NONCOMMUTATIVE RESIDUE INVARIANTS FOR CR AND CONTACT MANIFOLDS

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Abstract. In this paper we produce several new invariants for CR and contact manifolds by looking at the noncommutative residue traces of various geometric Ψ_HDO projections. In the CR setting these operators arise from the $\overline{\partial}_b$-complex and include the Szegö projections acting on forms. In the contact setting they stem from the generalized Szegö projections at arbitrary integer levels of Epstein-Melrose and from the contact complex of Rumin. In particular, we recover and extend recent results of Hirachi and Boutet de Monvel and we answer a question of Fefferman.

1. Introduction

Motivated by Fefferman's program in CR geometry [21], Hirachi [32] recently proved that the integral of the coefficient of the logarithmic singularity of the Szegö kernel on the boundary of a strictly pseudoconvex domain in $\mathbb{C}^{n+1}$ gives rise to a CR invariant. This was subsequently extended to the contact setting by Boutet de Monvel [12] in terms of the Szegö projections of [14]. As later shown by Boutet de Monvel [13] these invariants vanish, but it it was also asked by Fefferman whether there exist other examples of geometric operators such that the logarithmic singularities of their kernels give rise to CR or contact invariants.

The aim of this paper is to answer Fefferman's question by exhibiting various geometric projections on CR and contact manifolds such that the logarithmic singularities of their kernels give rise to invariants of the corresponding geometric structures. Furthermore, the framework that we used makes it possible to compute of the logarithmic singularities and the corresponding invariants by using techniques borrowed from index theory and Connes' noncommutative geometry.

The Szegö projection and its generalizations in [14] are Ψ_HDOs in the sense of the Heisenberg calculus of [8] and [45]. Moreover, it has been shown by the author ([37], [41]) that the integral of the logarithmic singularity of the kernel of a Ψ_HDO gives rise to a noncommutative residue trace for the Heisenberg calculus. Our invariants then appear as noncommutative residues of geometric Ψ_HDO projections on CR and contact manifolds. These projections can be classified into three families of operators.

The first family arises from the $\overline{\partial}_b$-complex on CR manifolds. Namely, under $Y(q)$-type conditions the Szegö projection on forms and the orthogonal projections onto the kernels of the operators $\overline{\partial}_b$ and $\overline{\partial}_b^*$ are Ψ_HDOs (see, e.g., [8]). We then show

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that their noncommutative residues are all CR diffeomorphism invariants (Theorem 4.2). In particular, in the strictly pseudoconvex case this allows us to recover Hirachi's result. The result further extends to include the projections associated to the $\overline{\partial}_b$-complex with coefficients in a CR holomorphic vector bundle (Theorem 4.4). In addition, we show that these invariants are not affected by deformations of the CR structure (see Proposition 4.6).

The Szegö projections of [14] on a contact manifold M have been further generalized by Epstein-Melrose [19] to arbitrary integer level and in such way to act on the sections of an arbitrary vector bundle $\mathcal{E}$ over M. These operators are Ψ_HDOs and we show that the value of the noncommutative residue of a generalized Szegö projection at a given integer level k is independent of the choice of the operator and is an invariant of the Heisenberg diffeomorphism class of M and of the K-theory class of $\mathcal{E}$ (Theorem 5.6). As a consequence this residue is independent of the choice of the contact form and is invariant under deformations of the contact structure. Moreover, when $k = 0$ and $\mathcal{E}$ is the trivial line bundle this allows us to recover Boutet de Monvel's result.

The last family of examples stems from the contact complex of Rumin [43]. The latter is a complex of horizontal differential forms on a contact manifold which is hypoelliptic in every degree. The orthogonal projections onto the kernels of the differentials of this complex are Ψ_HDOs and we show that their noncommutative residues are Heisenberg diffeomorphism invariants and are invariant under deformation of the contact structure (Theorem 6.1).

The proofs for the examples arising from the $\overline{\partial}_b$-complexe and the contact complex use simpler arguments than those of [32] and [12], as the results follow from the observation that two Ψ_HDO projections with same range or same kernel have same noncommutative residue (Lemma 3.2). The proof for the examples coming from generalized Szegö projections partly relies on the fact that two Ψ_HDO projections with homotopic principal symbols have same noncommutative residue (Lemma 3.7). This generalizes the homotopy arguments of [32] and [12].

Next, the computation of these invariants is rather difficult. They appear as the integrals of local noncommutative residue densities, for which we have explicit formulas in terms of Heisenberg symbols. However, the number of terms to compute increases dramatically with the dimension, so there is no hope to get explicit geometric formulas without further tools to organize the computation.

Furthermore, computating of the local noncommutative residue densities is even more important than the actual computating of the corresponding invariant, before the former implies the latter and could further provide us with some geometric information about the logarithmic singularities of the kernels of the corresponding Ψ_HDO projections. At least in the case of the Szegö kernel this would be of great interest in view of Fefferman's program. Therefore, even if the invariant may vanish it is interesting to compute the corresponding noncommutative residue densities.

In this paper we also allude to some new possible approaches for computing these densities and the corresponding invariants.

A first approach that we suggest is to make use of Getzler's rescaling techniques in the setting of the Heisenberg calculus. This comes in naturally with the framework of the paper. It is believed that this could allow us to compute local densities associated to generalized Szegö kernels, at least on strictly pseudoconvex CR manifolds (see Subsection 4.3 and Remark 5.7).

Another approach suggested in Appendix is to make use of global K-theoretic techniques similar to those involved the K-theoretic proof of the index theorem of Atiyah-Singer [6]. To this end we give a K-theoretic interpretation of the noncommutative residue of a Ψ_HDO projection on a *general* Heisenberg manifold (M, H). More precisely, if we let $K_0(S_0(\mathfrak{g}^*M))$ denote the first K-group of the (noncommutative) algebra of zero'th order symbols, then the noncommutative residue of a Ψ_HDO projection gives rise to an additive map $\rho_R : K_0(S_0(\mathfrak{g}^*M)) \rightarrow \mathbb{R}$ (see Proposition A.7).

Notice that due to the noncommutativity of $S_0(\mathfrak{g}^*M)$ we really have to rely on the K-theory of algebras rather on that of spaces. Therefore, computing the map ρ_R would definitely involve using tools from Connes' noncommutative geometry. As we also explain in Appendix two opposite interesting phenomena may occur:

(i) The map ρ_R is nontrivial and is computable in topological terms;

(ii) The map ρ_R vanishes identically.

Proving (i) could allow us to compute the invariants when we cannot use Getzler's rescaling techniques and this could be especially relevant for dealing with the invariants from the contact complex and with the CR invariants on CR manifolds with degenerate Levi form. However, the occurence of (ii) won't be too disappointing, because it would allow us to define the eta invariant of hypoelliptic selfadjoint Ψ_HDOs, which should be useful for dealing with index problems on complex manifolds with boundaries and on the asymptotically complex hyperbolic (ACH) manifolds.

Finally, the arguments used in this paper are fairly general and should hold in many other settings as well. In particular, it would be interesting to extend them to the setting of complex manifolds with boundary and ACH manifolds. In particular, it would be of special interest to get an analogue in this context of Hirachi's invariant defined in terms of the Bergman projection.

The rest of the paper is organized as follows. In Section 2 we recall the main facts about the Heisenberg calculus and the noncommutative residue for this calculus. In Section 3 we prove general results about noncommutative residues of Ψ_HDO projections. In Section 4 we deal with the invariants from the $\overline{\partial}_b$-complex on a CR manifold. Section 5 is devoted to the noncommutative residues of generalized Szegö projections on a contact manifold. In Section 6 we deal with the invariants arising from the contact complex. Finally, in Appendix we give a K-theoretic interpretation of the noncommutative residue of a Ψ_HDO projection.

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2. Heisenberg calculus and noncommutative residue

In this section we recall the main facts about the Heisenberg calculus and the noncommutative residue trace for this calculus. We also explain how the invariants of Hirachi and Boutet de Monvel can be interpreted as noncommutative residues.

2.1. **Heisenberg manifolds.** A Heisenberg manifold is a pair (M, H) consisting of a manifold M^{d+1} together with a distinguished hyperplane bundle $H \subset TM$.

This definition covers many examples: Heisenberg group, CR manifolds, contact manifolds, (codimension 1) foliations and the confoliations of [18]. In addition, given another Heisenberg manifold (M', H') we say that a diffeomorphism $\phi : M \rightarrow M'$ is a Heisenberg diffeomorphism when $\phi_* H = H'$.

The terminology Heisenberg manifold stems from the fact that the relevant tangent structure in this setting is that of a bundle GM of graded nilpotent Lie groups (see, e.g., [8], [20], [28], [39], [42]). This tangent Lie group bundle can be described as follows.

First, we can define an intrinsic Levi form as the 2-form $\mathcal{L} : H \times H \rightarrow TM/H$ such that, for any point $a \in M$ and any sections X and Y of H near a, we have

$$\mathcal{L}_a(X(a), Y(a)) = [X, Y](a) \mod H_a. \tag{2.1}$$

In other words the class of $[X, Y](a)$ modulo H_a depends only on $X(a)$ and $Y(a)$, not on the germs of X and Y near a (see [39]).

We define the tangent Lie algebra bundle $\mathfrak{g}M$ as the graded Lie algebra bundle consisting of $(TM/H) \oplus H$ together with the fields of Lie bracket and dilations such that, for sections X_0, Y_0 of TM/H and X', Y' of H and for $t \in \mathbb{R}$, we have

$$[X_0 + X', Y_0 + Y'] = \mathcal{L}(X', Y'), \qquad t.(X_0 + X') = t^2 X_0 + tX'. \tag{2.2}$$

Each fiber $\mathfrak{g}_a M$ is a two-step nilpotent Lie algebra, so by requiring the exponential map to be the identity the associated tangent Lie group bundle GM appears as $(TM/H) \oplus H$ together with the grading above and the product law such that, for sections X_0, Y_0 of TM/H and X', Y' of H, we have

$$(X_0 + X').(Y_0 + Y') = X_0 + Y_0 + \frac{1}{2}\mathcal{L}(X', Y') + X' + Y'. \tag{2.3}$$

Moreover, if ϕ is a Heisenberg diffeomorphism from (M, H) onto a Heisenberg manifold (M', H') then, as $\phi_* H = H'$ we get linear isomorphisms from TM/H onto TM'/H' and from H onto H' which together give rise to a linear isomorphism $\phi'_H : (TM/H) \oplus H \rightarrow (TM'/H') \oplus H'$. In fact ϕ'_H is a graded Lie group isomorphism from GM onto GM' (see [39]).

2.2. **Heisenberg calculus.** The Heisenberg calculus is the relevant pseudodifferential calculus to study hypoelliptic operators on Heisenberg manifolds. It was independently introduced by Beals-Greiner [8] and Taylor [45].

The initial idea in the Heisenberg calculus, which is due to Stein, is to construct a class of operators on a Heisenberg manifold (M^{d+1}, H), called Ψ_HDOs, which at any point $a \in M$ are modeled on homogeneous left-invariant convolution operators on the tangent group $G_a M$.

Locally the Ψ_HDOs can be described as follows. Let $U \subset \mathbb{R}^{d+1}$ be a local chart together with a frame $X_0, \ldots, X_d$ of TU such that $X_1, \ldots, X_d$ span H. Such a chart is called a Heisenberg chart. Moreover, on $\mathbb{R}^{d+1}$ we consider the dilations,

$$t.\xi = (t^2\xi_0, t\xi_1, \ldots, t\xi), \qquad \xi \in \mathbb{R}^{d+1}, \quad t > 0. \tag{2.4}$$

Definition 2.1. 1) $S_m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, is the space of functions $p(x, \xi)$ in $C^\infty(U \times \mathbb{R}^{d+1} \backslash 0)$ such that $p(x, t.\xi) = t^m p(x, \xi)$ for any $t > 0$.

2) $S^m(U \times \mathbb{R}^{d+1})$, $m \in \mathbb{C}$, consists of functions $p \in C^\infty(U \times \mathbb{R}^{d+1})$ with an asymptotic expansion $p \sim \sum_{j \geq 0} p_{m-j}$, $p_k \in S_k(U \times \mathbb{R}^{d+1})$, in the sense that, for

any integer N and for any compact $K \subset U$, we have

$$|\partial_x^\alpha \partial_\xi^\beta (p - \sum_{j<N} p_{m-j})(x,\xi)| \leq C_{\alpha\beta NK} \|\xi\|^{\Re m - \langle\beta\rangle - N}, \qquad x \in K, \quad \|\xi\| \geq 1, \tag{2.5}$$

where we have let $\langle\beta\rangle = 2\beta_0 + \beta_1 + \ldots + \beta_d$ and $\|\xi\| = (\xi_0^2 + \xi_1^4 + \ldots + \xi_d^4)^{1/4}$.

Next, for $j = 0, \ldots, d$ let $\sigma_j(x,\xi)$ denote the symbol (in the classical sense) of the vector field $\frac{1}{i}X_j$ and set $\sigma = (\sigma_0, \ldots, \sigma_d)$. Then for $p \in S^m(U \times \mathbb{R}^{d+1})$ we let $p(x, -iX)$ be the continuous linear operator from $C_c^\infty(U)$ to $C^\infty(U)$ such that

$$p(x,-iX)f(x) = (2\pi)^{-(d+1)} \int e^{ix.\xi} p(x, \sigma(x,\xi)) \hat{f}(\xi) d\xi, \qquad f \in C_c^\infty(U). \tag{2.6}$$

Definition 2.2. $\Psi_H^m(U)$, $m \in \mathbb{C}$, consists of operators $P : C_c^\infty(U) \to C^\infty(U)$ which are of the form $P = p(x,-iX) + R$ for some p in $S^m(U \times \mathbb{R}^{d+1})$, called the symbol of P, and some smoothing operator R.

