NONCOMMUTATIVE RESIDUE AND NEW INVARIANTS FOR CR MANIFOLDS

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ABSTRACT. In this note we produce new CR invariants by looking at the noncommutative residue traces of geometric Ψ_HDO projections on CR manifolds. In particular, we recover and extend a recent result of Hirachi and answer a question of Fefferman.

INTRODUCTION

Motivated by Fefferman's program in CR geometry [Fe2], Hirachi [Hi] proved that the integral of the coefficient of the logarithmic singularity of the Szegö kernel on the boundary of a strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ gives rise to a CR invariant. It was then asked by Fefferman whether they would exist other such invariants.

The aim of this note is to explain how the noncommutative residue trace of [Po1] and [Po5] allows us to construct several new CR invariants extending Hirachi's invariant. In particular, we obtain a positive answer to Fefferman's question.

The note is organized as follows. First, we recall some background about the main definitions and examples concerning Heisenberg and CR manifolds and the Heisenberg calculus of Beals-Greiner [BG] and Taylor [Ta] (Section 1), the noncommutative residue trace for the Heisenberg calculus of [Po1] and [Po5] (Section 2) and the invariant of Hirachi [Hi] (Section 3). Then, in Section 4 we present the construction of the new CR invariants.

We refer to [Po4] for complete proofs and for the extensions of the results to the contact setting (see also [Bo2]).

1. HEISENBERG CALCULUS

In this section we recall basic facts about Heisenberg and CR manifolds and on the Heisenberg calculus.

1.1. **Heisenberg and CR manifolds.** A Heisenberg manifold is a pair (M, H) consisting of a manifold M together with a distinguished hyperplane bundle $H \subset TM$. In addition, given another Heisenberg manifold (M', H') we say that a diffeomorphism $\phi : M \to M'$ is a Heisenberg diffeomorphism when $\phi_* H = H'$.

The terminology Heisenberg manifold stems from the fact that the relevant tangent structure in this setting is that of a bundle GM of graded nilpotent Lie groups (see [BG], [Be], [EM], [FS], [Gr], [Po2], [Ro]).

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Among the main examples of Heisenberg manifolds we have the following.

a) Heisenberg group. The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^{2n+1}$ is $\mathbb{R}^{2n+1} = \mathbb{R} \times \mathbb{R}^{2n}$ equipped with the group law,

$$x.y = (x_0 + y_0 + \sum_{1 \leq j \leq n} (x_{n+j}y_j - x_j y_{n+j}), x_1 + y_1, \ldots, x_{2n} + y_{2n}). \tag{1.1}$$

A left-invariant basis for its Lie algebra $\mathfrak{h}^{2n+1}$ is provided by the vector-fields,

$$X_0 = \frac{\partial}{\partial x_0}, \quad X_j = \frac{\partial}{\partial x_j} + x_{n+j}\frac{\partial}{\partial x_0}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - x_j \frac{\partial}{\partial x_0}, \tag{1.2}$$

with $j = 1, \ldots, n$. For $j, k = 1, \ldots, n$ and $k \neq j$ we have the relations,

$$[X_j, X_{n+k}] = -2\delta_{jk}X_0, \qquad [X_0, X_j] = [X_j, X_k] = [X_{n+j}, X_{n+k}] = 0. \tag{1.3}$$

In particular, the subbundle spanned by the vector fields $X_1, \ldots, X_{2n}$ defines a left-invariant Heisenberg structure on $\mathbb{H}^{2n+1}$.

b) CR manifolds. If $D \subset \mathbb{C}^{n+1}$ a bounded domain with boundary ∂D then the maximal complex structure, or CR structure, of $T(\partial D)$ is given by $T_{1,0} = T(\partial D) \cap T_{1,0}\mathbb{C}^{n+1}$, where $T_{1,0}$ denotes the holomorphic tangent bundle of $\mathbb{C}^{n+1}$. More generally, a CR structure on an orientable manifold M^{2n+1} is given by a complex rank n integrable subbundle $T_{1,0} \subset T_{\mathbb{C}}M$ such that $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$. Besides on boundaries of complex domains, and more generally such structures naturally appear on real hypersurfaces in $\mathbb{C}^{n+1}$, quotients of the Heisenberg group $\mathbb{H}^{2n+1}$ by cocompact lattices, boundaries of complex hyperbolic spaces, and circle bundles over complex manifolds.

