A NEW SHORT PROOF OF THE LOCAL INDEX FORMULA OF ATIYAH-SINGER

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Abstract. In this talk we present a new short proof of the local index formula of Atiyah-Singer for Dirac operators ([AS1], [AS2]) which, as a byproduct and unlike Getzler's short proof, allows us to compute the CM cyclic cocycle for Dirac spectral triples.

Introduction

The aim of the talk is to present the results of [Po] where a new short proof of the local index formula of Atiyah-Singer for Dirac operators ([AS1], [AS2]) was given. The point is that this proof is as simple as Getzler's short proof [Ge1] but, as a byproduct and unlike Getzler's proof, it allows us to compute the CM cocycle for Dirac spectral triples. This is interesting because:

- The probabilistic arguments in (part of) Getzler's short proof don't go through to compute the CM cocycle;
- The previous computations of the CM cocycle (Connes-Moscovici [CM], Chern-Hu [CH], Lescure [Le]) made use of the asymptotic ΨDO calculus used by Getzler in its first proof [Ge2], which we can bypass here.

The main ingredients of the proof are:

- Getzler's rescaling as in [Ge2];
- Greiner's approach of the heat kernel asymptotics [Gr].

The talk is divided into four parts. In the first one we recall Greiner's approach of the heat kernel asymptotics and in the second one we prove the Atiyah-Singer index formula. The third part reviews the framework for the local index formula in noncommutative geometry following [CM]. Then in the last part we compute the CM cocycle for Dirac spectral triples.

1. Greiner's approach of the heat kernel asymptotics

Here we let M^n be a compact Riemannian manifold, let $\mathcal{E}$ be a Hermitian bundle over M, and consider a second order selfadjoint elliptic differential operator $\Delta : C^\infty(M,\mathcal{E}) \to C^\infty(M,\mathcal{E})$ such that the principal symbol of Δ is positive definite. This has the effect that Δ is bounded from below and generates a bounded heat semi-group $e^{-t\Delta}$, $t \geq 0$, on $L^2(M,\mathcal{E})$. In fact, for $t > 0$ the operator $e^{-t\Delta}$ is smoothing, i.e. has a smooth distribution kernel.

On the other hand, the heat semigroup $e^{-t\Delta}$, $t \geq 0$, allows us to invert the heat equation, since if we consider the operator $Q_0 : C_c^\infty(M\times\mathbb{R}) \to \mathcal{D}'(M\times\mathbb{R})$ given by

$$Q_0u(x,t) = \int_0^\infty e^{-s\Delta}u(x,t-s)ds, \qquad u \in C_c^\infty(M\times\mathbb{R},\mathcal{E}), \tag{1.1}$$

then, for $u \in C_c^\infty(M\times\mathbb{R},\mathcal{E})$ we have

$$(\Delta+\partial_t)Q_0u = Q_0(\Delta+\partial_t)u = u \quad \text{in } \mathcal{D}'(M\times\mathbb{R},\mathcal{E}). \tag{1.2}$$

Observe also that at the level of distribution kernels (1.1) implies that:

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1. Q_0 has the *Volterra property*, i.e. it has a distribution kernel of the form $K_{Q_0}(x, y, t-s)$ with $K_{Q_0}(x, y, t) = 0$ for $t < 0$.

2. For $t > 0$ we have

$$K_{Q_0}(x, y, t) = k_t(x, y). \tag{1.3}$$

The equality (1.3) leads us to use ΨDO techniques for looking at the asymptotics of $k_t(x, x)$ as $t \to 0^+$. For achieving this aim the relevant ΨDO calculus is the Volterra calculus developed independently by Greiner [Gr] and Piriou [Pi].

In the sequel we let U be an open subset of $\mathbb{R}^n$ and let $\mathbb{C}_-$ be the complex halfplane $\{\Im\tau > 0\}$ with closure $\bar{\mathbb{C}}_-$. Then the Volterra symbols can be defined as follows.

Definition 1.1. *$S_{\mathrm{v},m}(U\times\mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of smooth functions $q_m(x, \xi, \tau)$ on $U_x \times (\mathbb{R}^{n+1}_{(\xi,\tau)} \setminus 0)$ so that $q_m(x, \xi, \tau)$ can be extended to a smooth function on $U_x \times [(\mathbb{R}^n_\xi \times \bar{\mathbb{C}}_-) \setminus 0]$ in such way to be analytic with respect to $\tau \in \mathbb{C}_-$ and to be homogeneous of degree m, i.e. $q_m(x, \lambda\xi, \lambda^2\tau) = \lambda^m q_m(x, \xi, \tau)$ for any $\lambda \in \mathbb{R} \setminus 0$.*

In fact, Definition 1.1 is intimately related to the Volterra property, for we have:

Lemma 1.2 ([BGS, Prop. 1.9])**.** *Any homogeneous Volterra symbol $q(x, \xi, \tau) \in S_{\mathrm{v},m}(U\times\mathbb{R}^{n+1})$ can be extended into a unique distribution $g(x, \xi, \tau) \in C^\infty(U)\hat{\otimes}\mathcal{S}'(\mathbb{R}^{n+1})$ in such way to be homogeneous with respect to the covariables (ξ, τ) and so that $\check{q}(x, y, t) := \mathcal{F}^{-1}_{(\xi,\tau)\to(y,t)}[g](x, y, t) = 0$ for $t < 0$.*