For any $a \in U$ there is exists a unique affine change of variable $\psi_a : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $\psi_a(a) = 0$ and $(\psi_a)_* X_j = \frac{\partial}{\partial x_j}$ at $x = 0$ for $j = 0, 1, \ldots, d+1$. Then, a continuous operator $P : C_c^\infty(U) \to C^\infty(U)$ is a Ψ_HDO of order m if, and only if, its kernel $k_P(x,y)$ has a behavior near the diagonal of the form

$$k_P(x,y) \sim \sum_{j \geq -(m+d+2)} \left(a_j(x, \psi_x(y)) - \sum_{\langle\alpha\rangle = j} c_\alpha(x) \psi_x(x)^\alpha \log \|\psi_x(y)\| \right), \tag{2.7}$$

with $c_\alpha \in C^\infty(U)$ and $a_j(x,y) \in C^\infty(U \times (\mathbb{R}^{d+1} \backslash 0))$ such that $a_j(x, \lambda.y) = \lambda^j a_j(x,y)$ for any $\lambda > 0$. Moreover, $a_j(x,y)$ and $c_\alpha(x)$, $\langle\alpha\rangle = j$, depend only on the symbol of P of degree $-(j+d+2)$.

The class of Ψ_HDOs is invariant under changes of Heisenberg chart (see [8], [40]), so we may extend the definition of Ψ_HDOs to an arbitrary Heisenberg manifold (M, H) and let them act on sections of a vector bundle $\mathcal{E}$ over M. We let $\Psi_H^m(M, \mathcal{E})$ denote the class of Ψ_HDOs of order m on M acting on sections of $\mathcal{E}$.

From now on we let (M^{d+1}, H) be a compact Heisenberg manifold and we let $\mathfrak{g}^*M$ denote the (linear) dual of the Lie algebra bundle $\mathfrak{g}M$ of GM with canonical projection $\mathrm{pr} : \mathfrak{g}^*M \to M$. As shown in [40] (see also [19]) the principal symbol of $P \in \Psi_H^m(M, \mathcal{E})$ can be intrinsically defined as a symbol $\sigma_m(P)$ of the class below.

Definition 2.3. $S_m(\mathfrak{g}^*M, \mathcal{E})$, $m \in \mathbb{C}$, consists of sections $p \in C^\infty(\mathfrak{g}^*M \backslash 0, \mathrm{End}\, \mathrm{pr}^*\mathcal{E})$ which are homogeneous of degree m with respect to the dilations in (2.2), i.e., we have $p(x, \lambda.\xi) = \lambda^m p(x,\xi)$ for any $\lambda > 0$.

Next, for any $a \in M$ the convolution on G_aM gives rise under the (linear) Fourier transform to a bilinear product for homogeneous symbols,

$$*^a : S_{m_1}(\mathfrak{g}_a^*M, \mathcal{E}_a) \times S_{m_2}(\mathfrak{g}_a^*M, \mathcal{E}_a) \longrightarrow S_{m_1+m_2}(\mathfrak{g}_a^*M, \mathcal{E}_a), \tag{2.8}$$

This product depends smoothly on a as much so to yield the product

$$* : S_{m_1}(\mathfrak{g}^*M, \mathcal{E}) \times S_{m_2}(\mathfrak{g}^*M, \mathcal{E}) \longrightarrow S_{m_1+m_2}(\mathfrak{g}^*M, \mathcal{E}), \tag{2.9}$$

$$p_{m_1} * p_{m_2}(a,\xi) = [p_{m_1}(a,.) *^a p_{m_2}(a,.)](\xi). \tag{2.10}$$

This provides us with the right composition for principal symbols, for we have

$$\sigma_{m_1+m_2}(P_1 P_2) = \sigma_{m_1}(P_1) * \sigma_{m_2}(P_2) \qquad \forall P_j \in \Psi_H^{m_j}(M, \mathcal{E}). \tag{2.11}$$

Notice that when G_aM is not commutative, i.e., $\mathcal{L}_a \neq 0$, the product $*^a$ is not anymore the pointwise product of symbols and, in particular, is not commutative. Consequently, unless when H is integrable, the product for Heisenberg symbols is not commutative and, while local, it is not microlocal.

If $P \in \Psi_H^m(M,\mathcal{E})$ then the transpose P^t belongs to $\Psi_H^m(M,\mathcal{E}^*)$, and if $\mathcal{E}$ is further endowed with a Hermitian metric then the adjoint P^* belongs to $\Psi_H^{\overline{m}}(M,\mathcal{E})$ (see [8]). Moreover, as shown in [40], Sect. 3.2, the principal symbols of P^t and P^* are

$$\sigma_m(P^t) = \sigma_m(P)(x,-\xi)^t \quad \text{and} \quad \sigma_{\overline{m}}(P^*) = \sigma_m(P)(x,\xi)^*. \tag{2.12}$$

When the principal symbol of $P \in \Psi_H^m(M,\mathcal{E})$ is invertible with respect to the product $*$, the symbolic calculus of [8] allows us to construct a parametrix for P in $\Psi_H^{-m}(M,\mathcal{E})$. In particular, although not elliptic, P is hypoelliptic with a controlled loss/gain of derivatives (see [8]).

In general, it may be difficult to determine whether the principal symbol of a given Ψ_HDO $P \in \Psi_H^m(m,\mathcal{E})$ is invertible with respect to the product $*$, but this can be completely determined in terms of a representation theoretic criterion on each tangent group G_aM, the so-called Rockland condition (see [40], Thm. 3.3.19). In particular, if $\sigma_m(P)(a,.)$ is *pointwise* invertible with respect to the product $*^a$ for any $a \in M$ then $\sigma_m(P)$ is *globally* invertible with respect to $*$.

2.3. **Noncommutative residue.** Let $P : C^\infty(M,\mathcal{E}) \rightarrow C^\infty(M,\mathcal{E})$ be a Ψ_HDO of integer order m. Then it follows from (2.7) that in a trivializing Heisenberg chart the kernel $k_P(x,y)$ of P has a behavior near the diagonal of the form

$$k_P(x,y) = \sum_{-(m+d+2)\leq j\leq 1} a_j(x,-\psi_x(y)) - c_P(x)\log\|\psi_x(y)\| + \mathrm{O}(1), \tag{2.13}$$

where $a_j(x,y)$ is homogeneous of degree j in y with respect to the dilations (2.2) and $c_P(x)$ is the smooth function given by

$$c_P(x) = |\psi_x'| \int_{\|\xi\|=1} p_{-(d+2)}(x,\xi)d\xi, \tag{2.14}$$

where $p_{-(d+2)}(x,\xi)$ is the homogeneous symbol of degree $-(d+2)$ of P.

Under the action of Heisenberg diffeomorphisms $c_P(x)$ behaves like a density (see [41], Prop. 3.11). Therefore, the coefficient $c_P(x)$ makes intrinsically sense on M as a section of $|\Lambda|(M) \otimes \operatorname{End}\mathcal{E}$, where $|\Lambda|(M)$ is the bundle of densities on M.

We can now define a functional on $\Psi_H^{\mathbb{Z}}(M,\mathcal{E}) = \cup_{m\in\mathbb{Z}}\Psi_H^m(M,\mathcal{E})$ by letting

$$\operatorname{Res} P = \int_M \operatorname{tr}_{\mathcal{E}} c_P(x), \qquad P \in \Psi_H^{\mathbb{Z}}(M,\mathcal{E}). \tag{2.15}$$

As shown in [41] this functional is the analogue for the Heisenberg calculus of the noncommutative residue of Wodzicki ([47], [48]) and Guillemin [29], since it also arises as the residual trace on integer order Ψ_HDOs induced by the analytic continuation of the usual trace to Ψ_HDOs of non-integer orders.

Proposition 2.4 ([37], [41]). *1)* Res *is a trace on the algebra* $\Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ *which vanishes on differential operators and on* Ψ_H*DOs of integer order* $\leq -(d+3)$*, In fact, when* M *is connected this is the unique trace up to constant multiple.*

2) For any $P \in \Psi_H^{\mathbb{Z}}(M,\mathcal{E})$ *we have* $\operatorname{Res} P^t = \overline{\operatorname{Res} P^*} = \operatorname{Res} P$.

3) Let ϕ be a Heisenberg diffeomorphism from (M, H) onto a Heisenberg manifold (M', H'). Then for any $P \in \Psi_H^{\mathbb{Z}}(M, \mathcal{E})$ we have $\operatorname{Res} \phi_* P = \operatorname{Res} P$.

2.4. **Logarithmic singularity of Szegö kernels.** Let S be a Szegö projection on a contact manifold M^{2n+1} as in [14]. This is a FIO with complex phase $q(x, y)$ and near the diagonal the kernel of S a behavior of the form

$$k_S(x, y) \sim \sum_{-(n+1) \leq j \leq -1} \alpha_j(x, y) q(x, y)^j + \sum_{j \geq 0} \beta_j(x, y) q(x, y)^j \log q(x, y), \tag{2.16}$$

where $\alpha_j(x, y)$ and $\beta_j(x, y)$ are smooth functions defined near the diagonal.

The coefficient $\beta_0(x, x)$ of the logarithmic singularity makes sense globally as a density on M and so we can define

$$L(S) = \int_M \beta_0(x, x). \tag{2.17}$$

This is this object which is shown to give rise to a global invariant in [32] and [12].

In fact, the phase $q(x, y)$ vanishes on the diagonal and is such that $id_x q = -id_y q$ is a nonzero annihilator of H on the diagonal and $\Re q(x, y) \gtrsim |x - y|^2$ near the diagonal. Therefore, the Taylor expansion of $q(x, y)$ near $y = x$ is of the form,

$$q(x, y) \sim \sum_{\langle \alpha \rangle \geq 2} a_\alpha(x) \psi_x(y)^\alpha, \tag{2.18}$$

where $q_2(x, y) := \sum_{\langle \alpha \rangle \geq 2} a_\alpha(x) \psi_x(y)^\alpha$ is nonzero for $y \neq 0$ and y close enough to x.

In fact, plugging (2.18) into (2.16) shows that the kernel of S has near the diagonal the singularity of the form (2.7) with $m = 0$ and so S is a zero'th order Ψ_HDO. Moreover, as near the diagonal we have $\log q(x, y) = 2 \log \|\psi_x(y)\| + \mathrm{O}(1)$, we see that $c_S(x) = -2\beta_0(x, x)$. Thus,

$$L(S) = -\frac{1}{2} \operatorname{Res} S. \tag{2.19}$$

This shows that the invariants considered by Hirachi and Boutet de Monvel can be interpreted as noncommutative residues.

Remark 2.5. Guillemin [30] has defined noncommutative residue traces for some algebras of FIOs, including the algebra of Töplitz operators on a contact manifold. The latter is an ideal of the algebra of Ψ_HDOs (see, e.g., [19]) and one can check that in this context Guillemin's trace is equal to $-\frac{1}{2} \operatorname{Res}$ on Töplitz operators. In particular, we see that $L(S)$ agrees with the noncommutative residue trace in Guillemin's sense of S.

3. Noncommutative residues of Ψ_HDO projections

This section we gather several general lemmas about noncommutative residues of a Ψ_HDO projections.

Throughout all the section we let (M^{d+1}, H) be a compact Heisenberg manifold equipped with a smooth density > 0 and let $\mathcal{E}$ be a Hermitian vector bundle.

Lemma 3.1. *Let $\Pi \in \Psi_H^0(M, \mathcal{E})$ be a $\Psi_H DO$ projection. Then the orthogonal projection Π_0 onto its range is a zero'th order $\Psi_H DO$ and we have* $\operatorname{Res} \Pi_0 = \operatorname{Res} \Pi$.

Proof. It is well known that any projection on $L^2(M,\mathcal{E})$ is similar to the orthogonal projection onto its range (see, e.g., [11], Prop. 4.6.2). Indeed, the operator $B = 1 + (\Pi - \Pi^*)(\Pi^* - \Pi)$ is invertible on $L^2(M,\mathcal{E})$ and we have $\Pi_0 = \Pi\Pi^* B^{-1}$. Moreover, if we let $A = \Pi_0 + (1-\Pi)(1-\Pi_0)$ then A is invertible with inverse $A^{-1} = \Pi + (1-\Pi_0)(1-\Pi)$ and we have $\Pi_0 = A^{-1}\Pi A$.

Observe that B is a zero'th order Ψ_HDO. Let $Q = \Pi - \Pi^*$ and for $a \in M$ let B^a and Q^a be the respective model operators at a of B as defined in [40]. Recall that the latter are bounded left invariant convolution operators on $L^2(G_aM, \mathcal{E}_a)$. Since $B = 1+QQ^*$ it follows from [40], Props. 3.2.9, 3.2.12, that, either $B^a = 1+Q^a(Q^a)^*$ if Q has order 0, or $B^a = 1$ otherwise.

In any case B^a is an invertible bounded operator on $L^2(G_aM,\mathcal{E})$, so it follows from [40], Thm. 3.3.10, that the principal symbol of B is invertible. Therefore, there exist $C \in \Psi_H^0(M,\mathcal{E})$ and smoothing operators R_1 and R_2 such that $CB = 1 - R_1$ and $BC = 1 - R_2$. From this we get $C = B^{-1} - R_1 B^{-1} = B^{-1} - B^{-1}R_2$. This implies that B^{-1} induces a continuous endomorphism of $C^\infty(M,\mathcal{E})$ and agrees with C modulo a smoothing operator, hence is a zero'th order Ψ_HDO.