A real hypersurface $M = \{r = 0\} \subset \mathbb{C}^{n+1}$ is strictly pseudoconvex when the Hessian $\partial\overline{\partial}r$ is positive definite. In general, to a CR manifold M we can associate a Levi form $L_\theta(Z, W) = -id\theta(Z, \overline{W})$ on the CR tangent bundle $T_{1,0}$ by picking a non-vanishing real 1-form θ annihilating $T_{1,0} \oplus T_{0,1}$. We then say that M is strictly pseudoconvex (resp. κ-strictly pseudoconvex) when we can choose θ so that L_θ is positive definite (resp. is nondegenerate with κ negative eigenvalues) at every point.

In addition, important examples of Heisenberg manifolds include contact manifolds, (codimension 1) foliations and the confoliations of Elyashberg-Thurston [ET].

1.2. **Heisenberg calculus.** The Heisenberg calculus is the relevant pseudodifferential calculus to study hypoelliptic operators on Heisenberg manifolds. It was independently introduced by Beals-Greiner [BG] and Taylor [Ta] (see also [Bo1], [Dy1], [Dy2], [EM], [FS], [RS]).

The initial idea in the Heisenberg calculus, which is due to Stein, is to construct a class of operators on a Heisenberg manifold (M^{d+1}, H), called Ψ_HDO's, which at each point $a \in M$ are modeled on homogeneous left-invariant convolution operators on the tangent group G_aM.

Locally the ΨDO's can be described as follows. Let $U \subset \mathbb{R}^{d+1}$ be a local chart together with a frame $X_0, \ldots, X_d$ of TU such that $X_1, \ldots, X_d$ span H. Such a chart is called a Heisenberg chart. Moreover, on $\mathbb{R}^{d+1}$ we consider the dilations,

$$t.\xi = (t^2\xi_0, t\xi_1, \ldots, t\xi), \qquad \xi \in \mathbb{R}^{d+1}, \quad t > 0. \tag{1.4}$$

Definition 1.1. *1) $S_m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of functions $p(x, \xi)$ in $C^\infty(U \times \mathbb{R}^{d+1} \backslash 0)$ such that $p(x, t.\xi) = t^m p(x, \xi)$ for any $t > 0$.*

2) $S^m(U\times\mathbb{R}^{d+1})$, $m\in\mathbb{C}$, *consists of functions* $p\in C^\infty(U\times\mathbb{R}^{d+1})$ *with an asymptotic expansion* $p\sim\sum_{j\geq 0}p_{m-j}$, $p_k\in S_k(U\times\mathbb{R}^{d+1})$, *in the sense that, for any integer* N *and for any compact* $K\subset U$, *we have*

$$|\partial_x^\alpha\partial_\xi^\beta(p-\sum_{j<N}p_{m-j})(x,\xi)|\leq C_{\alpha\beta NK}\|\xi\|^{\Re m-\langle\beta\rangle-N},\qquad x\in K,\quad \|\xi\|\geq 1, \tag{1.5}$$

where we have let $\langle\beta\rangle=2\beta_0+\beta_1+\ldots+\beta_d$ *and* $\|\xi\|=(\xi_0^2+\xi_1^4+\ldots+\xi_d^4)^{1/4}$.

Next, for $j=0,\ldots,d$ let $\sigma_j(x,\xi)$ denote the symbol (in the classical sense) of the vector field $\frac{1}{i}X_j$ and set $\sigma=(\sigma_0,\ldots,\sigma_d)$. Then for $p\in S^m(U\times\mathbb{R}^{d+1})$ we let $p(x,-iX)$ be the continuous linear operator from $C_c^\infty(U)$ to $C^\infty(U)$ such that

$$p(x,-iX)f(x)=(2\pi)^{-(d+1)}\int e^{ix.\xi}p(x,\sigma(x,\xi))\hat{f}(\xi)d\xi,\qquad f\in C_c^\infty(U). \tag{1.6}$$

Definition 1.2. $\Psi_H^m(U)$, $m\in\mathbb{C}$, *consists of operators* $P:C_c^\infty(U)\rightarrow C^\infty(U)$ *which are of the form* $P=p(x,-iX)+R$ *for some* p *in* $S^m(U\times\mathbb{R}^{d+1})$, *called the symbol of* P, *and some smoothing operator* R.