Definition 1.3. *$S^m_{\mathrm{v}}(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of the smooth functions $q(x, \xi, \tau)$ on $U_x \times \mathbb{R}^{n+1}_{(\xi,\tau)}$ that have an asymptotic expansion $q \sim \sum_{j\geq 0} q_{m-j}$, $q_{m-j} \in S_{\mathrm{v},m-j}(U\times\mathbb{R}^{n+1})$, where $\sim$ means that, for any integer N and for any compact $K \subset U$, there exists a constant $C_{NK\alpha\beta k} > 0$ such that*

$$|\partial^\alpha_x \partial^\beta_\xi \partial^k_\tau (q - \sum_{j<N} q_{m-j})(x, \xi, \tau)| \leq C_{NK\alpha\beta k}(|\xi| + |\tau|^{1/2})^{m-|\beta|-wk-N}, \tag{1.4}$$

for $x \in K$ and $|\xi| + |\tau|^{1/2} \geq 1$.

Definition 1.4. *A Volterra ΨDO of order m is a continuous operator Q from $C^\infty_c(M \times \mathbb{R}, \mathcal{E})$ to $C^\infty(M \times \mathbb{R}, \mathcal{E})$ such that:*

- Q has the Volterra property, i.e. has a distribution kernel of the form $K_Q(x, y, t-s)$ with $K_Q(x, y, t) = 0$ for $t < 0$;

- The distribution kernel of Q is smooth off the diagonal;

- Locally we can write $Q = q(x, D_x, D_t) + R$ for some Volterra symbol q of order m and some smoothing operator R.

Example 1.5. The heat operator $\Delta + \partial_t$ is a Volterra ΨDO of order 2 and its principal symbol is $p_2(x, \xi) + i\tau$ (where p_2 denotes the principal symbol of Δ).

Example 1.6. Let q_l be a homogeneous Volterra symbol on $U \times \mathbb{R}^n \times \mathbb{R}$ and define $q_l(x, D_x, D_t)$ to be the operator with kernel $\check{q}_l(x, x-y, t-s)$. Then $q_l(x, D_x, D_t)$ is a Volterra ΨDO of order l.

Using the Volterra ΨDO calculus we can prove:

Proposition 1.7 ([Gr], [Pi])**.** *The heat operator $\Delta + \partial_t$ is invertible on smooth sections and its inverse is a Volterra ΨDO of order -2.*

On the other hand, we have the following.

Lemma 1.8 ([Gr])**.** *Let $Q \in$ be a Volterra ΨDO of order m on $U \times \mathbb{R}$. Then, with asymptotics in $C^\infty(U)$, we have*

$$K_Q(x, x, t) \sim_{t\to 0^+} t^{-(\frac{n}{2}+[\frac{m}{2}]+1)} \sum t^l \check{q}_{2[\frac{m}{2}]-2l}(x, 0, 1), \tag{1.5}$$

where $q \sim \sum_{j\geq 0} q_{m-j}$ *denotes the symbol of* Q.

Applying this to $Q=(\Delta+\partial_t)^{-1}$ and using (1.3) we get:

Proposition 1.9 ([Gr]). *In* $C^\infty(M,|\Lambda|(M)\otimes \operatorname{End}\mathcal{E})$ *we have:*

$$k_t(x,x) \sim_{t\to 0^+} t^{\frac{-n}{2}} \sum t^j a_l(\Delta)(x), \qquad a_l(\Delta)(x)=\check{q}_{-2-2l}(x,0,1), \tag{1.6}$$

where $q \sim \sum j \geq 0 q_{-2-j}$ *denotes the symbol of any Volterra* ΨDO *parametrix for* $\Delta+\partial_t$.

The interest of this approach is twofold. First, to determine the asymptotics of $k_t(x,x)$ at a point x_0 we only need a Volterra parametrix Q for $\Delta+\partial_t$ near x_0, for we have:

$$k_t(x_0,x_0)=K_Q(x_0,x_0,t)+\mathrm{O}(t^\infty). \tag{1.7}$$

Second, we can differentiate the heat kernel asymptotics as follows. Let P be a differential operator of order m and let $h_t(x,y)$ denote the kernel of $Pe^{-t\Delta}$, $t>0$. Then:

$$h_t=P_xk_t=P_xK_{(\Delta+\partial_t)^{-1}}=K_{P(\Delta+\partial_t)^{-1}}. \tag{1.8}$$

Therefore by applying Lemma 1.8 to the operator $P(\Delta+\partial_t)^{-1}$ we get a differentiable version of Proposition 1.9 as follows.