Since B^{-1} is a zero'th order Ψ_HDO we deduce that $\Pi_0 = \Pi\Pi^* B^{-1}$ is a zero'th order Ψ_HDO as well. This implies that A and A^{-1} are also Ψ_HDOs, so as Res is a trace we get $\operatorname{Res}\Pi_0 = \operatorname{Res} A^{-1}\Pi A = \operatorname{Res}\Pi$. The lemma is thus proved. ☐

As a consequence of this lemma we will obtain:

Lemma 3.2. *For $j = 1,2$ let $\Pi_j \in \Psi_H^0(M,\mathcal{E})$ be a Ψ_HDO projection. If Π_1 and Π_2 have same range or have same kernel then $\operatorname{Res}\Pi_1 = \operatorname{Res}\Pi_2$.*

Proof. If Π_1 and Π_2 have same range then by Lemma 3.1 their noncommutative residues agree, since their are both equal to that of the orthogonal projection onto their common range.

If Π_1 and Π_2 have same kernel then $1-\Pi_1$ and $1-\Pi_2$ have same range, so we have $\operatorname{Res}(1-\Pi_2) = \operatorname{Res}(1-\Pi_2)$. As $\operatorname{Res}(1-\Pi_j) = -\operatorname{Res}\Pi_j$, $j=1,2$, it follows that we have $\operatorname{Res}\Pi_2 = \operatorname{Res}\Pi_1$. ☐

Another consequence of Lemma 3.1 is the following.

Lemma 3.3. *Let $\Pi_0 \in \Psi_H^0(M,\mathcal{E})$ be a Ψ_HDO projection. Then $\operatorname{Res}\Pi$ is a real number and we have $\operatorname{Res}\Pi^* = \operatorname{Res}\Pi^t = \operatorname{Res}\Pi$.*

Proof. By Proposition 2.4 we have $\overline{\operatorname{Res}\Pi^*} = \operatorname{Res}\Pi^t = \operatorname{Res}\Pi$, so we only have to check that $\operatorname{Res}\Pi$ is in $\mathbb{R}$. Let Π_0 be the orthogonal projection onto the range of Π. As Π_0 is a selfadjoint Ψ_HDO projection we have $\overline{\operatorname{Res}\Pi_0} = \operatorname{Res}\Pi_0^* = \operatorname{Res}\Pi_0$, so that $\operatorname{Res}\Pi_0$ is a real number. Since by Lemma 3.1 the latter agrees with $\operatorname{Res}\Pi$, we see that $\operatorname{Res}\Pi$ is in $\mathbb{R}$ as well. ☐

Next, we define C^1-paths of Ψ_HDOs as follows. For an open $V \subset \mathbb{R}^{d+1}$ we endow $S^m(V\times\mathbb{R}^{d+1})$, $m \in \mathbb{C}$, with the Fréchet space topology induced by the topology of $C^\infty(V\times\mathbb{R}^{d+1})$ and the sharpest constants in (2.5). We then let $S^m(V\times\mathbb{R}^{d+1})_t$ denote the space of C^1-paths from $I := [0,1]$ to $S^m(V\times\mathbb{R}^{d+1})$.

Similarly we endow $S_m(\mathfrak{g}^*M,\mathcal{E})$ with the Fréchet space topology inherited from that of $C^\infty(\mathfrak{g}^*M\setminus 0, \operatorname{End}\operatorname{pr}^*\mathcal{E})$ and we let $S_m(\mathfrak{g}^*M,\mathcal{E})_t$ denote the space of C^1-paths from I to $S_m(\mathfrak{g}^*M,\mathcal{E})$.

Definition 3.4. $\Psi_H^m(M,\mathcal{E})_t$ is the space of paths $(P_t)_{t\in I} \subset \Psi_H^m(M,\mathcal{E})$ which are C^1 in the sense that:

(i) The kernel of P_t is given outside the diagonal by a C^1-path of smooth kernels;

(ii) For any Heisenberg chart $\kappa : U \to V \subset \mathbb{R}^{d+1}$ with a H-frame $X_0, \ldots, X_d$ and any trivialization $\tau : \mathcal{E}_{|_U} \to U \times \mathbb{C}^r$ we can write

$$\kappa_* \tau_* (P_{t|_U}) = p_t(x, -iX) + R_t, \tag{3.1}$$

for some C^1-path $p_t \in M_r(S^m(U \times \mathbb{R}^{d+1})_t)$ and some C^1-path R_t of smoothing operators, i.e., R_t is given by a C^1-path of smooth kernels.

We gather the main properties of C^1-paths of Ψ_HDOs in the following.

Lemma 3.5 ([40], Chap. 4)**.** *1) If $P_t \in \Psi_H^m(M, \mathcal{E})_t$ then $\sigma_m(P_t)$ is a C^1-path with values in $S_m(\mathfrak{g}^*M, \mathcal{E})$ and, in fact, in a local trivializing chart all the homogeneous components of the symbol of P_t yield C^1-paths of homogeneous symbols.*

2) If $P_{j,t} \in \Psi_H^{m_j}(M, \mathcal{E})_t$, $j = 1, 2$, then $P_{1,t} P_{2,t} \in \Psi_H^{m_1+m_2}(M, \mathcal{E})_t$.

3) Let ϕ be a Heisenberg diffeomorphism from a Heisenberg manifold (M', H') onto (M, H). Then for any $P_t \in \Psi_H^m(M, \mathcal{E})_t$ the path $\phi^* P_t$ is in $\Psi_H^m(M', \phi^*\mathcal{E})_t$.

Remark 3.6. In [40] the proofs are actually carried out for holomorphic families of Ψ_HDOs, but they remain valid *mutatis mutandis* for C^1-paths of Ψ_HDOs.

Bearing this in mind we have:

Lemma 3.7. *Let Π_0 and Π_1 be projections in $\Psi_H^0(M, \mathcal{E})$ such that their principal symbols can be joined to each other by means of a C^1 path of idempotents in $S_0(\mathfrak{g}^*M, \mathcal{E})$. Then we have* $\operatorname{Res} \Pi_0 = \operatorname{Res} \Pi_1$.

Proof. For $j = 0, 1$ let $F_j = 2\Pi_j - 1$. Then $F_j^2 = 1$ and the principal symbol of F_0 can be connected to that of F_1 by means of a C^1-path $f_{0,t} \in S_0(\mathfrak{g}^*M, \mathcal{E})_t$ such that $f_{0,t} * f_{0,t} = 1$ $\forall t \in [0, 1]$. We can construct a path $G_t \in \Psi_H^0(M, \mathcal{E})_t$ so that we have $G_t^2 = 1 \bmod \Psi_H^{-(d+3)}(M, \mathcal{E})_t$ and $G_j = F_j \bmod \Psi_H^{-(d+3)}(M, \mathcal{E})$ for $j = 0, 1$ as follows.

Let $(\varphi_k)_{k \geq 0} \subset C^\infty(M)$ be a partition of the unity subordinated to an open covering $(U_k)_{k \geq 0}$ of domains of Heisenberg charts $\kappa_k : U_k \to V_k$ with H-frame $X_0^{(k)}, \ldots, X_d^{(k)}$ and over which there are trivializations $\tau_k : \mathcal{E}_{|_{U_k}} \to U_k \times \mathbb{C}^r$. Let $q_{0,t}^{(k)}(x, \xi) = (1 - \chi(\xi))(\kappa_{\alpha *} \tau_{\alpha *} f_t)(x, \xi)$ where $\chi \in C_c^\infty(\mathbb{R}^{d+1})$ is such that $\chi(\xi) = 1$ near the origin. Then $q_{0,t}^{(k)}$ is a C^1-path with values in $S^0(U_k \times \mathbb{R}^{d+1}, \mathbb{C}^r)$ and we obtain a C^1-path $Q_t \in \Psi_H^0(M, \mathcal{E})_t$ with principal symbol $f_{0,t}$ by letting $Q_t = \sum_{k \geq 0} \varphi_k (\tau_k^* \kappa_k^* q_{0,t}(x, -iX^{(k)})) \psi_k$, where $\psi_k \in C_c^\infty(U_k)$ is so that $\psi_k = 1$ near $\operatorname{supp} \varphi_k$.

Since Q_t has principal symbol $f_{0,t}$ we see that $F_j - Q_j \in \Psi_H^{-1}(M, \mathcal{E})$ for $j = 0, 1$. Consider now the C^1-path, $P_t = Q_t + (1 - t)(F_0 - Q_0) + t(F_1 - Q_1)$. Then P_t has principal symbol $f_{0,t}$ for every $t \in [0, 1]$ and for $j = 0, 1$ we have $P_j = F_j$.

Next, since $f_{0,t} * f_{0,t} = 1$ we can write $P_t^2 = 1 - R_t$ with $R_t \in \Psi_H^{-1}(M, \mathcal{E})_t$. Let $\sum_{k \geq 0} a_k z^k$ be the Taylor series at $z = 0$ of $(1 - z)^{-\frac{1}{2}}$. In particular, for any integer N we can write $(1 - z)(\sum_{0 \leq k \leq N} a_k z^k)^2 = 1 + P_N(z)$, where $P_N(z)$ is a polynomial of the form $P_N(z) = \sum_{N+1 \leq k+l \leq 2N+1} b_{N,k} z^k$.

Let $G_t = P_t \sum_{0 \leq k \leq d+2} a_k R_t^k$. Then $G_t \in \Psi_H^0(M, \mathcal{E})_t$ and as $R_t = 1 - P_t^2$ commutes with P_t we get

$$G_t^2 = P_t^2 \Big(\sum_{0 \leq k \leq d+2} a_k R_t^k\Big)^2 = (1 - R_t)\Big(\sum_{0 \leq k \leq d+2} a_k R_t^k\Big)^2 = 1 + P_N(R_t). \tag{3.2}$$

Since $P_{d+2}(R_t)$ is in $\Psi_H^{-(d+3)}(M,\mathcal{E})_t$ this shows that $G_t^2 = 1 \bmod \Psi_H^{-(d+3)}(M,\mathcal{E})_t$. Moreover, as for $j=0,1$ we have $R_j = 1 - F_j^2 = 0$ we see that $G_j = P_j = F_j$.

Now, the equality $G_t^2 = 1 \bmod \Psi_H^{-(d+3)}(M,\mathcal{E})_t$ implies that $\dot{G}_t G_t + G_t \dot{G}_t = 0$ modulo $\Psi_H^{-(d+3)}(M,\mathcal{E})_t$. Therefore $-G_t \dot{G}_t G_t = G_t^2 \dot{G}_t = \dot{G}_t \bmod \Psi_H^{-(d+3)}(M,\mathcal{E})_t$, and so we get $\dot{G}_t = \frac{1}{2}(G_t^2 \dot{G}_t - G_t \dot{G}_t G_t) = \frac{1}{2}[G_t, G_t \dot{G}_t] \bmod \Psi_H^{-(d+3)}(M,\mathcal{E})_t$.

On the other hand, it follows from (2.14) and Lemma 3.5 that Res commutes with the differentiation of C^1-paths. Since Res is a trace and vanishes on $\Psi_H^{-(d+3)}(M,\mathcal{E})$, we obtain $\frac{d}{dt} \operatorname{Res} G_t = \operatorname{Res} \dot{G}_t = \frac{1}{2} \operatorname{Res}[G_t, G_t \dot{G}_t] = 0$. Hence we have $\operatorname{Res} G_0 = \operatorname{Res} G_1$. As $\operatorname{Res} G_j = \operatorname{Res} F_j = \operatorname{Res}(2\Pi_j - 1) = 2 \operatorname{Res} \Pi_j$ it follows that $\operatorname{Res} \Pi_0 = \operatorname{Res} \Pi_1$ as desired. □

4. Invariants from the $\overline{\partial}_b$-complex

Throughout all this section we let M^{2n+1} be a compact orientable CR manifold with CR bundle $T_{1,0} \subset T_{\mathbb{C}}M$, so that $H = \Re(T_{1,0} \oplus T_{0,1}) \subset TM$ is a hyperplane bundle of TM admitting an (integrable) complex structure.

4.1. Construction of the CR invariants.

Since M is orientable and H is orientable by means of its complex structure, there exists a global non-zero real 1-form θ annihilating H. Associated to θ is the Hermitian Levi form,

$$L_\theta(Z,W) = -id\theta(Z,\overline{W}) = i\theta([Z,\overline{W}]), \qquad Z,W \in C^\infty(M,T_{1,0}). \tag{4.1}$$

We then say that M is strictly pseudoconvex (resp. κ-strictly pseudoconvex) if for some choice of θ the Levi form is everywhere positive definite (resp. has everywhere κ negative eigenvalues and $n-\kappa$ positive eigenvalues).

Let $\mathcal{N}$ be a supplement of H in TM. This is an orientable line bundle which gives rise to the splitting,

$$T_{\mathbb{C}}M = T_{1,0} \oplus T_{0,1} \oplus (\mathcal{N} \otimes \mathbb{C}). \tag{4.2}$$

For $p,q = 0,\ldots,n$ let $\Lambda^{p,q} = (\Lambda^{1,0})^p \wedge (\Lambda^{0,1})^q$ be the bundle of (p,q)-covectors, where $\Lambda^{1,0}$ (resp. $\Lambda^{0,1}$) denotes the annihilator in $T_{\mathbb{C}}^*M$ of $T_{0,1} \oplus (\mathcal{N} \otimes \mathbb{C})$ (resp. of $T_{1,0} \oplus (\mathcal{N} \otimes \mathbb{C})$). Then we have the decomposition:

$$\Lambda^* T_{\mathbb{C}}^* M = (\bigoplus_{p,q=0}^{n} \Lambda^{p,q}) \oplus \theta \wedge \Lambda^* T_{\mathbb{C}}^* M. \tag{4.3}$$

Notice that this decomposition does not depend on the choice of θ, but it does depend on that of $\mathcal{N}$.