For any $a\in U$ there is exists a unique affine change of variable $\psi_a:\mathbb{R}^{d+1}\rightarrow\mathbb{R}^{d+1}$ such that $\psi_a(a)=0$ and $(\psi_a)_*X_j=\frac{\partial}{\partial x_j}$ at $x=0$ for $j=0,1,\ldots,d+1$. Then, a continuous operator $P:C_c^\infty(U)\rightarrow C^\infty(U)$ is a Ψ_HDO of order m if, and only if, its kernel $k_P(x,y)$ has a behavior near the diagonal of the form,

$$k_P(x,y)\sim\sum_{j\geq-(m+d+2)}(a_j(x,\psi_x(y))-\sum_{\langle\alpha\rangle=j}c_\alpha(x)\psi_x(x)^\alpha\log\|\psi_x(y)\|), \tag{1.7}$$

with $c_\alpha\in C^\infty(U)$ and $a_j(x,y)\in C^\infty(U\times(\mathbb{R}^{d+1}\backslash 0))$ such that $a_j(x,\lambda.y)=\lambda^j a_j(x,y)$ for any $\lambda>0$. Moreover, $a_j(x,y)$ and $c_\alpha(x)$, $\langle\alpha\rangle=j$, depend only on the symbol of P of degree $-(j+d+2)$.

The class of Ψ_HDO's is invariant under changes of Heisenberg chart (see [BG, Sect. 16], [Po3, Appendix A]), so we may extend the definition of Ψ_HDO's to an arbitrary Heisenberg manifold (M,H) and let them act on sections of a vector bundle $\mathcal{E}$ over M. We let $\Psi_H^m(M,\mathcal{E})$ denote the class of Ψ_HDO's of order m on M acting on sections of $\mathcal{E}$.

2. Noncommutative residue

Let (M^{d+1},H) be a Heisenberg manifold equipped with a smooth positive density and let $\mathcal{E}$ be a Hermitian vector bundle over M. We let $\Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ denote the space of Ψ_HDO of integer order acting on sections of $\mathcal{E}$.

2.1. **Logarithmic singularity.** Let $P:C^\infty(M,\mathcal{E})\rightarrow C^\infty(M,\mathcal{E})$ be a Ψ_HDO of integer order m. Then it follows from (1.7) that in a trivializing Heisenberg chart the kernel $k_P(x,y)$ of P has a behavior near the diagonal of the form,

$$k_P(x,y)=\sum_{-(m+d+2)\leq j\leq 1}a_j(x,-\psi_x(y))-c_P(x)\log\|\psi_x(y)\|+\mathrm{O}(1), \tag{2.1}$$

where $a_j(x,y)$ is homogeneous of degree j in y with respect to the dilations (1.4). Furthermore, we have

$$c_P(x)=|\psi_x'|\int_{\|\xi\|=1}p_{-(d+2)}(x,\xi)d\xi, \tag{2.2}$$

where $p_{-(d+2)}(x,\xi)$ is the homogeneous symbol of degree $-(d+2)$ of P.

Let $|\Lambda|(M)$ be the bundle of densities on M. Then we have:

Proposition 2.1 ([Po1], [Po5]). *The coefficient $c_P(x)$ makes sense intrinsically on M as a section of $|\Lambda|(M)\otimes \operatorname{End}\mathcal{E}$.*

2.2. **Noncommutative residue.** From now on we assume M compact. Therefore, for any $P\in\Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ we can let

$$\operatorname{Res} P=\int_M \operatorname{tr}_{\mathcal{E}} c_P(x). \tag{2.3}$$

If P is in $\Psi_H^m(M,\mathcal{E})$ with $\Re m<-(d+2)$ then P is trace-class. It can be shown that we have an analytic continuation of the trace to Ψ_HDO's of non-integer orders which is analogous to that for classical ΨDO's in [KV]. Moreover, on Ψ_HDO's of integer orders this analytic extension of the trace induces a residual functional agreeing with (2.3), so that we have:

Proposition 2.2. *Let $P\in\Psi_H^{\mathbb{Z}}(M,\mathcal{E})$. Then for any family $(P(z))_{z\in\mathbb{C}}\subset\Psi_H^*(M,\mathcal{E})$ which is holomorphic in the sense of* [Po3] *and such that $P(0)=P$ and $\operatorname{ord}P(z)=z+\operatorname{ord}P$ we have*

$$\operatorname{Res} P=-\operatorname{res}_{z=0}\operatorname{Trace} P(z). \tag{2.4}$$

Thus the functional (2.3) is the analogue for the Heisenberg calculus of the noncommutative residue of Wodzicki ([Wo1], [Wo2]) and Guillemin [Gu]. Furthermore, we have:

Proposition 2.3 ([Po1], [Po5]). *1) Let ϕ be a Heisenberg diffeomorphism from (M,H) onto a Heisenberg manifold (M',H'). Then for any $P\in\Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ we have $\operatorname{Res}\phi_*P=\operatorname{Res}P$.*

2) Res is a trace on the algebra $\Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ which vanishes on differential operators and on Ψ_HDO's of integer order $\leq -(d+3)$.

3) If M is connected then Res is the unique trace up to constant multiple.

3. Hirachi's invariant

Let $D\subset\mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with boundary ∂D. Let θ be a pseudohermitian contact form on ∂D, i.e., if near a point of ∂D we let $\rho(z,\overline{z})$ be a local defining function for D with $\partial\overline{\partial}\rho>0$ then θ agrees up to a conformal factor with $i(\partial-\overline{\partial})\rho$.

We endow ∂D with the Levi metric defined by the Levi form associated to θ and we let $S_\theta: L^2(\partial D)\rightarrow L^2(\partial D)$ be the Szegö projection associated to this metric, i.e., the orthogonal projection onto the L^2-closure of boundary values of holomorphic functions on D. For instance, when $\partial D=S^1$ we recover the Cauchy formula,

$$Su(z)=\frac{1}{2i\pi}\int_{|w|=1}\frac{u(w)}{w-z}dw. \tag{3.1}$$

Let $k_{S_\theta}(z,\overline{w})d\theta^n\wedge\theta$ be the Schwartz kernel of S_θ. As shown by Fefferman [Fe1] and Boutet de Monvel-Sjöstrand [BS] near the diagonal $w=z$ we can write

$$k_{S_\theta}(z,\overline{w})=\varphi_\theta(z,\overline{w})\rho(z,\overline{w})^{-(n+1)}+\psi_\theta(z,\overline{w})\log\rho(z,\overline{w}), \tag{3.2}$$

where $\varphi_\theta(z,\overline{w})$ and $\psi_\theta(z,\overline{w})$ are smooth functions. Then Hirachi defined

$$L(S_\theta):=\int_M\psi_\theta(z,\overline{z})d\theta^n\wedge\theta. \tag{3.3}$$

Theorem 3.1 (Hirachi [Hi]). *1) $L(S_\theta)$ is a CR invariant, i.e., it does not depend on the choice of θ. In particular, this is a biholomorphic invariant of D.*

2) $L(S_\theta)$ is invariant under smooth deformations of the domain D.

It has been asked by Fefferman whether there would exist other invariants like $L(S_\theta)$, i.e., invariants arising from the integrals of the log singularities of geometric operators. A first observation is that S_θ is a Ψ_HDO of order 0 and with the notation of the previous sections we have $c_{S_\theta}(z) = -\frac{1}{2}\psi_\theta(z,\overline{z})d\theta^n \wedge \theta$. Thus,

$$\operatorname{Res} S_\theta = -\frac{1}{2}L(S_\theta). \tag{3.4}$$

Therefore, answering Fefferman's question boils down to construct geometric Ψ_HDO's on CR manifolds whose noncommutative residues give rise to CR invariants.