Proposition 1.10. *With asymptotics in* $C^\infty(M,|\Lambda|(M))$ *we have*

$$h_t(x,x) \sim_{t\to 0^+} t^{-(\frac{n}{2}+[\frac{m}{2}])} \sum_{l\geq 0} t^l b_l(P,\Delta)(x), \qquad b_l(P,\Delta)(x)=\check{q}_{2[\frac{m}{2}]-2-2l}(x,0,1), \tag{1.9}$$

where $q\sim \sum_{j\geq 0} q_{m-2-j}$ *denotes the symbol of* $P(\Delta+\partial_t)^{-1}$.

2. Proof of the Atiyah-Singer index formula

Let M^n be a compact Riemannian spin manifold with spin bundle $\mathcal{S}=\displaystyle{\not}S^+\oplus {\not}S^-$, let $\mathcal{E}$ be a Hermitian bundle equipped with a unitary connection and consider the Dirac operator with coefficients in $\mathcal{E}$,

$$\not{D}_{\mathcal{E}}: C^\infty(M,{\not}S\otimes\mathcal{E}) \to C^\infty(M,{\not}S\otimes\mathcal{E}). \tag{2.1}$$

Then the Atiyah-Singer index formula is the following.

Theorem 2.1 ([AS1], [AS2]). *We have:*

$$\operatorname{ind}\not{D}_{\mathcal{E}} = (2i\pi)^{-\frac{n}{2}} \int_M [\hat{A}(R_M)\wedge \operatorname{Ch}(F^{\mathcal{E}})]^{(n)}, \tag{2.2}$$

where $\hat{A}(R^M)=\det{}^{\frac{1}{2}}(\frac{R^M/2}{\sinh(R^M/2)})$ *is the* $\hat{A}$*-form of the Riemann curvature* R^M *and* $\operatorname{Ch}(F^{\mathcal{E}})=\operatorname{Tr}\exp(-F^{\mathcal{E}})$ *is the Chern form of the curvature* $F^{\mathcal{E}}$ *of* $\mathcal{E}$.

Let $k_t(x,y)$ denote the heat kernel of $\not{D}_{\mathcal{E}}^2$. Then by the McKean-Singer formula we have

$$\operatorname{ind}\not{D}_{\mathcal{E}} = \int \operatorname{Str} k_t(x,x) \qquad \text{for any } t>0, \tag{2.3}$$

where $\operatorname{Str}=\operatorname{Tr}_{{\not}S^+\otimes\mathcal{E}}-\operatorname{Tr}_{{\not}S^-\otimes\mathcal{E}}$ denotes the supertrace. Therefore, the index formula of Atiyah-Singer follows from:

Theorem 2.2. *As* $t\to 0^+$ *we have*

$$\operatorname{Str} k_t(x,x) = [\hat{A}(R_M)\wedge \operatorname{Ch}(F^{\mathcal{E}})]^{(n)}+\mathrm{O}(t), \tag{2.4}$$

with $\mathrm{O}(t)$ *in* $C^\infty(M,|\Lambda|(M))$.

This theorem, also called Local Index Theorem, was first proved by Patodi, Gilkey and Atiyah-Bott-Patodi ([ABP], [Gi]), and then in a purely analytic fashion by Getzler ([Ge1], [Ge2]) and Bismut [Bi] (see also [BGV], [Ro]). Moreover, as it is a purely local statement it holds *verbatim* for (geometric) Dirac operators acting on a Clifford bundle. Thus it allows us to recover, on the one hand, the Gauss-Bonnet, signature and Riemann-Roch theorems ([ABP], [BGV], [LM], [Ro]) and, on the other hand, the full index theorem of Atiyah-Singer ([ABP], [LM]).

Proof of Theorem 2.2. First, it is enough to prove Theorem 2.2 at a point x_0. Furthermore, to this end we only need a Volterra parametrix Q for $D\!\!\!/_{\mathcal{E}}^2 + \partial_t$ in local trivializing coordinates near x_0, for we have

$$k_t(x_0, x_0) = K_Q(x_0, x_0, t) + \mathrm{O}(t^\infty). \tag{2.5}$$

In fact, we can use normal coordinates centered at x_0 and trivializations of $S\!\!\!/$ and $\mathcal{E}$ by means of parallel translation along the geodesics out of the origin. This allows us to replace $D\!\!\!/_{\mathcal{E}}$ by a Dirac operator $D\!\!\!/$ on $\mathbb{R}^n$ acting on the trivial bundle with fiber $S\!\!\!/_n \otimes \mathbb{C}^p$, where $S\!\!\!/_n$ is the space of spinors on $\mathbb{R}^n$.

Now, recall that as algebra $\operatorname{End} S\!\!\!/_n$ can be identified with the Clifford algebra $\mathrm{Cl}(n)$. The latter is isomorphic to the exterior algebra $\Lambda(n) = \oplus_{i=0}^n \Lambda^i(n)$,

$$\Lambda(n) \xrightarrow{c} \mathrm{Cl}(n) \xrightarrow{\sigma = c^{-1}} \Lambda(n), \tag{2.6}$$

so that for $\xi, \eta \in \Lambda(n)$ we have

$$\sigma[c(\xi^{(i)})c(\eta^{(j)})] = \xi^{(i)} \wedge \eta^{(j)} \bmod \Lambda^{i+j-2}, \tag{2.7}$$

$$\zeta^{(l)} = \text{component of } \zeta \in \Lambda(n) \text{ in } \Lambda^l(n). \tag{2.8}$$

Here c is called the Clifford quantification map and σ the Clifford symbol map.

