{Sp } P \cap \Lambda_{\theta, \theta'}$  is finite and we have

$$(A.10) \quad \Pi_{\theta, \theta'}(P) = \sum_{\lambda \in \text{Sp } P \cap \Lambda_{\theta, \theta'}} \Pi_{\lambda}(P).$$

Hence  $\Pi_{\theta, \theta'}(P)$  has range  $E_{\theta, \theta'}(P)$ .

*Proof.* Since  $\Pi_{\theta, \theta'}(P)$  is a (bounded)  $\Psi$ DO projection, either it has order zero or it is smoothing. Thus  $\Pi_{\theta, \theta'}(P)$  is a smoothing operator if, and only if, its zero'th order symbol is zero. By Proposition 3.1 the latter is the Riesz projection  $\Pi_{\theta, \theta'}(p_m(x, \xi))$  onto the root space associated to eigenvalues of  $p_m(x, \xi)$  in  $\Lambda_{\theta, \theta'}$ . Therefore  $\Pi_{\theta, \theta'}(P)$  is smoothing if, and only if, for any  $(x, \xi) \in T^*M \setminus 0$  there are no eigenvalues of  $p_m(x, \xi)$  within  $\overline{\Lambda_{\theta, \theta'}}$ .

Assume now that for any  $(x, \xi) \in T^*M \setminus 0$  there are no eigenvalues of  $p_m(x, \xi)$  within  $\overline{\Lambda_{\theta, \theta'}}$ . Then there is an open angular sector  $\Lambda$  containing  $\overline{\Lambda_{\theta, \theta'}} \setminus 0$  such that no eigenvalue of  $p_m(x, \xi)$  lies in  $\Lambda$ . Then (2.2) tells us that  $\text{Sp } P \cap \Lambda_{\theta, \theta'}$  is finite and for  $R$  large enough there exists  $C_{R\theta\theta'} > 0$  such that we have

$$(A.11) \quad \|(P - \lambda)^{-1}\|_{\mathcal{L}(L^2(M, \mathcal{E}))} \leq C_{R\theta\theta'} |\lambda|^{-1}, \quad \lambda \in \overline{\Lambda_{\theta, \theta'}}, \quad |\lambda| \geq R.$$

It follows that in (3.1) we may replace the integration contour  $\Gamma_{\theta, \theta'}$  by a *bounded* smooth contour  $\Gamma$  which has index  $-1$  and enlaces  $\text{Sp } P \cap \Lambda_{\theta, \theta'}$  but not the origin. Therefore, using (3.4) we see that  $\Pi_{\theta, \theta'}(P)$  is equal to

$$(A.12) \quad \frac{1}{2i\pi} \int_{\Gamma} \frac{P}{\lambda(P - \lambda)} d\lambda = \sum_{\mu \in \text{Sp } P \cap \Lambda_{\theta, \theta'}} \frac{-1}{2i\pi} \int_{\Gamma_{\mu}} \frac{P}{\lambda(P - \lambda)} d\lambda = \sum_{\mu \in \text{Sp } P \cap \Lambda_{\theta, \theta'}} \Pi_{\mu}(P).$$

The proof is thus achieved.  $\square$

Next, recall that  $P$  is said to have a *complete system of root vectors* when the total root space  $\dot{+}_{\lambda \in \text{Sp } P} E_{\lambda}(P)$  is dense in  $L^2(M, \mathcal{E})$ .