The complex $\overline{\partial}_b : C^\infty(M,\Lambda^{0,*}) \to C^\infty(M,\Lambda^{0,*+1})$ of Kohn-Rossi ([36], [35]) is defined as follows. For any $\eta \in C^\infty(M,\Lambda^{0,q})$ we can uniquely decompose $d\eta$ as

$$d\eta = \overline{\partial}_{b;q}\eta + \partial_{b;q}\eta + \theta \wedge \mathcal{L}_{X_0}\eta, \tag{4.4}$$

where $\overline{\partial}_{b;q}\eta$ and $\partial_{b;q}\eta$ are sections of $\Lambda^{0,q+1}$ and $\Lambda^{1,q}$ respectively and X_0 is the section of $\mathcal{N}$ such that $\theta(X_0) = 1$. Thanks to the integrability of $T_{1,0}$ we have $\overline{\partial}_{b;q+1} \circ \overline{\partial}_{b;q} = 0$, so we get a cochain complex.

The $\overline{\partial}_b$-complex depends only on the CR structure of M and on the choice of $\mathcal{N}$. The dependence on the latter can be determined as follows. Let $\mathcal{N}'$ be another supplement of H and let us assign the superscipt $'$ to objects defined using $\mathcal{N}'$, e.g., $\overline{\partial}'_{b;q}$ is the $\overline{\partial}_b$-operator associated to $\mathcal{N}'$.

Let X_0' be the section of $\mathcal{N}'$ such that $\theta(X_0') = 1$ and let $\varphi = \varphi_{X_0, X_0'}$ be the vector bundle isomorphism of $T_{\mathbb{C}}M$ onto itself such that φ is identity on $T_{1,0} \oplus T_{0,1}$ and $\varphi(X_0) = X_0'$. By duality this defines a vector bundle isomorphism φ^t from $\Lambda^* T_{\mathbb{C}}^* M$ onto itself. Then φ^t induces an isomorphism from $\Lambda^{p,q}$ onto $\Lambda'^{p,q}$ and restricts to the identity on $\theta \wedge \Lambda^* T_{\mathbb{C}}^* M$. Thus, if $\eta \in C^\infty(M, \Lambda^{p,q})$ then $\varphi^t(\eta)$ is the component in $\Lambda'^{p,q}$ of η with respect to the decomposition (4.3) associated to $\mathcal{N}'$. In fact, we can check that we have

$$\varphi^t(\eta) = \eta - \theta \wedge \iota_{X_0'} \eta \qquad \forall \eta \in C^\infty(M, \Lambda^{p,q}). \tag{4.5}$$

Lemma 4.1. *For $q = 0, \ldots, n$ we have $\overline{\partial}'_{b;q} = \varphi^t \overline{\partial}_{b;q} (\varphi^t)^{-1}$.*

Proof. Let $\eta \in C^\infty(M, \Lambda^{0,q})$ and let us compute $\overline{\partial}'_{b;q}[\varphi^t(\eta)]$. Thanks to (4.5) we have $d[\varphi^t(\eta)] = d\eta - d\theta \wedge \iota_{X_0'}\eta + \theta \wedge d\iota_{X_0'}\eta$. Moreover, we have $\theta \wedge d\iota_{X_0'}\eta = \theta \wedge \mathcal{L}_{X_0'}\eta - \theta \wedge \iota_{X_0'} d\eta$, so using (4.4) we get $\theta \wedge d\iota_{X_0'}\eta = \theta \wedge \beta - \theta \wedge \iota_{X_0'} \overline{\partial}_{b;q}\eta - \theta \wedge \iota_{X_0'} \partial_{b;q}\eta$ for some form β. Therefore, we see that $d\varphi^t(\eta)$ is equal to

$$\begin{aligned} \theta \wedge \beta' + (\overline{\partial}_{b;q}\eta - \theta \wedge \iota_{X_0'} \overline{\partial}_{b;q}\eta) + (\partial_{b;q}\eta - \theta \wedge \iota_{X_0'} \partial_{b;q}\eta) - d\theta \wedge \iota_{X_0}\eta \\ = \theta \wedge \beta' + \varphi^t(\overline{\partial}_{b;q}\eta) + \varphi^t(\partial_{b;q}\eta) - d\theta \wedge \iota_{X_0}\eta, \end{aligned} \tag{4.6}$$

for some β' in $C^\infty(M, \Lambda^* T_{\mathbb{C}}^* M)$.

Since $T_{0,1}$ is integrable for any sections $\overline{Z}$ and $\overline{W}$ of $T_{0,1}$ the Lie bracket $[\overline{Z}, \overline{W}]$ is again a section of $T_{0,1}$ and so we have $d\theta(\overline{Z}, \overline{W}) = -\theta([\overline{Z}, \overline{W}]) = 0$. This means that $d\theta$ has no component in $\Lambda'^{2,0}$. Therefore, in (4.6) the form $d\theta \wedge \iota_{X_0}\eta$ cannot have a component in $\Lambda'^{0,q+1}$. In view of the definition of $\overline{\partial}'_{b;q}$ it follows that $\overline{\partial}'_{b;q}(\varphi^t(\eta)) = \varphi^t(\partial_{b;q}\eta)$. Hence the lemma. $\square$

Assume now that M is endowed with a Hermitian metric h on $T_{\mathbb{C}}M$ which commutes with complex conjugation and makes the splitting (4.2) become orthogonal. Let $\Box_{b;q} = \overline{\partial}^*_{b;q+1}\overline{\partial}_{b;q} + \overline{\partial}_{b;q-1}\overline{\partial}^*_{b;q}$ be the Kohn Laplacian and let $S_{b;q}$ be the Szegö projection on $(0,q)$-forms, i.e., the orthogonal projection onto $\ker \Box_{b;q}$.

We also consider the orthogonal projections $\Pi_0(\overline{\partial}_{b;q})$ and $\Pi_0(\overline{\partial}^*_{b;q})$ onto $\ker \overline{\partial}_{b;q}$ and $\ker \overline{\partial}^*_{b;q} = (\operatorname{im} \overline{\partial}_{b;q-1})^\perp$. In fact, as $\ker \overline{\partial}_{b;q} = \ker \Box_{b;q} \oplus \operatorname{im} \overline{\partial}_{b;q-1}$ we have $\Pi_0(\overline{\partial}_{b;q}) = S_{b;q} + 1 - \Pi_0(\overline{\partial}^*_{b;q})$, that is,

$$S_{b;q} = \Pi_0(\overline{\partial}_{b;q}) + \Pi_0(\overline{\partial}^*_{b;q}) - 1. \tag{4.7}$$

Let $N_{b;q}$ be the partial inverse of $\Box_{b;q}$, so that $N_{b;q}\Box_{b;q} = \Box_{b;q}N_{b;q} = 1 - S_{b;q}$. Then it can be shown (see, e.g., [8], pp. 170–172) that we have

$$\Pi_0(\overline{\partial}_{b;q}) = 1 - \overline{\partial}^*_{b;q+1} N_{b;q+1} \overline{\partial}_{b;q}, \qquad \Pi_0(\overline{\partial}^*_{b;q}) = 1 - \overline{\partial}_{b;q-1} N_{b;q-1} \overline{\partial}^*_{b;q-1}. \tag{4.8}$$

The principal symbol of $\Box_{b;q}$ is invertible if, and only if, the condition $Y(q)$ holds at every point $x \in M$ (see [8], [40]). If we let $\kappa_+(x)$ and $\kappa_-(x)$ denote the number of positive and negative eigenvalues of L_θ at x, then the condition $Y(q)$ at x requires to have

$$q \notin \{\kappa_+(x), \ldots, n - \kappa_-(x)\} \cup \{\kappa_-(x), \ldots, n - \kappa_+(x)\}. \tag{4.9}$$

When the condition $Y(q)$ holds at every point the operator $\Box_{b;q}$ is hypoelliptic and admits a parametrix in $\Psi_H^{-2}(M, \Lambda^{0,q})$ and then $S_{b;q}$ is a smoothing operator and $N_{b;q}$ is a Ψ_HDO of order -2. Therefore, using (4.8) we see that if the condition $Y(q+1)$ (resp. $Y(q-1)$) holds everywhere then $\Pi_0(\overline{\partial}_{b;q})$ (resp. $\Pi_0(\overline{\partial}^*_{b;q})$) is a Ψ_HDO.

Furthermore, in view of (4.7) we also see that if at every point the condition $Y(q)$ fails, but the conditions $Y(q-1)$ and $Y(q+1)$ hold, then the Szegö projection $S_{b;q}$ is a zero'th order Ψ_HDO projection. Notice that this may happen if, and only if, M is κ-strictly pseudoconvex with $\kappa = q$ or $\kappa = n - q$.

Bearing all this in mind we have:

Theorem 4.2. *The following noncommutative residues are CR diffeomorphism invariants of M:*

(i) $\operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$ *when the condition $Y(q+1)$ holds everywhere;*

(ii) $\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;q})$ *when the condition $Y(q-1)$ holds everywhere;*

(iii) $\operatorname{Res} S_{b;\kappa}$ *and* $\operatorname{Res} S_{b;n-\kappa}$ *when M is κ-strictly pseudoconvex.*

In particular, they depend neither on the choice of the line bundle $\mathcal{N}$, nor on that of the Hermitian metric h.

Proof. Let us first show that the noncommutative residues in (i) and (ii) don't depend on the metric h. As the range of $\Pi_0(\overline{\partial}_{b;q})$ and the kernel of $\Pi_0(\overline{\partial}^*_{b;q})$ are $\ker\overline{\partial}_{b;q}$ and $(\ker\overline{\partial}^*_{b;q})^\perp = \operatorname{im}\overline{\partial}_{b;q}$ they don't depend on h. Therefore, if the $Y(q+1)$ holds everywhere then $\Pi_0(\overline{\partial}_{b;q})$ is a Ψ_HDO projection whose range is independent of h, so the same its true for $\operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$ by Lemma 3.2. Similarly, when the condition $Y(q-1)$ holds everywhere the value of $\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;q})$ is also independent of the choice of the Hermitian metric.

Next, let $\mathcal{N}'$ be a supplement of H in TM and let h' be a Hermitian metric on $T_{\mathbb{C}}M$ which commutes with complex conjugation and makes the splitting (4.2) associated to $\mathcal{N}'$ becomes orthogonal. We shall assign the superscript $'$ to objects associated to the data $(\mathcal{N}', h')$.

Let X_0' be the section of $\mathcal{N}'$ such that $\theta(X_0') = 1$ and let $\varphi = \varphi_{X_0,X_{0'}}$ be the vector bundle isomorphism of $T_{\mathbb{C}}M$ onto itself such that φ is identity on $T_{1,0}\oplus T_{0,1}$ and $\varphi(X_0) = X_0'$. Since $\Pi_0(\overline{\partial}_{b;q})$ and $\Pi_0(\overline{\partial}^*_{b;q})$ don't depend on the choice of h' we may assume that $h' = \varphi_* h$, so that φ is a unitary isomorphism from $(T_{\mathbb{C}}M, h)$ onto $(T_{\mathbb{C}}, h')$ and φ^t is a unitary vector bundle isomorphism from $\Lambda^{0,q}$ onto $\Lambda'^{0,q}$.

Assume that the condition $Y(q+1)$ holds everywhere. Thanks to Lemma 4.1 we know that $\overline{\partial}'_{b;q}$ and $\varphi^t\overline{\partial}_{b;q}(\varphi^t)^{-1}$ agree. Since φ^t is unitary we also see that $\overline{\partial}^{*'}_{b;q}$ and $\varphi^t\overline{\partial}^*_{b;q}(\varphi^t)^{-1}$ too agree and so we have $\Box'_{b;q} = \varphi^t\overline{\partial}^*_{b;q}(\varphi^t)^{-1}$. Combining this with (4.8) we see that the projections $\Pi_0(\overline{\partial}'_{b;q})$ and $\varphi\Pi_0(\overline{\partial}_{b;q})(\varphi^t)^{-1}$ agree and so they have same noncommutative residue.

On the other hand, we have $\operatorname{Res}\varphi\Pi_0(\overline{\partial}_{b;q})(\varphi^t)^{-1} = \operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$, since Res is a trace. Hence $\operatorname{Res}\Pi_0(\overline{\partial}'_{b;q}) = \operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$, that is, the value of $\operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$ does not depend on $\mathcal{N}$. In the same way we can show that when the condition $Y(q-1)$ holds everywhere the residue $\operatorname{Res}\Pi_0(\overline{\partial}^*_{b;q})$ is independent of the choice made for $\mathcal{N}$.

Now, let $\phi : M \to M'$ be a CR diffeomorphism from M onto a CR manifold M'. Let $\mathcal{N}'$ be a supplement of H in TM and let h' be a Hermitian metric on $T_{\mathbb{C}}M'$ which commutes with complex conjugation and makes the splitting (4.2) of $T_{\mathbb{C}}M'$ associated to $\mathcal{N}'$ becomes orthogonal. We will assign the superscript $'$ to objects related to M'.

Since the values of the noncommutative residues (i)–(ii) related to M' are independent of the data $(\mathcal{N}, h)$, we may assume that $\mathcal{N} = \phi_* \mathcal{N}$ and $h' = \phi_* h$, so that ϕ gives rise to a unitary isomorphism from $L^2(M, \Lambda^{0,q})$ onto $L^2(M', \Lambda'^{0,q})$. As the fact that ϕ is a CR diffeomorphism implies that $\phi_* \overline{\partial}_{b;q} = \overline{\partial}'_{b;q}$, we see that $\Pi_0(\overline{\partial}'_{b;q}) = \phi_* \Pi_0(\overline{\partial}_{b;q})$. Therefore, if the condition $Y(q+1)$ holds everywhere then we have $\operatorname{Res} \Pi_0(\overline{\partial}'_{b;q}) = \operatorname{Res} \phi_* \Pi_0(\overline{\partial}_{b;q}) = \operatorname{Res} \Pi_0(\overline{\partial}_{b;q})$.