Lemma 2.3 (Getzler [Ge1])**.** *For any a in $\operatorname{End} S\!\!\!/_n$ we have:*

$$\operatorname{Str} a = (-2i)^{\frac{n}{2}} \sigma[a]^{(n)}. \tag{2.9}$$

From all this we see that as $t \to 0^+$ we have:

$$\operatorname{Str} k_t(0,0) = (-2i)^{\frac{n}{2}} \sigma[K_Q(0,0,t)]^{(n)} + \mathrm{O}(t^\infty), \tag{2.10}$$

where Q is any Volterra ΨDO parametrix for $D\!\!\!/^2 + \partial_t$.

Now, recall that the Getzler rescaling [Ge2] assigns the following degrees:

$$\deg \partial_j = \frac{1}{2} \deg \partial_t = \deg c(dx^j) = -\deg x^j = 1, \tag{2.11}$$

while $\deg B = 0$ for any $B \in M_p(\mathbb{C})$. It defines a filtration of Volterra ΨDO's with coefficients in $\operatorname{End}(S\!\!\!/_n \otimes \mathbb{C}^p) \simeq \mathrm{Cl}(\mathbb{R}^n) \otimes M_p(\mathbb{C})$ as follows.

Let $q \sim \sum_{l \leq m'} q_l$ be a Volterra symbol with values in $\operatorname{End} S\!\!\!/_n \otimes M_p(\mathbb{C})$. Then taking components in each subspace $\Lambda^{(j)}(n)$ and using Taylor expansions near $x = 0$ allows us to get formal expansions,

$$\sigma[q] \sim \sum_{l \leq m'} \sigma[q_l] \sim \sum_{l,j} \sigma[q_l]^{(j)} \sim \sum_{l,j,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial_x^\alpha q_l(0,\xi,\tau)]^{(j)}. \tag{2.12}$$

Observe that each symbol $\frac{x^\alpha}{\alpha!}\sigma[\partial_x^\alpha q_l(0,\xi,\tau)]^{(j)}$ is Getzler homogeneous of degree $-|\alpha|+l+j$. Therefore, we have

$$\sigma[q(x,\xi,\tau)] \sim \sum_{j \geq 0} q_{(m-j)}(x,\xi,\tau), \qquad q_{(m)} \neq 0, \tag{2.13}$$

where $q_{(m-j)}$ is a Getzler homogeneous symbol of degree $m-j$.

Definition 2.4. *Using (2.13) we set-up the following definitions:*
- *The integer m is the Getzler order of Q,*
- *The symbol $q_{(m)}$ is the principal Getzler homogeneous symbol of Q,*
- *The operator $Q_{(m)} = q_{(m)}(x, D_x, D_t)$ is the model operator of Q.*

Example 2.5. The operator $\not{D}^2$ has Getzler order 2 and its model operator is

$$\not{D}^2_{(2)} = H_R + F^{\mathcal{E}}(0), \qquad H_R = -\sum_{i=1}^{n}(\partial_i - \frac{1}{4}R^M_{ij}(0)x^j)^2, \tag{2.14}$$

$$R^M_{ij}(0) = \frac{1}{2}\langle R^M(\partial_i, \partial_j)\partial_k, \partial_l\rangle(0)dx^k \wedge dx^l, \qquad F^{\mathcal{E}}(0) = \frac{1}{2}F^{\mathcal{E}}(\partial_k, \partial_l)(0)dx^k \wedge dx^l. \tag{2.15}$$

This uses the behavior near $x = 0$ in normal coordinates of the metric and of the coefficients of the Levi-Civita connection in the synchronous frame (*cf.* [Ge2], [Po]).

The interest of the above definitions stems from the two results below.

Lemma 2.6 ([Po]). *For $j = 1, 2$ let Q_j be a Volterra ΨDO of Getzler order m_j. Then*

$$Q_1Q_2 = c(Q_{(m_1)}Q_{(m_2)}) + \mathrm{O}_G(m_2 + m_2 - 1), \tag{2.16}$$

where $\mathrm{O}_G(m_2 + m_2 - 1)$ has Getzler order $\leq m_2 + m_2 - 1$.

Lemma 2.7 ([Po]). *Let Q have Getzler order m and model operator $Q_{(m)}$.*
- *If $m - j$ is odd, then $\sigma[K_Q(0, 0, t)]^{(j)} = \mathrm{O}(t^{\frac{j-m-n-1}{2}})$.*
- *If $m - j$ is even, then $\sigma[K_Q(0, 0, t)]^{(j)} = t^{\frac{j-m-n}{2}-1}K_{Q_{(m)}}(0, 0, 1)^{(j)} + \mathrm{O}(t^{\frac{j-m-n}{2}})$.*

In particular, for $m = -2$ and $j = n$ we get:

$$\sigma[K_Q(0, 0, t)]^{(n)} = K_{Q_{(-2)}}(0, 0, 1)^{(n)} + \mathrm{O}(t). \tag{2.17}$$

In particular, using Lemma 2.6 we can easily get:

Lemma 2.8 ([Po]). *Let Q be a Volterra ΨDO parametrix for $\not{D}^2 + \partial_t$. Then:*

1) Q has Getzler order 2 and model operator $Q_{(-2)} = (H_R + F^{\mathcal{E}}(0) + \partial_t)^{-1}$.