4. New CR invariants

Let M^{2n+1} be a compact orientable CR manifold with CR tangent bundle $T_{1,0} \subset T_{\mathbb{C}}M$, so that $H = \Re(T_{1,0} \oplus T_{0,1}) \subset TM$ is a hyperplane bundle of TM admitting an (integrable) complex structure. Let θ be a global non-zero real 1-form annihilating H and let L_θ be the associated Levi form,

$$L_\theta(Z,W) = -id\theta(Z,\overline{W}) = i\theta([Z,\overline{W}]), \qquad Z,W \in C^\infty(M,T_{1,0}). \tag{4.1}$$

Let $\mathcal{N}$ be a supplement of H in TM. This is an orientable line bundle which gives rise to the splitting,

$$T_{\mathbb{C}}M = T_{1,0} \oplus T_{0,1} \oplus (\mathcal{N} \otimes \mathbb{C}). \tag{4.2}$$

Let $\Lambda^{1,0}$ and $\Lambda^{0,1}$ denote the annihilators in $T^*_{\mathbb{C}}M$ of $T_{0,1}\oplus(\mathcal{N}\otimes\mathbb{C})$ and $T_{1,0}\oplus(\mathcal{N}\otimes\mathbb{C})$ respectively and for $p,q = 0,\ldots,n$ let $\Lambda^{p,q} = (\Lambda^{1,0})^p \wedge (\Lambda^{0,1})^q$ be the bundle of (p,q)-forms. Then we have the splitting,

$$\Lambda^* T^*_{\mathbb{C}}M = (\bigoplus_{p,q=0}^{n} \Lambda^{p,q}) \oplus \theta \wedge \Lambda^* T^*_{\mathbb{C}}M. \tag{4.3}$$

Notice that this decomposition does not depend on the choice of θ, but it does depend on that of $\mathcal{N}$.

The complex $\overline{\partial}_b : C^\infty(M,\Lambda^{0,*}) \to C^\infty(M,\Lambda^{0,*+1})$ of Kohn-Rossi ([KR], [Ko]) is defined as follows. For any $\eta \in C^\infty(M,\Lambda^{0,q})$ we can uniquely decompose $d\eta$ as

$$d\eta = \overline{\partial}_{b,q}\eta + \partial_{b,q}\eta + \theta \wedge \mathcal{L}_{X_0}\eta, \tag{4.4}$$

where $\overline{\partial}_{b,q}\eta$ and $\partial_{b,q}\eta$ are sections of $\Lambda^{0,q+1}$ and $\Lambda^{1,q}$ respectively and X_0 is the section of $\mathcal{N}$ such that $\theta(X_0) = 1$. Thanks to the integrability of $T_{1,0}$ we have $\overline{\partial}_{b,q+1} \circ \overline{\partial}_{b,q} = 0$, so that we get a chain complex. Notice that this complex depends on the CR structure of M and on the choice of $\mathcal{N}$.

Assume now that M is endowed with a Hermitian metric h on $T_{\mathbb{C}}M$ which commutes with complex conjugation and makes the splitting (4.2) become orthogonal. The associated Kohn Laplacian is

$$\Box_{b,q} = \overline{\partial}^*_{b,q+1}\overline{\partial}_{b,q} + \overline{\partial}_{b,q-1}\overline{\partial}^*_{b,q}. \tag{4.5}$$

For $x \in M$ let $\kappa_+(x)$ (resp. $\kappa_-(x)$) be the number of positive (resp. negative) eigenvalues of L_θ at x. We then say that the condition $Y(q)$ holds when at every point $x \in M$ we have

$$q \notin \{\kappa_-(x),\ldots,n-\kappa_+(x)\} \cup \{\kappa_+(x),\ldots,n-\kappa_-(x)\}. \tag{4.6}$$

For instance, when M is κ-strictly pseudoconvex we have $\kappa_-(x) = n - \kappa_+(x) = \kappa$, so the condition $Y(q)$ exactly means that we must have $q \neq \kappa$ and $q \neq n - \kappa$.