Similarly, when the condition $Y(q+1)$ holds everywhere we see that $\operatorname{Res} \Pi_0({\overline{\partial}^*_{b;q}}')$ and $\operatorname{Res} \Pi_0(\overline{\partial}^*_{b;q})$ agree. Thus, the noncommutative residues (i) and (ii) are CR diffeomorphisms invariants.

Finally, assume that M is κ-strictly pseudoconvex and that $q = \kappa$ or $q = n - \kappa$. At every point of M the condition $Y(q)$ fails, but the condition $Y(q-1)$ and $Y(q+1)$ hold, so $S_{b;q}$ is a Ψ_HDO projection and $\operatorname{Res} \Pi_0(\overline{\partial}_{b;q})$ and $\operatorname{Res} \Pi_0(\overline{\partial}^*_{b;q})$ are CR diffeomorphism invariants. Since (4.7) implies that $\operatorname{Res} S_{b;q} = \operatorname{Res} \Pi_0(\overline{\partial}_{b;q}) + \operatorname{Res} \Pi_0(\overline{\partial}^*_{b;q})$, it follows that $\operatorname{Res} S_{b;q}$ too is an invariant of the CR diffeomorphism class of M. The proof is thus achieved. □

Theorem 4.2 allows us to get CR invariants for CR manifolds that are not necessarily strictly pseudoconvex or have not necessarily a nondegenerate Levi form. However, specializing it to the strictly pseudoconvex case yields:

Theorem 4.3. *Suppose that M is a compact strictly pseudoconvex CR manifold. Then $\operatorname{Res} S_{b;k}$, $k = 0, n$, and $\operatorname{Res} \Pi_0(\overline{\partial}_{b;q})$, $q = 1, \ldots, n-1$, are CR diffeomorphism invariants of M. In particular, when M is the boundary of a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ they give rise to biholomorphism invariants of D.*

Finally, we can get further CR invariants by using the $\overline{\partial}_b$-complex with coefficients in a CR holomorphic vector bundle as follows.

A complex vector vector bundle $\mathcal{E}$ over M is a CR holomorphic vector bundle when there exists a patching of trivializations such that the transition maps are given by invertible matrices with CR function entries. For $q = 0, \ldots, n$ let $\Lambda^{0,q}(\mathcal{E}) = \Lambda^{0,q} \otimes \mathcal{E}$. Then there exists a unique first order differential operator $\overline{\partial}_{b,\mathcal{E}} : C^\infty(M, \Lambda^{0,*}(\mathcal{E})) \to C^\infty(M, \Lambda^{0,*+1}(\mathcal{E}))$ such that, for any local CR frame $e_1, \ldots, e_r$ of $\mathcal{E}$ and any local section $\omega = \sum \omega_i \otimes e_i$ of $\Lambda^{0,q}(\mathcal{E})$, we have

$$\overline{\partial}_{b,\mathcal{E}} \omega = \sum_i (\overline{\partial}_b \omega_i) \otimes e_i. \tag{4.10}$$

We have $\overline{\partial}^2_{b,\mathcal{E}} = 0$ and the Leibniz's rule holds, i.e., we have

$$\overline{\partial}_{b,\mathcal{E}}(\eta \wedge \omega) = (\overline{\partial}_b \eta) \wedge \omega + (-1)^q \eta \wedge \overline{\partial}_{b,\mathcal{E}} \omega, \tag{4.11}$$

for any $(0,q)$-form η and section ω of $\Lambda^{0,*}(\mathcal{E})$. Thus this yields a chain complex called the $\overline{\partial}_b$-complex with coefficients in $\mathcal{E}$.

We equip $\mathcal{E}$ with a Hermitian metric and let $\Box_{b,\mathcal{E}} = \overline{\partial}^*_{b,\mathcal{E}} \overline{\partial}_{b,\mathcal{E}} + \overline{\partial}_{b,\mathcal{E}} \overline{\partial}^*_{b,\mathcal{E}}$ be the Kohn Laplacian with coefficients in $\mathcal{E}$. It follows from (4.10) that in any CR trivialization $\Box_{b,\mathcal{E}}$ has the same principal symbol as $\Box_b \otimes 1_{\mathcal{E}}$, so its principal symbol is invertible if, and only if, the condition $Y(q)$ holds. Therefore, we can define the Szegö projection $S_{b,\mathcal{E};q}$ and the projections $\Pi_0(\overline{\partial}_{b,\mathcal{E};q})$ and $\Pi_0(\overline{\partial}^*_{b,\mathcal{E};q})$ as before and we see that:

- $\Pi_0(\overline{\partial}_{b,\mathcal{E};q})$ is a zero'th order projection under condition $Y(q+1)$;

- $\Pi_0(\overline{\partial}^*_{b,\mathcal{E};q})$ is a zero'th order projection under condition $Y(q-1)$;

- $S_{b,\mathcal{E};q}$ is a smoothing operator under the condition $Y(q)$, but it's a zero'th order projection when M is κ-strictly pseudoconvex and $q=\kappa$ or $q=n-\kappa$.

Then it is not difficult to modify the proof of Theorem 4.2 to get:

Theorem 4.4. *The following residues depend only on the CR diffeomorphism class of M and of the CR holomorphic bundle isomorphism class of $\mathcal{E}$:*

(i) $\operatorname{Res}\Pi_0(\overline{\partial}_{b,\mathcal{E};q})$ *when the condition $Y(q+1)$ holds everywhere;*

(ii) $\operatorname{Res}\Pi_0(\overline{\partial}^*_{b,\mathcal{E};q})$ *when the condition $Y(q-1)$ holds everywhere;*

(iii) $\operatorname{Res} S_{b,\mathcal{E};\kappa}$ *and* $\operatorname{Res} S_{b,\mathcal{E};n-\kappa}$ *when M is κ-strictly pseudoconvex.*

In particular, their values depend neither on the choice of the line bundle $\mathcal{N}$, nor on that of the Hermitian metrics on $T_{\mathbb{C}}M$ and $\mathcal{E}$.

4.2. **Invariance by deformation of the CR structure.** We now look at the behavior of the CR invariants under deformations of the CR structure. For sake of simplicity the results are proved for the invariants of Theorem 4.2, but they can be extended to the invariants of Theorem 4.4 with coefficients in a CR holomorphic vector bundle $\mathcal{E}$, provided that we consider deformations of the CR holomorphic structure of $\mathcal{E}$ compatible with the deformation of the CR structure of M.

We shall focus on deformations of the CR structure given by the deformation of the complex structure of H, that is, families $(J_t)_{t\in\mathbb{R}}\subset C^\infty(M,\operatorname{End}_{\mathbb{R}}H)$ such that for any $t\in\mathbb{R}$ we have $J_t^2=-1$ and $T_{1,0,t}=\ker(J_t-i)$ is an integrable subbundle of $T_{\mathbb{C}}M$. When M is κ-strictly pseudoconvex this will allows us to deal with all the deformations of the CR structure, because deformations of the contact structure are always locally trivial.

Lemma 4.5. *The signature of the Levi form L_θ in (4.1) is invariant under deformations of the complex structure of H.*

Proof. First, observe that the rank L_θ is half that of $d\theta$. Indeed, let $X_1,\ldots,X_{2n}$ be a local frame of H such that $X_{n+j}=JX_j$ for $j=1,\ldots,n$ and set $Z_j=X_j-iX_{n+1}$, so that $Z_1,\ldots,Z_n$ is a local frame $T_{1,0}$. Let A be the matrix of L_θ with respect to $Z_1,\ldots,Z_n$. Then the matrix of $d\theta$ with respect to $X_1,\ldots,X_{2n}$ is $B=\begin{pmatrix}0 & -\frac{1}{2}A\\ \frac{1}{2}A & 0\end{pmatrix}$, which shows that $\operatorname{rk}d\theta=2\operatorname{rk}L_\theta$.

Next, let $(J_t)_{t\in\mathbb{R}}$ be a smooth family of complex structures on H and for each t let $L_{\theta,t}$ be the Levi form (4.1) on $T_{1,0,t}=\ker(J_t-i)$. For $j=1,\ldots,n$ let $Z_{j,t}=X_j-iJX_{n+j}$, where $X_1,\ldots,X_{2n}$ is a local frame of H as above, and let A_t denote the matrix of $L_{\theta,t}$ with respect to $Z_{1,t},\ldots,Z_{n,t}$. Since $\operatorname{rk}L_{\theta,t}$ is equal to $\frac{1}{2}\operatorname{rk}d\theta$, which does not depend on t, we see that the rank of $L_{\theta,t}$ and A_t is independent of t.

In addition, the polar decomposition of A_t takes the form $A_t=U_t|A_t|$ where U_t is a smooth family of purely imaginary antisymmetric matrices such that U_t is unitary on $\operatorname{im}A_t$. Because of that U_t has only the eigenvalues ±1 on $\operatorname{im}A_t$ and we have the splitting $\operatorname{im}A_t=\ker(U_t-1)\oplus\ker(U_t+1)$. As A_t agrees with $\pm|A_t|$ on $\ker(U_t\mp1)$, hence is definite there, we see that the numbers of positive (resp. negative) eigenvalues of A_t is equal to $\dim\ker(U_t-1)$ (resp. $\dim\ker(U_t+1)$) which does not depend on t.

Since A_t is the matrix of L_θ with respect to the frame $Z_{1,t},\ldots,Z_{n,t}$, it follows from all this that the signature of $L_{\theta,t}$ is independent of t. Hence the result. □

It follows from this lemma that the condition $Y(q)$ is invariant under deformations of the complex structure of H. We are now in position to prove:

Proposition 4.6. *The invariants (i)–(iii) of Theorem 4.2 are invariant under deformation of the complex structure of H, and there are invariant under a general deformation of the CR structure when M is κ-strictly pseudoconvex.*

Proof. We will prove the result for $\operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$ only since the proofs for the other residues follow along similar lines.

Let $(J_t)_{t\in\mathbb{R}} \subset C^\infty(M, \operatorname{End}_{\mathbb{R}} H)$ be a smooth family of complex structures on H. We construct a smooth family $(h_t)_{t\in\mathbb{R}}$ of admissible Hermitian metrics on $T_{\mathbb{C}}M$ as follows.

Let g be a Riemannian metric on H and let us extend it into the Hermitian metric h on $H \otimes \mathbb{C}$ such that, for any sections X_1, X_2, Y_1, Y_2 of H, we have

$$h(X_1 + iY_1, X_2 + iY_2) = g(X_1, X_2) + g(Y_1, Y_2) + i(g(Y_1, X_2) - g(X_1, Y_2)). \tag{4.12}$$

Notice that h commutes with complex conjugation.

Let X_0 be the global section of $\mathcal{N}$ such that $\theta(X_0) = 1$. Then for $t \in \mathbb{R}$ we let h_t denote the Hermitian metric on $T_{\mathbb{C}}M$ such that, for sections Z, W of $H \otimes \mathbb{C}$ and functions λ, μ on M, we have

$$h_t(Z + \lambda X_0, W + \mu X_0) = h(Z, W) + h(J_t Z, J_t W) + \lambda\overline{\mu}, \tag{4.13}$$

This metric commutes with complex conjugation. Moreover, as J_t is unitary with respect to $h_{t|_H}$ the subbundles $T_{1,0;t} = \ker(J_t - i)$ and $T_{0,1;t} = \ker(J_t + i)$ are perpendicular to each other with respect to h_t, and so the splitting $T_{1,0;t} \oplus T_{0,1;t} \oplus (\mathcal{N} \otimes \mathbb{C})$ is orthogonal with respect to h_t. Thus $(h_t)_{t\in\mathbb{R}}$ is a smooth family of admissible Hermitian metrics on $T_{\mathbb{C}}M$.

We will use the subscript t to denote operators related to the Hermitian metric h_t and the CR structure defined by J_t. In addition, we extend J_t into a section of $\operatorname{End} T_{\mathbb{C}}M$ such that $J_t X_0 = 0$. Then J_t commutes with its adjoint $J_t^* = -J_t$ with respect to h_t and the orthogonal projection $\pi_{0,1;t}$ onto $T_{0,1;t} = \ker(J_t + i)$ gives rise to a smooth family with values in $C^\infty(M, \operatorname{End} T_{\mathbb{C}}M)$, for we have $\pi_{0,1,t} = \frac{1}{2i\pi}\int_{|\lambda+i|=1/2}(\lambda - J_t)^{-1}d\lambda$. Therefore, the operator $\overline{\partial}_{b,t;q} = \pi_t^{0,q} \circ d$, its adjoint $\overline{\partial}_{b,t;q}^*$ and the Kohn Laplacian $\Box_{b,t;q}$ give rise to smooth families of differential operators, hence their principal symbols depend smoothly on t.

Assume now that the condition $Y(q+1)$ holds everywhere. Then $(\sigma_2(\Box_{b,t;q+1}))_{t\in\mathbb{R}}$ is is a smooth family of symbols which are invertible for every t, so the results of [40], Chap. 3, imply that the family inverses $(\sigma_2(\Box_{b,t;q+1})^{*-1})_{t\in\mathbb{R}}$ is a smooth family of symbols. Since the principal symbol of $N_{b,t;q+1}$ is $\sigma_2(\Box_{b,t;q+1})^{*-1}$, using (4.8) we see that the principal symbol of $\Pi_0(\overline{\partial}_{b,t;q})$ depends smoothly on t. It then follows from Lemma 3.7 that $\operatorname{Res}\Pi_0(\overline{\partial}_{b,t;q})$ is independent of t. Hence $\operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$ is invariant under deformations of the complex structure of H.