2) $K_{Q_{(-2)}}(x, 0, t) = G_R(x, t) \wedge \exp(-tF^{\mathcal{E}}(0))$, where $G_R(x, t)$ is the fundamental solution of $H_R + \partial_t$, i.e. $(H_R + \partial_t)G_R(x, t) = \delta(x, t)$, where $\delta(x, t)$ is the Dirac distribution at the origin.

Noticing that H_R is a harmonic oscillator associated to the curvature at $x = 0$ we get:

Lemma 2.9 (Melher's Formula). *We have:*

$$G_R(x, t) = \chi(t)(4\pi t)^{-\frac{n}{2}}\hat{A}(tR^M(0)) \exp(-\frac{1}{4t}\langle\frac{tR^M(0)/2}{\tanh(tR^M(0)/2)}x, x\rangle), \tag{2.18}$$

where $\chi(t)$ is the characteristic function of the interval $(0, +\infty)$.

Now, from (2.10) and Lemma 2.7 we deduce

$$\operatorname{Str} k_t(0, 0) = (-2i)^{\frac{n}{2}}\sigma[K_Q(0, 0, t)]^{(n)} + \mathrm{O}(t^\infty) = (-2i)^{\frac{n}{2}}K_{Q_{(-2)}}(0, 0, 1)^{(n)} + \mathrm{O}(t). \tag{2.19}$$

Thus using Lemma 2.8 and Lemma 2.9 we get:

$$\begin{aligned}\operatorname{Str} k_t(0, 0) &= (-2i)^{\frac{n}{2}}[G_R(0, 1) \wedge \exp(-F^{\mathcal{E}}(0))]^{(n)} + \mathrm{O}(t),\\ &= (2i\pi)^{-\frac{n}{2}}[\hat{A}(R_M(0)) \wedge \mathrm{Ch}(F^{\mathcal{E}}(0))]^{(n)} + \mathrm{O}(t).\end{aligned} \tag{2.20}$$

Comparing this with the heat kernel asymptotics (1.6) for $\not{D}^2$ we get Theorem 2.2, and so complete the proof of the Atiyah-Singer index formula. □

The main new feature in the previous proof of the Atiyah-Singer index formula is the use of Lemma 2.7 which, by very elementary considerations on Getzler orders shows that the convergence of the supertrace of the heat kernel is a consequence of a general fact about Volterra ΨDO's. It incidentally gives a differentiable version of Theorem 2.2 as follows.

Proposition 2.10 ([Po]). *Let $\not{P}$ be a differential operator of Getzler order m and let $h_t(x,y)$ denote the distribution kernel of $\not{P}e^{-t\not{D}_M}$.*

- If m is odd, then $\operatorname{Str} h_t(x,x) = \mathrm{O}(t^{\frac{-m+1}{2}})$.

- If m is even, then $\operatorname{Str} h_t(x,x) = t^{\frac{-m}{2}} B_0(\not{D}_{\mathcal{E}}^2, \not{P})(x) + \mathrm{O}(t^{\frac{-m}{2}+1})$, where in synchronous normal coordinates centered at x_0 we have $B_0(\not{D}_{\mathcal{E}}^2, \not{P})(0) = (-2i\pi)^{\frac{n}{2}}[(\not{P}_{(m)}G_R)(0,1) \wedge \mathrm{Ch}(F^{\mathcal{E}}(0))]^{(n)}$, with $\not{P}_{(m)}$ denoting the model operator of P.

Proof. By Proposition 1.10 we have

$$h_t(x,x) = K_{\not{P}(\not{D}_{\mathcal{E}}^2+\partial_t)^{-1}}(x,x,t) \sim_{t\to 0^+} \sum_{j\geq 0} t^{-(\frac{n}{2}+[\frac{m'}{2}])} b_j(P,\Delta)(x), \tag{2.21}$$

with asymptotics in $C^\infty(M, |\Lambda| \otimes \operatorname{End} \not{S} \otimes \operatorname{End} \mathcal{E})$. On the other hand, using Lemma 2.6 we see that $\not{P}(\not{D}_{\mathcal{E}}^2 + \partial_t)^{-1}$ has Getzler order $m-2$. Combining all this with Lemma 2.7 gives the result. □

3. The local index formula in noncommutative geometry

Rather than at topological or geometrical levels the local index formula holds in full generality in a purely operator theoretic setting ([CM]; see also [Hi]). This uses two main tools, namely, spectral triples [CM] and cyclic cohomology [Co].