**Proposition A.4.** *If  $P$  has a complete system of root vectors then  $\Pi_{\theta, \theta'}(P)$  is the projection onto  $\overline{E_{\theta, \theta'}(P)}$  and along  $E_0(P) \dot{+} \overline{E_{\theta', \theta+2\pi}(P)}$ .*

*Proof.* Let us first prove that  $\text{ran } \Pi_{\theta, \theta'}(P)$  is equal to  $\overline{E_{\theta, \theta'}(P)}$ . We already know that the latter is contained in the former. Conversely, let  $\xi$  be in  $\text{ran } \Pi_{\theta, \theta'}(P)$ , so that  $\Pi_{\theta, \theta'}(P)\xi = \xi$ . Since  $P$  has a complete system of root vectors there exists a sequence  $(\xi_k)_{k \geq 0} \subset \dot{+}_{\lambda \in \text{Sp } P} E_{\lambda}(P)$  which converges to  $\xi$  in  $L^2(M, \mathcal{E})$ . As  $\xi_k$  is the sum of finitely many root vectors we have  $\xi_k = \sum_{\lambda \in \text{Sp } P} \Pi_{\lambda}(P)\xi_k$ , where the sum is actually finite. Combining this with (3.13) gives

$$(A.13) \quad \Pi_{\theta, \theta'}(P)\xi_k = \sum_{\lambda \in \text{Sp } P} \Pi_{\theta, \theta'}(P)\Pi_{\lambda}(P)\xi_k = \sum_{\lambda \in \text{Sp } P \cap \Lambda_{\theta, \theta'}} \Pi_{\lambda}(P)\xi_k,$$

so that  $\Pi_{\theta, \theta'}\xi_k$  belongs to  $E_{\theta, \theta'}(P)$ . Since  $\xi = \Pi_{\theta, \theta'}(P)\xi = \lim_{k \rightarrow \infty} \Pi_{\theta, \theta'}\xi_k$  it follows that  $\xi$  is in the closure of  $E_{\theta, \theta'}(P)$ . Hence  $\text{ran } \Pi_{\theta, \theta'}(P) = \overline{E_{\theta, \theta'}(P)}$ .

Similarly, the projection  $\Pi_{\theta', \theta+2\pi}(P)$  has range  $\overline{E_{\theta', \theta+2\pi}(P)}$ . Observe also that as in (5.9) we have  $\Pi_{\theta, \theta'}(P) + \Pi_{\theta', \theta+2\pi}(P) = 1 - \Pi_0(P)$ . Thus,

$$(A.14) \quad \text{ran}(1 - \Pi_{\theta, \theta'}(P)) = \text{ran } \Pi_0(P) + \text{ran } \Pi_{\theta', \theta+2\pi}(P) = E_0(P) \dot{+} \overline{E_{\theta', \theta+2\pi}(P)}.$$

Hence  $\Pi_{\theta, \theta'}(P)$  is the projection onto  $\overline{E_{\theta, \theta'}(P)}$  and along  $E_0(P) \dot{+} \overline{E_{\theta', \theta+2\pi}(P)}$ .  $\square$

**Proposition A.5.** *If  $P$  is normal then  $\Pi_{\theta, \theta'}(P)$  is the orthogonal projection onto  $\oplus_{\lambda \in \text{Sp } P \cap \Lambda_{\theta, \theta'}} \ker(P - \lambda)$ , where  $\oplus$  denotes the Hilbertian direct sum on  $L^2(M, \mathcal{E})$ .*

*Proof.* Since  $P$  is normal it diagonalizes on a Hilbert basis, that is, we have

$$(A.15) \quad L^2(M, \mathcal{E}) = \oplus_{\lambda \in \text{Sp } P} \ker(P - \lambda),$$

where  $\oplus$  denotes the Hilbertian direct sum on  $L^2(M, \mathcal{E})$  (see [Kat, Thm. V.2.10]). In particular,  $P$  has a complete system of root vectors, so by Proposition A.4 the sectorial projection  $\Pi_{\theta, \theta'}(P)$  projects onto  $\overline{E_{\theta, \theta'}(P)}$  and along  $E_0(P) \dot{+} \overline{E_{\theta', \theta+2\pi}(P)}$ .