Next, assume that M is κ-strictly pseudoconvex and consider a deformation of the CR structure defined by a smooth family of contact forms $(\theta_t)_{t\in\mathbb{R}}$ (so that $d\theta_t$ is nondegenerate on $H_t = \ker\theta_t$) together with the datum for each $t \in \mathbb{R}$ of integrable complex structure on H_t depending smoothly on t.

Let $t_0 \in \mathbb{R}$. As $(\theta_t)_{t\in B}$ is a deformation of the contact structure of M, by a result of Gray [26], Sect. 5.1, there exists an open interval I containing t_0 and a smooth family $(\phi_t)_{t\in I}$ of diffeomorphisms of M onto itself such that $\phi_t^* H_t = H_{t_0}$. In addition, for $t \in \mathbb{R}$ set $T_{1,0;t} = \ker(J_t + i)$ and for $t \in I$ set $T'_{1,0;t} = \ker(\phi_t^* J_t + i)$.

For $q\neq \kappa, n-\kappa$ let us denote $\operatorname{Res}\Pi_{0;t}(\overline{\partial}_{b,t;p,q})$ (resp. $\operatorname{Res}\Pi'_{0;t}(\overline{\partial}'_{b,t;p,q})$) the invariant (i) from Theorem 4.2 in bidegree $(0,q)$ associated to the CR structure defined by $T_{1,0;t}$ (resp. $T'_{1,0;t}$).

As ϕ_t is a CR diffeomorphism from $(M,T'_{1,0;t})$ onto $(M,T_{1,0;t})$, by Theorem 4.2 we have $\operatorname{Res}\Pi_0(\overline{\partial}_{b;q})'_t = \operatorname{Res}\Pi_{0;t}(\overline{\partial}_{b,t;q})$. Observe also that $(\phi_t^*J_t)$ is a smooth deformation of the complex structure of H_{t_0}, so by the first part of the proof we have $\operatorname{Res}\Pi_{0;t_0}(\overline{\partial}_{b,t_0;q}) = \operatorname{Res}\Pi_0(\overline{\partial}_{b;q})'_t = \operatorname{Res}\Pi_{0;t}(\overline{\partial}_{b,t;q})$. Hence $\operatorname{Res}\Pi_{0;t}(\overline{\partial}_{b,t;q})$ is independent of t. This proves that $\operatorname{Res}\Pi_0(\overline{\partial}_{b;q})$ is invariant under deformations of the CR structure. □

4.3. **Computation of the invariants.** Let us now make some comments about the computation the densities $c_\Pi(x)$ whose integrals yield the invariants $\operatorname{Res}\Pi$ from Theorems 4.2 and 4.2 (here Π denotes any of the Ψ_HDO projection involved in these Theorems).

As explained in Introduction the computation of the densities $c_\Pi(x)$ is interesting even if $\operatorname{Res}\Pi$ may vanish, because it could provide us with geometric information about the logarithmic singularity of the kernel of the geometric projection Π. However, the direct computation of $c_\Pi(x)$ in local coordinates is rather involved: it amounts to determine the symbol of degree $-(2n+2)$ of Π, so that we have more and more terms to compute as the dimension increases. Therefore, we need additional tools to deal with the computation.

When the bundle $\mathcal{E}$ is trivial and the CR manifold M is strictly pseudoconvex and endowed with the Levi metric defined by a pseudohermitian contact form θ, we can extend the arguments of [9] to show that the densities $c_\Pi(x)$ are of the form $\tilde{c}_\Pi(x)d\theta^n\wedge\theta$, where $\tilde{c}_\Pi(x)$ is a local pseudohermitian invariant of weight $n+1$. This means that $\tilde{c}_\Pi(x)$ is a universal polynomial in complete contractions of the covariant derivatives of the curvature and torsion tensors of the Tanaka-Webster connection and the polynomial is homogeneous of degree $-(n+1)$ under scalings $\theta\rightarrow\lambda\theta$, $\lambda>0$, of the pseudohermitian contact form. Thus the residues $\operatorname{Res}\Pi$ are *geometric* global CR invariants. In dimension 3 there are no non-zero such invariants (see [10]), but to date there no known obstruction to the existence of global geometric CR invariant.

In conformal geometry the conjecture of Deson-Swimmer [17], partially proved by Alexakis ([1], [2]), predicts the local form of the Riemannian invariants whose integrals yield global conformal invariants. It would be very interesting to prove an analogue of this conjecture in CR geometry, but to date it is not even clear what could be the conjecture, so we cannot use it to predict the form of the densities $c_\Pi(x)$. However, the computation of some $c_\Pi(x)$ by other means would certainly shed some light on some of the pseudohermtian invariants that should enter in the conjecture.

On the other hand, in the case of the Szegö projection $S_{b,0}$ on functions the density $c_{S_{b,0}}(x)$ is not a CR invariant, but it transforms conformally under the conformal changes $\theta\rightarrow e^{2f}\theta$ of pseudohermitian contact forms that come from CR pluriharmonic functions f, i.e., functions that are locally real parts of CR functions. Therefore, it would be natural to try to extend the CR invariant theory of [7] and [31] to deal with this class of invariants and to get information about the logarithmic singularity of the Szegö kernel. It seems that Hirachi [33] has made recent progress in this direction.

We could like to suggest another approach which comes in naturally with the framework of the paper and would allows us to deal with invariants with coefficients in CR holomorphic vector bundles as well. Namely, it would be natural to make use of a version of rescaling of Getzler [23] to simplify the computation of the noncommutative residues densities. The latter is a powerful trick which, by taking into account the supersymmetry of the Dirac operator, allows us to get a short proof of the small time convergence to the Atiyah-Singer integrand of the local supertrace of the heat kernel of the square of the Dirac operator. This bypasses the invariant theory of [4] and [25] and provides us with a purely analytical proof of the Atiyah-Singer index theorem for Dirac operators.

Let M be a strictly pseudoconvex CR manifold endowed with the Levi metric defined by pseudohermitian contact form θ and let $\mathcal{E}$ be a Hermitian CR vector bundle over M. It is believed that implementing a version of Getzler's rescaling into the Heisenberg calculus would allow us to compute the density:

$$\operatorname{Str}_{\Lambda^{0,*}(\mathcal{E})} c_{S_{b,\mathcal{E}}}(x) = c_{S_{b,\mathcal{E};0}}(x) + (-1)^n \operatorname{Tr}_{\Lambda^{0,n}(\mathcal{E})} c_{S_{b,\mathcal{E};n}}(x), \tag{4.14}$$

where $\operatorname{Str} := \sum_{q=0}^{n} (-1)^q \operatorname{Tr}_{\Lambda^{0,q}(\mathcal{E})}$ is the supertrace on $\Lambda^{0,*}(\mathcal{E})$ and $S_{b,\mathcal{E}}$ denotes the Szegö projection acting on all sections of $\Lambda^{0,*}(\mathcal{E})$.

It should be apparent at least from [38] that Getzler's rescaling techniques could be used in the setting of the Heisenberg calculus. The upshot is that the Getzler's rescaling would yield near any point of the manifold a refinement of the filtration of the Heisenberg calculus, so that determining $\operatorname{Str} c_{S_{b,\mathcal{E}}}(x)$ would boil down to computing the second subleading symbol of $S_{b,\mathcal{E}}$ with respect to this new filtration. This would be *infinitely* better than to have to compute the symbol of order $-(2n+2)$ in the usual sense of the Heisenberg calculus. We expect to carry out the explicit calculation in a future paper.

5. Invariants of generalized Szegö projections

Let (M^{2n+1}, H) be an orientable contact manifold, i.e., a Heisenberg manifold admitting a real 1-form θ, called contact form, such that θ annihilates H and $d\theta_{|_H}$ is nondegenerate. Given a contact form θ on M we let X_0 be the Reeb vector field of θ, i.e., the unique vector field X_0 such that $\iota_{X_0}\theta = 1$ and $\iota_{X_0} d\theta = 0$.

In addition, we let J be an almost complex structure on H which is *calibrated* in the sense that J preserves $d\theta_{|H}$ and we have $d\theta(X, JX) > 0$ for any non-vanishing section X of H. Extending J to TM by requiring to have $JX_0 = 0$, we then can equip TM with the Riemannian metric $g_{\theta,J} := d\theta(., J.) + \theta^2$.

In this context Szegö projections have been defined by Boutet de Monvel and Guillemin in [14] as FIOs with complex phase. This construction has been further generalized by Epstein-Melrose [19] as follows.

Let $\mathbb{H}^{2n+1}$ be the Heisenberg group of dimension $2n+1$ consisting of $\mathbb{R}^{2n+1}$ together with the group law,

$$x.y = (x_0 + y_0 + \frac{1}{2} \sum_{1 \leq j \leq n} (x_{n+j} y_j - x_j y_{n+j}), x_1 + y_1, \ldots, x_{2n} + y_{2n}). \tag{5.1}$$

Let $\theta^0 = dx_0 + \frac{1}{2}\sum_{j=1}^{n} (x_j dx_{n+j} - x_{n+j} dx_j)$ be the standard left-invariant contact form of $\mathbb{H}^{2n+1}$. Its Reeb vector field is $X_0^0 = \frac{\partial}{\partial x_0}$ and if for $j = 1, \ldots, n$ we let $X_j^0 = \frac{\partial}{\partial x_j} + \frac{1}{2} x_{n+j} \frac{\partial}{\partial x_0}$ and $X_{n+j}^0 = \frac{\partial}{\partial x_{n+j}} - \frac{1}{2} x_j \frac{\partial}{\partial x_0}$ then $X_1^0, \ldots, X_{2n}^0$ form a left-invariant frame of $H^0 = \ker \theta^0$. Note that for $j, k = 1, \ldots, n$ and $k \neq j$ we have

$$[X_j^0, X_{n+k}^0] = -\delta_{jk} X_0^0, \qquad [X_0^0, X_j^0] = [X_j^0, X_k^0] = [X_{n+j}^0, X_{n+k}^0] = 0. \tag{5.2}$$

The standard CR structure of $\mathbb{H}^{2n+1}$ is given by the complex structure J^0 on H^0 such that $J^0 X_j^0 = X_{n+j}^0$ and $J^0 X_{n+j} = -X_j^0$. It follows from (5.2) that J^0 is calibrated with respect to θ^0 and that $X_0^0, X_1^0, \ldots, X_{2n}^0$ form an orthonormal frame of $T\mathbb{H}^{2n+1}$ with respect to the metric g_{θ^0, J^0}.

The scalar Kohn Laplacian on $\mathbb{H}^{2n+1}$ is equal to

$$\Box_{b,0}^0 = -\frac{1}{2}((X_1^0)^2 + \ldots + (X_{2n}^0)^2) + i\frac{n}{2} X_0^0. \tag{5.3}$$

For $\lambda \in \mathbb{C}$ the operator $-\frac{1}{2}((X_1^0)^2 + \ldots + (X_{2n}^0)^2) + i\lambda X_0^0$ is invertible if, and only if, we have $\lambda \not\in \pm(\frac{n}{2} + \mathbb{N})$ (see [22], [8]). For $k = 0, 1, \ldots$ the orthogonal projection $\Pi_0(\Box_b + ikX_0^0)$ onto the kernel of $\Box_b + ikX_0^0$ is a left-invariant homogeneous Ψ_HDO of order 0 (see [8], Thm. 6.61). We let $s_k^0 \in S_0((\mathfrak{h}^{2n+1})^*)$ denote its symbol (where $(\mathfrak{h}^{2n+1})^*$ denotes the dual of the Lie algebra $\mathfrak{h}^{2n+1}$ of $\mathbb{H}^{2n+1}$). We then have $\Pi_0(\Box_b + ikX_0^0) = s_k^0(-iX^0)$.

Now, since the existence of a contact structure implies that the Levi form (2.1) of (M, H) is everywhere nondegenerate, the tangent Lie group bundle GM is a fiber bundle with typical fiber $\mathbb{H}^{2n+1}$ (see [39]). A local trivialization near a given point $a \in M$ is obtained as follows.

Let $X_1, \ldots, X_{2n}$ be a local orthonormal frame of H on an open neighborhood U of a and which is admissible in the sense that $X_{n+j} = JX_j$ for $j = 1, \ldots, n$. In addition, let $\underline{X_0(a)}$ denote the class of $X_0(a)$ in T_aM/H_a. Then as shown in [39] the map $\phi_{X,a} : (T_aM/H_a) \oplus H_a \to \mathbb{R}^{2n+1}$ such that

$$\phi_{X,a}(x_0 \underline{X_0(a)} + x_1 X_1(a) + \ldots + x_{2n} X_{2n}(a)) = (x_0, \ldots, x_{2n}), \qquad x_j \in \mathbb{R}, \tag{5.4}$$

gives rise to a Lie group isomorphism from G_aM onto $\mathbb{H}^{2n+1}$. In fact, as $\phi_{X,a}$ depends smoothly on a we get a fiber bundle trivialization of $GM|_U \simeq U \times \mathbb{H}^{2n+1}$.

For $j = 0, \ldots, 2n$ let X_j^a be the model vector field of X_j at a as defined in [39]. This is the unique left-invariant vector field on G_aM which, in the coordinates provided by $\phi_{X,a}$, agrees with $\frac{\partial}{\partial x_j}$ at $x = 0$. Therefore, we have $X_j^a = \phi_{X,a}^* X_j^0$ and so we get $\phi_{X,a}^* \Box_b^0 = -\frac{1}{2}((X_1^a)^2 + \ldots + (X_{2n}^a)^2) + i\frac{n}{2} X_0^a$.