A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ where:
- $\mathcal{H}$ is a Hilbert space together with a $\mathbb{Z}_2$-grading $\gamma : \mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{H}_- \oplus \mathcal{H}_+$;
- $\mathcal{A}$ is an involutive unital algebra represented in $\mathcal{H}$ and commuting with the $\mathbb{Z}_2$-grading γ;
- D is a selfadjoint unbounded operator on $\mathcal{H}$ s.t. $[D,a]$ is bounded $\forall a \in \mathcal{A}$ and of the form,

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D_\pm : \mathcal{H}^\mp \to \mathcal{H}^\pm. \tag{3.1}$$

In addition we assume that $\mathcal{A}$ is *smooth* in the sense that $\mathcal{A}$ is contained in $\cap_{k\geq 0} \operatorname{dom} \delta^k$.

Recall that the datum of D above defines an index map $\operatorname{ind}_D : K_*(\mathcal{A}) \to \mathbb{Z}$ so that

$$\operatorname{ind}_D[e] = \operatorname{ind} eD^+e, \tag{3.2}$$

for any selfadjoint idempotent $e \in M_q(\mathcal{A})$.

Example 3.1 (Dirac spectral triple). A typical spectral triple is a triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$, where M is a compact Riemannian spin manifold of even dimension, $\not{S}$ is the spin bundle of M and $\not{D}_M : C^\infty(M,\not{S}) \to C^\infty(M,\not{S})$ is the Dirac operator on M. Moreover, under the Serre-Swan isomorphism $K_0(C^\infty(M)) \simeq K^0(M)$ for any Hermitian bundle over M we have:

$$\operatorname{ind}_{\not{D}_M}[\mathcal{E}] = \operatorname{ind} \not{D}_{\mathcal{E}}. \tag{3.3}$$

Next, the *cyclic cohomology groups* $HC^*(\mathcal{A})$ are obtained from the spaces,

$$C^k(\mathcal{A}) = \{(k+1)\text{-linear forms on } \mathcal{A}\}, \qquad k \in \mathbb{N}, \tag{3.4}$$

by restricting the Hochschild coboundary,

$$b\psi(a^0, \cdots, a^{k+1}) = (-1)^{k+1}\psi(a^{k+1}a^0, \cdots, a^k) + \sum (-1)^j \psi(a^0, \cdots, a^j a^{j+1}, \cdots, a^{k+1}), \tag{3.5}$$

to cyclic cochains, i.e. those such that

$$\psi(a^1, \cdots, a^k, a^0) = (-1)^k \psi(a^0, a^1, \cdots, a^k). \tag{3.6}$$

Equivalently, $HC^*(\mathcal{A})$ can be obtained as the second filtration of the (b,B)-bicomplex of (arbitrary) cochains, where the horizontal differential B is given by

$$B = AB_0 : C^{m+1}(\mathcal{A}) \to C^m(\mathcal{A}), \qquad (A\phi)(a^0,\cdots,a^m) = \sum(-1)^{mj}\psi(a^j,\cdots,a^{j-1}), \tag{3.7}$$

$$B_0\psi(a^0,\cdots,a^m) = \psi(1,a^0,\cdots,a^m). \tag{3.8}$$

The *periodic cyclic cohomology* is the inductive limit of the groups $HC^k(\mathcal{A})$ with respect to the the cup product with the generator of $HC^2(\mathbb{C})$. In terms of the (b,B)-bicomplex this is the cohomology of the short complex,

$$C^{\mathrm{ev}}(\mathcal{A}) \overset{b+B}{\leftrightarrows} C^{\mathrm{odd}}(\mathcal{A}), \qquad C^{\mathrm{ev/odd}}(\mathcal{A}) = \bigoplus_{k \text{ even/odd}} C^k(\mathcal{A}), \tag{3.9}$$

the cohomology groups of which are denoted $HC^{\mathrm{ev}}(\mathcal{A})$ and $HC^{\mathrm{odd}}(\mathcal{A})$.

We have a pairing between $HC^{\mathrm{ev}}(\mathcal{A})$ and $K_0(\mathcal{A})$ such that, for any cocycle $\varphi = (\varphi_{2k})$ in $C^{\mathrm{ev}}(\mathcal{A})$ and any selfadjoint idempotent e in $M_q(\mathcal{A})$, we have

$$\langle[\varphi],[e]\rangle = \sum_{k\geq 0}(-1)^k\frac{(2k)!}{k!}\varphi_{2k}\#\operatorname{Tr}(e,\cdots,e), \tag{3.10}$$

where $\varphi_{2k}\#\operatorname{Tr}$ is the$(2k+1)$-linear map on $M_q(\mathcal{A}) = M_q(\mathbb{C})\otimes\mathcal{A}$ given by

$$\varphi_{2k}\#\operatorname{Tr}(\mu^0\otimes a^0,\cdots,\mu^{2k}\otimes a^{2k}) = \operatorname{Tr}(\mu^0\ldots\mu^{2k})\varphi_{2k}(a^0,\cdots,a^{2k}), \tag{3.11}$$

for $\mu^j \in M_q(\mathbb{C})$ and $a^j \in \mathcal{A}$.