Proposition 4.1 (see [BG, Sect. 21], [Po3, Sect. 3.5])**.** *The Kohn Laplacian $\Box_{b,q}$ admits a parametrix in $\Psi_H^{-2}(M, \Lambda^{0,q})$ iff the condition $Y(q)$ is satisfied.*

Let $S_{b,q}$ be the Szegö projection on $(0,q)$-forms, i.e., the orthogonal projection onto $\ker \Box_{b,q}$. We also consider the orthogonal projections $\Pi_0(\overline{\partial}_{b,q})$ and $\Pi_0(\overline{\partial}_{b,q}^*)$ onto $\ker \overline{\partial}_{b,q}$ and $\ker \overline{\partial}_{b,q}^* = (\operatorname{im} \overline{\partial}_{b,q-1})^\perp$. In fact, as $\ker \overline{\partial}_{b,q} = \ker \Box_{b,q} \oplus \operatorname{im} \overline{\partial}_{b,q-1}$ we have $\Pi_0(\overline{\partial}_{b,q}) = S_b + 1 - \Pi_0(\overline{\partial}_{b,q}^*)$, that is,

$$S_{b,q} = \Pi_0(\overline{\partial}_{b,q}) + \Pi_0(\overline{\partial}_{b,q}^*) - 1. \tag{4.7}$$

Let $N_{b,q}$ be the partial inverse of $\Box_{b,q}$, so that $N_{b,q}\Box_{b,q} = \Box_{b,q}N_{b,q} = 1 - S_{b,q}$. Then it can be shown (see, e.g., [BG, pp. 170–172]) that we have

$$\Pi_0(\overline{\partial}_{b,q}) = 1 - \overline{\partial}_{b,q}^* N_{b,q+1} \overline{\partial}_{b,q}, \qquad \Pi_0(\overline{\partial}_{b,q}^*) = 1 - \overline{\partial}_{b,q-1} N_{b,q-1} \overline{\partial}_{b,q-1}^*. \tag{4.8}$$

By Proposition 4.1 when the condition $Y(q)$ holds at every point the operator $\Box_{b,q}$ admits a parametrix in $\Psi_H^{-2}(M, \Lambda^{0,q})$ and then $S_{b,q}$ is a smoothing operator and $N_{b,q}$ is a Ψ_HDO of order -2. Therefore, using (4.8) we see that if the condition $Y(q+1)$ (resp. $Y(q-1)$) holds everywhere then $\Pi_0(\overline{\partial}_{b,q})$ (resp. $\Pi_0(\overline{\partial}_{b,q}^*)$) is a Ψ_HDO.

Furthermore, in view of (4.7) we also see that if at every point the condition $Y(q)$ fails, but the conditions $Y(q-1)$ and $Y(q+1)$ hold, then the Szegö projection $S_{b,q}$ is a zero'th order Ψ_HDO projection. Notice that this may happen if, and only if, M is κ-strictly pseudoconvex with $\kappa = q$ or $\kappa = n - q$.

Bearing all this in mind we have:

Theorem 4.2 ([Po4])**.** *1) The following noncommutative residues are CR diffeomorphism invariants of M:*

(i) $\operatorname{Res} \Pi_0(\overline{\partial}_{b,q})$ *when the condition $Y(q+1)$ holds everywhere;*

(ii) $\operatorname{Res} \Pi_0(\overline{\partial}_{b,q}^*)$ *when the condition $Y(q-1)$ holds everywhere;*

(iii) $\operatorname{Res} S_{b,\kappa}$ *and* $\operatorname{Res} S_{b,n-\kappa}$ *when M is κ-strictly pseudoconvex.*

In particular, they depend neither on the choice of the line bundle $\mathcal{N}$, nor on that of the Hermitian metric h.

2) The noncommutative residues (i)–(iii) are invariant under deformations of the CR structure coming from deformations of the complex structure of H.

Specializing Theorem 4.2 to the strictly pseudoconvex case we get:

Theorem 4.3 ([Po4])**.** *Suppose that M is a compact strictly pseudoconvex CR manifold. Then:*

1) $\operatorname{Res} S_{b,k}$, $k = 0, n$, *and* $\operatorname{Res} \Pi_0(\overline{\partial}_{b,q})$, $q = 1, \ldots, n-1$, *are CR diffeomorphism invariants of M. In particular, when M is the boundary of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ they give rise to bilholomorphism invariants of D.*

2) The above residues are invariant under deformations of the CR structure.

This provides us with an affirmative answer to Fefferman's question.

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