If $\tilde{X}_1, \ldots, \tilde{X}_{2n}$ is another admissible orthonormal frame of H near a, then we pass from $(\tilde{X}_1^a, \ldots, \tilde{X}_{2n}^a)$ to $(X_1^a, \ldots, X_{2n}^a)$ by an orthogonal linear transformation, which leaves the expression $(X_1^a)^2 + \ldots + (X_{2n}^a)^2$ unchanged. Therefore, the differential operator $\Box_b^a := \phi_{X,a}^* \Box_b^0$ makes sense independently of the choice of the admissible frame $X_1, \ldots, X_{2n}$ near a.

On the other hand, as $\phi_{X,a}$ induces a unitary transformation from $L^2(G_aM)$ onto $L^2(\mathbb{H}^{2n+1})$ we have $\Pi_0(\Box_b^a + ikX_0^a) = \Pi_0(\phi_{X,a}^*(\Box_b^0 + ikX_0^0)) = \phi_{X,a}^* \Pi_0(\Box_b^0 + ikX_0^0))$. Hence $\Pi_0(\Box_b^a + ikX_0^a)$ is a zero'th order left-invariant homogeneous Ψ_HDO on G_aM with symbol $s_k^a(\xi) = \phi_{X,a}^* s_k^0(\xi) = s_k^0((\phi_{X,a}^{-1})^t \xi)$. In fact, since $\phi_{X,a}$ depends smoothly on a we obtain:

Proposition 5.1 ([19], Chap. 6)**.** *For $k = 0, 1, \ldots$ there is a uniquely defined symbol $s_k \in S_0(\mathfrak{g}^*M)$ such that, for any admissible orthonormal frame $X_1, \ldots, X_d$ of H near a point $a \in M$, we have $s_k(a, \xi) = \phi_{X,a}^* s_k^0(\xi)$ for any $(a, \xi) \in \mathfrak{g}^*M \setminus 0$.*

We call s_k the *Szegö symbol at level k*. This definition *a priori* depends on the contact form θ and the almost complex structure J. As we shall now see changing θ or J has minor effects on s_k, but first we need the following.

Lemma 5.2. *The space of calibrated almost complex structures on H is path-connected.*

Proof. Let $J' \in C^\infty(M, \operatorname{End} H)$ be a calibrated almost complex structure on H and set $g_0 = g_{\theta,J}$ and $g_1 = g_{\theta,J'}$. In the sequel the transpose superscript refers to tranposition with respect to g_0. For any sections X and Y of H we have $g_1(X,Y) = d\theta(X, J'Y) = g_0(JX, J'Y) = g_0({J'}^t JX, Y)$. In particular, we see that $A_1 := {J'}^t J$ is a symmetric and positive definite section of $\operatorname{End} H$. Furthermore, as J' preserves $d\theta_{|H}$ we have $g_1(X,Y) = d\theta(X, J'Y) = -d\theta(J'X, Y) = -g_0(JJ'X, Y)$. Hence ${J'}^t J' = -JJ'$, which gives ${J'}^t = JJ'J$.

For $t \in [0,1]$ let $g_t = (1-t)g_0 + tg_1$. This is a smooth family of Riemannian metrics on TM. On H we have $g_t(X,Y) = g_0(A_t X, Y)$ with $A_t = (1-t).1 + t{J'}^t J$ and we can write $d\theta(X,Y) = g_0(JX, Y) = g_t(B_t X, Y)$ with $B_t = A_t^{-1} J$. Notice that B_t is antisymmetric with respect to g_t. Moreover, for $t = 0$ we have $B_0 = J$ and since ${J'}^t = JJ'J$ for $t = 1$ we have $B_1 = ({J'}^t J)^{-1} J = J{J'}^t J = J'$.

Since B_t is antisymmetric with respect to g_t its modulus and its phase with respect to g_t are $|B_t|_t = \sqrt{-B_t^2}$ and $J_t = B_t(\sqrt{-B_t^2})^{-1}$. Thus $(J_t)_{t\in[0,1]}$ is a smooth path in $C^\infty(M, \operatorname{End} H)$ such that J_t is orthogonal with respect to g_t for any $t \in [0,1]$. Notice that $B_0 = J$ is already orthogonal with respect to g_0, so we have $J_0 = J$. Similarly, we have $J_1 = J'$. As B_t is antisymmetric with respect to g_t the same is true for J_t. Together with the orthogonality this implies that we have $J_t^{-1} = -J_t$, i.e., J_t is an almost complex structure on H. Moreover, for sections X and Y of H with X non-vanishing, we have $d\theta(X, J_t X) = g_t(B_t X, J_t X) = g_t(J_t^{-1} B_t X, X) = g_t(|B_t|_t X, X) > 0$ and $d\theta(J_t X, J_t Y) = g_t(B_t J_t X, J_t Y) = g_t(J_t B_t X, J_t Y) = g_t(B_t X, Y) = d\theta(X,Y)$. Therefore $(J_t)_{t\in[0,1]}$ is a smooth path of calibrated almost complex structures on H connecting J to J'. Hence the lemma. □

Granted this we can now prove:

Lemma 5.3 ([19], Chap. 6)**.** *(i) s_k is invariant under conformal changes of contact form.*

(ii) The change $(\theta, J) \to (-\theta, -J)$ transforms s_k into $s_k(x, -\xi)$.

(iii) s_k depends on J only up to homotopy of idempotents in $S_0(\mathfrak{g}^*M)$.

Proof. Throughout the proof we let $X_1, \ldots, X_{2n}$ be an admissible orthonormal frame of H near a point $a \in M$.

Let θ' be a contact form which is conformal to θ, that is, $\theta' = e^{-2f}\theta$ with $f \in C^\infty(M, \mathbb{R})$, and let s'_k be the Szegö symbol at level k with respect to θ' and J. For $j = 1, \ldots, 2n$ let $X'_j = e^f X_j$. Then $X'_1, \ldots, X'_{2n}$ is an admissible orthonormal frame of H with respect to $g_{\theta',J|_H} = e^{2f} d\theta(., J.)$. Moreover, as the Reeb vector field of θ' is such that $X'_0 = e^{2f} X_0 \bmod H$, we have $\underline{X'_0}(a) = e^{2f(a)} \underline{X_0}(a)$. Thus,

$$\begin{aligned}(5.5)\quad \phi^{-1}_{X',a}(x_0, \ldots, x_{2n}) &= x_0 \underline{X'_0}(a) + x_1 X'_1(a) + \ldots + x_{2n} X'_{2n}(a) \\ &= x_0 \lambda^2 \underline{X_0}(a) + x_1 \lambda X_1(a) + \ldots + x_{2n} \lambda X_{2n}(a)) = \phi^{-1}_{X,\alpha} \circ \delta_\lambda(x_0, \ldots, x_{2n})\end{aligned}$$

where $\lambda = e^{f(a)}$ and $\delta_\lambda(x) = \lambda.x$ for any $x \in \mathbb{H}^{2n+1}$.

On the other hand, as s_k^0 is homogeneous of degree 0 we have $\delta_{\lambda*}s_k^0 = s_k^0$. Therefore, we get $s'_k(a,.) = \phi^*_{X',a}s_k^0 = \phi^*_{X,a}\delta_{\lambda*}s_k^0 = \phi^*_{X,a}s_k^0 = s_k(a,.)$. Hence s_k is invariant under conformal changes of contact form.

Let s'_k be the Szegö symbol at level k with respect to $-\theta$ and $-J$. Define $X'_0 = -X_0$ and for $j = 1,\ldots,n$ let $X'_j = X_j$ and $X'_{n+j} = -X_{n+j}$. Then X'_0 is the Reeb vector field of $-\theta$ and $X'_1,\ldots,X'_{2n}$ form an admissible orthonormal frame with respect to $g_{-\theta,-J}$. Moreover, we have $\phi_{X',a} = \tau \circ \phi_{X,a}$, where we have let $\tau(x) = (-x_0, x_1, \ldots, x_n, -x_{n+1}, \ldots, x_{2n})$. Hence $s'_k(a,.) = \phi_{X',a}s_k^0 = \phi^*_{X,a}\tau^* s_k^0$.

We have $\tau^*[s_k^0(-iX^0)] = \tau^*\Pi_0(\Box_b^0 + ikX_0^0) = \Pi_0(\tau^*(\Box_b^0 + ikX_0^0))$, since the action of τ on $L^2(\mathbb{H}^{2n+1})$ is unitary. Moreover, as $\tau^* X_j^0$ is equal to X_j^0 if $j = 1,\ldots,n$ and to $-X_j^0$ otherwise, we see that $\tau^*(\Box_b^0 + ikX_0^0)$ is equal to

$$-\frac{1}{2}((X_1^0)^2 + \ldots + (X_{2n}^0)^2) - i(\frac{n}{2} + k)X_0^0 = (\Box_b^0 + ikX_0^0)^t. \tag{5.6}$$

Thus $\tau^*[s_k^0(-iX^0)] = \Pi_0((\Box_b^0 + ikX_0^0)^t) = [\Pi_0(\Box_b^0 + ikX_0^0)]^t = s_k^0(-iX^0)^t$, using (2.12) we obtain $(\tau^* s_k^0)(\xi) = s_k^0(-\xi)$. Hence we have $s'_k(a,\xi) = (\phi^*_{X,a}\tau^* s_k^0)(\xi) = \phi^*_{X,a}(s_k^0(-\xi)) = s_k(a,-\xi)$.

Let J' be another almost complex structure on H calibrated with respect to θ and let s'_k be the Szegö symbol at level k with respect to θ and J'. Then by Lemma 5.2 there exists a smooth path $(J_t)_{0\leq t\leq 0}$ of calibrated almost complex structures such that $J_0 = J$ and $J_1 = J'$. For $t \in [0,1]$ let $X_{j,t} = X_j$ if $j = 0,1,\ldots,n$ and $X_{j,t} = J_t X_j$ otherwise. Then $X_{1,t},\ldots,X_{2n,t}$ is an admissible orthonormal frame of H with respect to g_{θ,J_t} and the isomorphism $\phi_{X_t,a} : G_aM \to \mathbb{H}^{2n+1}$ depends smoothly on a and t. Therefore, $s_{k,t}(a,\xi) = \phi^*_{X_t,a}s_k^0(\xi)$ a smooth path in $S_0((\mathfrak{h}^{2n+1})^*)$ connecting s_k to s'_k. The proof is now complete. ☐

From now on we let $\mathcal{E}$ be a Hermitian vector bundle over M.

Definition 5.4 ([19], Chap. 6)**.** For $k = 0,1,\ldots$ a generalized Szegö projection at level k is a Ψ_HDO projection $S_k \in \Psi_H^0(M,\mathcal{E})$ with principal symbol $s_k \otimes \mathrm{id}_{\mathcal{E}}$.

Generalized Szegö projections at level k always exist (see [19] and Lemma A.6). Moreover, when $k = 0$ and $\mathcal{E}$ is the trivial line bundle the above definition allows us to recover the Szegö projections of [14], for we have:

Lemma 5.5. *Let $S : C^\infty(M) \to C^\infty(M)$ be a Szegö projection in the sense of* [14]*. Then S is a generalized Szegö projection at level* 0*.*

Proof. We saw in Section 2 that S is a zero'th order Ψ_HDO. Moreover, if $q(x,y)$ is the complex phase of S then it follows from (2.18) that at a point $a \in M$ the model operator S^a of S in the sense of [40], Def. 3.2.7, is a Szegö projection, whose complex phase is given by the leading term at $x = a$ in (2.18). In particular, under the identification $G_aM \simeq \mathbb{H}^{2n+1}$ provided by a map $\phi_{X,a}$ as in (5.5) we see that $(\phi_{X,a})_*S^a$ is a Szegö projection on $\mathbb{H}^{2n+1}$. In fact, as $(\phi_{X,a})_*S^a$ is left-invariant and homogeneous this is the Szegö projection $\Pi_0(\Box_b)$ considered above, so that $(\phi_{X,a})_*S^a$ has symbol s_0^0. Since by definition S^a has symbol $\sigma_0(S)(a,.)$ we see that $\sigma_0(S) = s_0$. Hence S is a generalized Szegö projection at level 0. ☐

In particular, when M is strictly pseudoconvex the Szegö projection $S_{b,0}$ is a generalized Szegö projection at level 0.

If S_k and S'_k are generalized Szegö projection at level k in $\Psi_H^0(M,\mathcal{E})$ then they have same principal symbol, so by Lemma 3.7 they have same noncommutative residue. We then let

$$L_k(M,\mathcal{E}) := \operatorname{Res} S_k. \tag{5.7}$$

Recall that the K-group $K^0(M)$ can be described as the group of formal differences of stable homotopy classes of (smooth) vector bundles over M, where a stable homotopy between vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ is given by an auxiliary vector bundle $\mathcal{F}$ and a vector bundle isomorphism $\phi : \mathcal{E}_1 \oplus \mathcal{F} \simeq \mathcal{E}_2 \oplus \mathcal{F}$. Then we have:

Theorem 5.6. *$L_k(M,\mathcal{E})$ depends only on the Heisenberg diffeomorphism class of M and on the K-theory class of $\mathcal{E}$, hence is invariant under deformations of the contact structure. In particular, $L_k(M,\mathcal{E})$ depends neither on the contact form θ, nor on the almost complex structure J.*

Proof. Throughout the proof we let $S_k \in \Psi_H^0(M,\mathcal{E})$ be generalized Szegö projection at level k, so that $L_k(M,\mathcal{E}) = \operatorname{Res} S_k$.