Example 3.2. Given a compact manifold M let $\mathcal{A}$ be the algebra $C^\infty(M)$ and let $\mathcal{D}_k(M)$ denote the space of k-dimensional de Rham currents. Then we have a morphism from $\mathcal{D}_{\mathrm{ev/odd}}(M)$ to $C^{\mathrm{ev/odd}}(\mathcal{A})$,

$$C = (C_k) \to \varphi_C = (\frac{1}{k!}\psi_{C_k}), \qquad \psi_{C_k}(f^0,f^1,\ldots,f^n) = \langle C_k, f^0df^1\wedge\ldots\wedge df^k\rangle. \tag{3.12}$$

such that $b\varphi_C = 0$ and $B\varphi_C = \varphi_{d^tC}$, where d^t denotes the de Rham boundary for currents. Therefore, we get a morphism from the even and odd de Rham's homology groups $H^{\mathrm{ev/odd}}(M)$ to $HC^{\mathrm{ev/odd}}(C^\infty(M))$. This even gives rise to an isomorphism if we restrict ourselves to continuous cyclic cochains.

Furthermore, we have:

$$\langle[\varphi_C],\mathcal{E}\rangle = \langle[C],\operatorname{Ch}^*\mathcal{E}\rangle \quad \forall\mathcal{E}\in K^0(M), \tag{3.13}$$

where $\operatorname{Ch}^* : K^0(M) \to H^{\mathrm{ev}}(M)$ is the Chern character in cohomology.

Now, we can compute the index map by pairing the K-theory a cyclic cocycle as follows. First, we say that the spectral triple $(\mathcal{A},\mathcal{H},D)$ is *p-summable* when we have

$$\mu_k(D^{-1}) = \mathrm{O}(k^{-1/p}) \quad \text{as } k\to+\infty, \tag{3.14}$$

where $\mu_k(D^{-1})$ denotes the $(k+1)$'th eigenvalue of D^{-1}.

Next, let $\Psi^0_D(\mathcal{A})$ be the algebra generated by the $\mathbb{Z}_2$-grading γ and the $\delta^k(a)$'s, $a\in\mathcal{A}$, where δ is the derivation $\delta(T) = [|D|,T]$.

Definition 3.3. *The* dimension spectrum of $(\mathcal{A},\mathcal{H},D)$ *is the union set of the singularities of all the zeta functions* $\zeta_b(z) = \operatorname{Tr} b|D|^{-z}$, $b\in\Psi^0_D(\mathcal{A})$.

Assuming p-summability and simple and discrete dimension spectrum we define a residual trace on $\Psi^0_D(\mathcal{A})$ by letting:

$$⨍ b = \operatorname{Res}_{z=0}\operatorname{Tr} b|D|^{-z} \qquad \text{for any } b\in\Psi^0_D(\mathcal{A}). \tag{3.15}$$

This trace is an algebraic analogue of the Wodzicki-Guillemin noncommutative residue trace ([Wo], [Gu]) for the algebra $\Psi^0_D(\mathcal{A})$. Moreover, it is local in the sense of noncommutative geometry since it vanishes on those elements of $\Psi^0_D(\mathcal{A})$ that are traceable.

Theorem 3.4 ([CM])**.** *Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is p-summable and has a discrete and simple dimension spectrum. Then:*

1) The following formulas define an even cocycle $\varphi_{\mathrm{CM}} = (\varphi_{2k})$ in the (b, B)-complex of $\mathcal{A}$.

- For $k = 0$, we let

$$\varphi_0(a^0) = \textit{finite part of } \operatorname{Tr} \gamma a^0 e^{-tD^2} \textit{ as } t \to 0^+, \tag{3.16}$$

- For $k \neq 0$, we let

$$\varphi_{2k}(a^0, \ldots, a^{2k}) = \sum_{\alpha} c_{k,\alpha} ⨍ \gamma P_{k,\alpha} |D|^{-2(|\alpha|+k)}, \tag{3.17}$$

$$P_{k,\alpha} = a^0 [D, a^1]^{[\alpha_1]} \ldots [D, a^{2k}]^{[\alpha_{2k}]}, \tag{3.18}$$

where $\Gamma(|\alpha| + k) c^{-1}_{k,\alpha} = 2(-1)^{|\alpha|} \alpha! (\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2k} + 2k)$ and the symbol $T^{[j]}$ denotes the j'th iterated commutator with D^2.

2) We have $\operatorname{ind}_D[\mathcal{E}] = \langle [\varphi_{\mathrm{CM}}], \mathcal{E} \rangle$ for any $\mathcal{E} \in K_0(\mathcal{A})$.

Remark 3.5. Here we have assumed that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ was $\mathbb{Z}_2$-graded. In the terminology of [CM] such a spectral triple is *even* and an ungraded spectral triple is said to be *odd*. In the latter case we can similarly compute the Fredholm index with coefficients in $K_1(\mathcal{A})$ by pairing $K_1(\mathcal{A})$ with an *odd* cyclic cocycle, also called CM cocycle, and which is given by formulas of the same type as (3.17)–(3.18) (see [CM] for a more detailed account).