On the other hand, the orthogonal decomposition (A.15) implies that for every  $\lambda \in \text{Sp } P$  we have  $E_\lambda(P) = \ker(P - \lambda) = \ker(P^* - \bar{\lambda})$ . Thus,

$$(A.16) \quad \overline{E_{\theta, \theta'}(P)} = \oplus_{\lambda \in \text{Sp } P \cap \Lambda_{\theta, \theta'}} \ker(P - \lambda).$$

Similarly  $E_0(P) \dot{+} \overline{E_{\theta', \theta+2\pi}(P)}$  is equal to

$$(A.17) \quad \ker P \oplus [\oplus_{\lambda \in \text{Sp } P \cap \Lambda_{\theta', \theta+2\pi}} \ker(P - \lambda)] = E_{\theta, \theta'}(P)^\perp.$$

Hence  $\Pi_{\theta, \theta'}(P)$  is the orthogonal projection onto  $\oplus_{\lambda \in \text{Sp } P \cap \Lambda_{\theta, \theta'}} \ker(P - \lambda)$ .  $\square$

As an immediate consequence we get:

**Corollary A.6.** *When  $P$  is selfadjoint the sectorial projection  $\Pi_{\uparrow\downarrow}(P)$  is the orthogonal projection onto the negative eigenspace of  $P$ .*

There are well known examples due to Seeley [Se2] and Agranovich-Markus [AM] of elliptic differential operators without a complete system root vectors. In these examples the principal symbol does not admit a spectral cutting. However, even when the principal symbol does admit a spectral cutting the best positive result about completeness result seems to be the following.

**Proposition A.7** ([Ag, Thm. 3.2], [Bu1, Appendix], [Agr, Thm. 6.4.3]). *Assume that the principal symbol of  $P$  admits spectral cuttings  $L_{\theta_1}, \dots, L_{\theta_N}$  dividing the complex planes into angular sectors of apertures  $< \frac{2n\pi}{m}$ . Then the system of root vectors of  $P$  is complete.*

This result follows from a criterion due to Dunford-Schwartz [DS, Cor. XI.9.31] for closed operators on a Hilbert spaces with a resolvent in some Schatten ideal. Combining it with Proposition A.4 thus gives:

**Proposition A.8.** *If the principal symbol of  $P$  admits spectral cuttings dividing the complex plane into angular sectors of apertures  $< \frac{2n\pi}{m}$ , then  $\Pi_{\theta, \theta'}(P)$  is the projection onto  $\overline{E_{\theta, \theta'}(P)}$  and along  $E_0(P) \dot{+} \overline{E_{\theta', \theta+2\pi}(P)}$ .*

Finally, if we only want to determine the range of  $\Pi_{\theta, \theta'}(P)$  then we have:

**Proposition A.9.** *If the principal symbol of  $P$  admits spectral cuttings  $L_{\theta_1}, \dots, L_{\theta_N}$  dividing the angular sector  $\Lambda_{\theta, \theta'}$  into angular sectors of apertures  $< \frac{2n\pi}{m}$ . Then the range of  $\Pi_{\theta, \theta'}(P)$  is equal to  $\overline{E_{\theta, \theta'}(P)}$ .*

*Proof.* The operator  $\tilde{P}$  induced by  $P$  on  $\text{ran } \Pi_{\theta, \theta'}(P)$  has spectrum  $\text{Sp } P \cap \Lambda_{\theta, \theta'}$  and its resolvent is also in the Schatten ideal  $\mathcal{L}^{\frac{m}{n} + \epsilon}$  for any  $\epsilon > 0$ . Moreover, the condition on the principal symbol implies that  $P$  has finitely many rays of minimal growth  $L_{\theta'_1}, \dots, L_{\theta'_N}$  dividing  $\Lambda_{\theta, \theta'}$  into angular sectors of aperture  $< \frac{2n\pi}{m}$ .

Henceforth  $\tilde{P}$  admits a finite sequence of rays of minimal growth dividing  $\mathbb{C}$  into angular sectors of aperture  $< \frac{2n\pi}{m}$ . It then follows from [DS, Cor. XI.9.31] that the total root space of  $\tilde{P}$ , that is,  $E_{\theta, \theta'}(P)$ , is dense in  $\text{ran } \Pi_{\theta, \theta'}(P)$ .  $\square$

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