4. Computation of the CM cocycle for Dirac spectral triples.

Here M^n is a compact Riemannian spin manifold of even dimension and $\not{D}_M : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ is the Dirac operator on M. Then $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$ is an even n-summable spectral triple with dimension spectrum $\{k \in \mathbb{Z}; k \leq n\}$. Therefore, the CM cocycle is well defined and allows us to compute the index map, so that for any hermitian bundle $\mathcal{E}$ equipped with a unitary connection we have

$$\langle [\varphi_{CM}], [\mathcal{E}] \rangle = \operatorname{ind}_{\not{D}_M}[\mathcal{E}] = \operatorname{ind} \not{D}_{\mathcal{E}}. \tag{4.1}$$

Proposition 4.1. *The components of the CM cyclic cocycle $\varphi_{\mathrm{CM}} = (\varphi_{2k})$ associated to the spectral triple $(C^\infty(M), L^2(M, \not{S}), \not{D}_M)$ are given by*

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \frac{1}{(2k)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R_M)^{(n-2k)}, \tag{4.2}$$

for $f^0, f^1, \ldots, f^n$ in $C^\infty(M)$.

Proof. First, using the Mellin transform,

$$|\not{D}_M|^{-2s} = \frac{1}{\Gamma(2s)} \int_0^\infty t^{s-1} e^{-t\not{D}_M^2} dt, \quad \Re s > 0, \tag{4.3}$$

we can rewrite $\varphi_{CM} = (\varphi_{2k})$ as

$$\varphi_{2k}(f^0, \ldots, f^{2k}) = \sum_{\alpha} \frac{2c_{k,\alpha}}{\Gamma(|\alpha| + k)} . \{\text{coeff. of } t^{-(|\alpha|+k)} \text{ in} \operatorname{Str} \not{P}_{k,\alpha} e^{-t\not{D}_M^2} \text{ as } t \to 0^+\}. \tag{4.4}$$

Notice also that $\not{P}_{k,\alpha} = f^0 [\not{D}_M, f^1]^{[\alpha_1]} \ldots [\not{D}_M, f^{2k}]^{[\alpha_{2k}]} = f^0 c(df^1)^{[\alpha_1]} \ldots c(df^{2k})^{[\alpha_{2k}]}$. Therefore, we can make use of Proposition 2.10 to compute the CM cocycle. Indeed, let $h_{\alpha,t}(x, y)$ denote the kernel of $\not{P}_{k,\alpha} e^{-t\not{D}_M^2}$. Then using Lemma 2.6 and Proposition 2.10 we obtain:

- If $\alpha = 0$, then $\not{P}_{k,0} = f^0 c(df^1)\ldots c(df^{2k})$ has Getzler order $2k$ and model operator $f^0 df^1 \wedge \ldots \wedge df^{2k}$, so that we have

$$\operatorname{Str} h_{0,t}(x,x) = \frac{t^{-k}}{(2i\pi)^{\frac{n}{2}}} f^0 df^1 \wedge \ldots \wedge df^{2k} \wedge \hat{A}(R_M)^{(n-2k)} + \mathrm{O}(t^{-k+1}). \tag{4.5}$$

- If $\alpha \neq 0$, then $\not{P}_{k,\alpha}$ has Getzler order $\leq 2(|\alpha| + k) - 1$, and so

$$\operatorname{Str} h_{\alpha,t}(x,x) = \mathrm{O}(t^{-(\alpha+k)+1}). \tag{4.6}$$

Combining this with (4.4) we get the formula (4.2)) for $\varphi_{2k}(f^0, \ldots, f^{2k})$. □

Finally, let us explain how this enables us to recover the local index formula of Atiyah-Singer. Notice that Formula (4.2) means that φ_{CM} is the image under the map (3.12) of the current that is the Poincaré dual of $\hat{A}(R^M)$. Therefore, using (3.13) we see that for any Hermitian bundle $\mathcal{E}$ with curvature $F^{\mathcal{E}}$ we have:

$$\langle [\varphi_{CM}], [\mathcal{E}] \rangle = \langle [\varphi_{\hat{A}(R^M)}], [\mathcal{E}] \rangle = (2i\pi)^{-\frac{n}{2}} \int_M [\hat{A}(R_M) \wedge \mathrm{Ch}(F^{\mathcal{E}})]^{(n)}. \tag{4.7}$$

On the other hand, we have

$$\langle [\varphi_{CM}], [\mathcal{E}] \rangle = \operatorname{ind}_{\not{D}_M} [\mathcal{E}] = \operatorname{ind} \not{D}_{\mathcal{E}}. \tag{4.8}$$

Thus, we obtain

$$\operatorname{ind} \not{D}_{\mathcal{E}} = (2i\pi)^{-\frac{n}{2}} \int_M [\hat{A}(R_M) \wedge \mathrm{Ch}(F^{\mathcal{E}})]^{(n)}, \tag{4.9}$$

which is precisely the Atiyah-Singer index formula.

Remark 4.2. When $\dim M$ is odd we can also compute the odd CM cocycle associated to the corresponding Dirac spectral triple (*cf.* [Po]). In this setting we recover the spectral flow formula of Atiyah-Patodi-Singer [APS].

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