Let us first show that $L_k(M,\mathcal{E})$ is independent from θ and J. To this end let θ' be a contact form on M, let J' be an almost complex structure on H calibrated with respect to θ' and let $S'_k \in \Psi_H^0(M,\mathcal{E})$ be a generalized Szegö projection at level k with respect to θ' and J'.

If θ and θ' are in the same conformal class, then by Lemma 5.3 the principal symbols of S_k and S'_k are homotopic, so by Lemma 3.7 we have $\operatorname{Res} S'_k = \operatorname{Res} S_k$.

Let $\tau : \mathcal{E} \to \mathcal{E}^*$ be the antilinear isomorphism provided by the Hermitian metric of $\mathcal{E}$ and define $S''_k = \tau^{-1} S_k^t \tau$. Then S''_k is a Ψ_HDO projection and by (2.12) its principal symbol is $\tau^{-1}(s_k(x,-\xi) \otimes \mathrm{id}_{\mathcal{E}^*})\tau = s_k(x,-\xi) \otimes \mathrm{id}_{\mathcal{E}}$. Therefore, by Lemma 5.3 this is a generalized Szegö projection at level k with respect to $-\theta$ and $-J$. Thus, if θ' is not in the conformal class of θ then it is in that of $-\theta$ and as above we get $\operatorname{Res} S'_k = \operatorname{Res} S''_k$. As $\operatorname{Res} S''_k = \operatorname{Res} S_k^t$ and by Lemma 3.3 we have $\operatorname{Res} S_k^t = \operatorname{Res} S_k$, we see that $\operatorname{Res} S'_k = \operatorname{Res} S_k$. Hence $L_k(M,\mathcal{E})$ does not depend on the choices of θ and J.

Next, let (M',θ') be a contact manifold together with a calibrated almost complex structure on $H' = \theta'$ such that there exists a Heisenberg diffeomorphism ϕ from (M',H') onto (M,H). Define $\mathcal{E}' = \phi^*\mathcal{E}$ and let $S'_k \in \Psi_{H'}^0(M',\mathcal{E}')$ be a generalized Ψ_HDO projection at level k with respect θ' and a calibrated almost complex structure J' on H'. Since $\operatorname{Res} S'_k$ depends neither on θ, nor on J, we may assume that we have $\theta' = \phi^*\theta$ and $J' = \phi^* J$.

By the results of [40], Sect. 3.2, the operator $\phi^* S_k$ is a projection in $\Psi_{H'}^0(M',\mathcal{E}')$ with principal symbol $\phi^* s_k(x,\xi) \otimes \mathrm{id}_{\mathcal{E}'}$, where $\phi^* s_k(x,\xi) = s_k(\phi(x), (\phi'_H(x)^{-1})^t \xi)$ and ϕ'_H is the the vector bundle isomorphism from $\mathfrak{g}M' = (TM'/H') \oplus H'$ onto $\mathfrak{g}M = (TM/H) \oplus H$ induced by ϕ'.

Let s'_k be the Szegö symbol on M' at level k with respect to θ' and J'. We claim that $s'_k = \phi^* s_k$. To see this let $X_1,\ldots,X_{2n}$ be an admissible orthonormal frame near a point $a \in M$ and for $j = 0,..,2n$ let $X'_j = \phi^* X_j$. Then X'_0 is the Reeb vector field of θ' and $X'_1,\ldots,X'_{2n}$ is an admissible orthonormal frame of H' near $a' = \phi^{-1}(a)$ with respect to $g_{\theta',J'} = \phi^* g_{\theta,J}$.

Moreover, by the results of [39] for $j = 0,1,\ldots,2n$ we have $(X'_j)^{a'} = \phi'_H(a)^* X_j^a$. Therefore, $\phi_{X',a'} = \phi_{X,a} \circ \phi'_H(a)$ and so $\phi^*_{X',a'} s_{k^0} = \phi'_H(a)^* \phi_{X,a} s_k^0 = (\phi^* s_k)(a,.)$. Hence $s'_k = \phi^* s_k$ as claimed.

It follows from this that $\phi^* S_k$ is a generalized Szegö projection at level k on M' with respect to θ' and J', so by the first part of the proof $\operatorname{Res} S'_k = \operatorname{Res} \phi_* S_k$. Since $\operatorname{Res} \phi_* S_k = \operatorname{Res} S_k$, we see that $\operatorname{Res} S'_k = \operatorname{Res} S_k$. Hence $L_k(M, \mathcal{E})$ is a Heisenberg diffeomorphism invariant of M.

Let us now prove that $L_k(M, \mathcal{E})$ is an invariant of the K-theory class of $\mathcal{E}$. Let ϕ be a vector bundle isomorphism from $\mathcal{E}$ onto a vector bundle $\mathcal{E}'$ over M and let $S'_k \in \Psi^0_H(M, \mathcal{E}')$ be a generalized Szegö projection of level k. Then $\phi_* S_k$ is a projection in $\Psi^0_H(M, \mathcal{E}')$ with principal symbol $s_k \otimes \mathrm{id}_{\mathcal{E}'}$, hence is a generalized Szegö projection of level k. Thus $\operatorname{Res} S'_k = \operatorname{Res} \phi_* S_k = \operatorname{Res} S_k$.

Next, for $j = 1, 2$ let $\mathcal{E}_j$ be a vector bundle over M and let $S_{k,\mathcal{E}_j} \in \Psi^0_H(M, \mathcal{E}_j)$ be a generalized Szegö projection at level k acting on the section of $\mathcal{E}_j$. In addition, let $S_{k,\mathcal{E}_1 \oplus \mathcal{E}_2} \in \Psi^0_H(M, \mathcal{E}_1 \oplus \mathcal{E}_2)$ be a generalized Szegö projection at level k acting on the section of $\mathcal{E}_1 \oplus \mathcal{E}_1$. Then $S_{k,\mathcal{E}_1} \oplus S_{k,\mathcal{E}_2}$ is a Ψ_HDO projection acting on the sections of $\mathcal{E}_1 \oplus \mathcal{E}_2$ with principal symbol $s_k \otimes \mathrm{id}_{\mathcal{E}_1 \oplus \mathcal{E}_2}$, hence is a generalized Szegö projection at level k. Thus $\operatorname{Res} S_{k,\mathcal{E}_1 \oplus \mathcal{E}_2} = \operatorname{Res}(S_{k,\mathcal{E}_1} \oplus S_{k,\mathcal{E}_2}) = \operatorname{Res} S_{k,\mathcal{E}_1} + \operatorname{Res} S_{k,\mathcal{E}_2}$, i.e., we have $L_k(M, \mathcal{E}_1 \oplus \mathcal{E}_2) = L_k(M, \mathcal{E}_1) + L_k(M, \mathcal{E}_2)$.

Bearing this is in mind, let $\mathcal{E}'$ be a (smooth) vector bundle in the K-theory class of $\mathcal{E}$, so that there exist an auxiliary vector bundle $\mathcal{F}$ and a vector bundle isomorphism ϕ from $\mathcal{E} \oplus \mathcal{F}$ onto $\mathcal{E}' \oplus \mathcal{F}$. Then we have $L_k(M, \mathcal{E} \oplus \mathcal{F}) = L_k(M, \mathcal{E}' \oplus \mathcal{F})$. As $L_k(M, \mathcal{E} \oplus \mathcal{F}) = L_k(M, \mathcal{E}) + L_k(M, \mathcal{F})$ and $L_k(M, \mathcal{E}' \oplus \mathcal{F}) = L_k(M, \mathcal{E}') + L_k(M, \mathcal{F})$, it follows that $L_k(M, \mathcal{E}) = L_k(M, \mathcal{E}')$. Hence $L_k(M, \mathcal{E})$ is an invariant of the K-theory class of $\mathcal{E}$.

Finally, since $L_k(M, \mathcal{E})$ is a Heisenberg diffeomorphism invariant of M arguing as in the proof of Proposition 4.6 shows its invariance under deformation of the contact structure of M. □

Remark 5.7. The almost complex structure J and the Reeb vector field X_0 give rise to splittings as in (4.2) and (4.3), so that (p, q)-forms make sense. We then can define generalized Szegö projections on $(0, q)$-forms with coefficients in a vector bundle $\mathcal{E}$. This can be done at any integer level $k = 0, 1, \ldots$, but for $1 \leq q \leq n - 1$ and $k \leq 2q - 1$ the corresponding Szegö symbol vanishes and we get a smoothing projection with vanishing noncommutative residue. Then arguing as in the proof of Theorem 5.6 shows that the corresponding noncommutative residues don't depend on the choice of the operator and yield contact invariants.

If (M, H) is CR, i.e., if J defines a complex structure on H, then M is strictly pseudoconvex, the contact form θ defines a pseudohermitian structure and $g_{\theta,J}$ is the associated Levi metric. Let $\mathcal{E}$ be a Hermitian CR vector bundle equipped with a compatible CR connection ∇ and let $\nabla^{0,q}_{X_0}$ denote the covariant derivative $\mathcal{L}_{X_0} \otimes 1_{\mathcal{E}} + 1_{\Lambda^{0,q}} \otimes \nabla_{X_0}$. We cannot make use of the formulas (4.7)–(4.8) to prove that the projections $\Pi_0(\Box_{b,\mathcal{E};q} + ik\nabla^{0,q}_{X_0})$ are Ψ_HDOs, but the arguments of [8], §25, can be extended to prove this result ([27]). Therefore, we get higher level versions of the invariants from Theorem 4.4. Furthermore, in this case it should be possible to apply the Getzler's rescaling techniques alluded to in Subsection 4.3 to similarly compute the supersymmetric densities,

$$\operatorname{Str}_{\Lambda^{0,*}(\mathcal{E})} c_{\Pi_0(\Box_{b,\mathcal{E}} + ik\nabla^{0,q}_{X_0})}(x) = \sum_{q=0}^{n} (-1)^q \operatorname{Tr}_{\Lambda^{0,q}(\mathcal{E})} c_{\Pi_0(\Box_{b,\mathcal{E};q} + ik\nabla^{0,q}_{X_0})}(x). \tag{5.8}$$

We hope to be able to deal with this computation in a subsequent paper.

6. Invariants from the contact complex

Let (M^{2n+1}, H) be an orientable contact manifold. Let θ be a contact form on M and let X_0 be its Reeb vector field of θ. We also let J be a calibrated almost complex structure on H and as in the previous section we endow TM with the Riemannian metric $g_{\theta,J} = d\theta(.,J.) + \theta^2$.

Observe that the splitting $TM = H \oplus \mathbb{R}X_0$ allows us to identify H^* with the annihilator of X_0 in T^*M. More generally, identifying $\Lambda^k_{\mathbb{C}}H^*$ with $\ker \iota_{X_0}$, where ι_{X_0} denotes the contraction operator by X_0, gives the splitting

$$\Lambda^*_{\mathbb{C}}TM = (\bigoplus_{k=0}^{2n} \Lambda^k_{\mathbb{C}}H^*) \oplus (\bigoplus_{k=0}^{2n} \theta \wedge \Lambda^k_{\mathbb{C}}H^*). \tag{6.1}$$

For any horizontal form $\eta \in C^\infty(M, \Lambda^k_{\mathbb{C}}H^*)$ we can write $d\eta = d_b\eta + \theta \wedge \mathcal{L}_{X_0}\eta$, where $d_b\eta$ is the component of $d\eta$ in $\Lambda^k_{\mathbb{C}}H^*$. This does not provide us with a complex, for we have $d_b^2 = -\mathcal{L}_{X_0}\varepsilon(d\theta) = -\varepsilon(d\theta)\mathcal{L}_{X_0}$ where $\varepsilon(d\theta)$ denotes the exterior multiplication by $d\theta$.

The contact complex of Rumin [43] is an attempt to get a complex of horizontal differential forms by forcing the equalities $d_b^2 = 0$ and $(d_b^*)^2 = 0$.

A natural way to modify d_b to get the equality $d_b^2 = 0$ is to restrict d_b to the subbundle $\Lambda^*_2 := \ker \varepsilon(d\theta) \cap \Lambda^*_{\mathbb{C}}H^*$, since the latter is closed under d_b and is annihilated by d_b^2.

Similarly, we get the equality $(d_b^*)^2 = 0$ by restricting d_b^* to the subbundle $\Lambda^*_1 := \ker \iota(d\theta) \cap \Lambda^*_{\mathbb{C}}H^* = (\operatorname{im} \varepsilon(d\theta))^\perp \cap \Lambda^*_{\mathbb{C}}H^*$, where $\iota(d\theta)$ denotes the interior product with $d\theta$. This amounts to replace d_b by $\pi_1 \circ d_b$, where π_1 is the orthogonal projection onto Λ^*_1.

In fact, since $d\theta$ is nondegenerate on H the operator $\varepsilon(d\theta) : \Lambda^k_{\mathbb{C}}H^* \to \Lambda^{k+2}_{\mathbb{C}}H^*$ is injective for $k \leq n-1$ and surjective for $k \geq n+1$. This implies that $\Lambda^k_2 = 0$ for $k \leq n-1$ and $\Lambda^k_1 = 0$ for $k \geq n+1$. Therefore, we only have two halves of complexes.

As observed by Rumin [43] we get a full complex by connecting the two halves by means of the operator $D_{R,n} : C^\infty(M, \Lambda^n_{\mathbb{C}}H^*) \to C^\infty(M, \Lambda^n_{\mathbb{C}}H^*)$ such that

$$D_{R,n} = \mathcal{L}_{X_0} + d_{b,n-1}\varepsilon(d\theta)^{-1}d_{b,n}, \tag{6.2}$$

where $\varepsilon(d\theta)^{-1}$ is the inverse of $\varepsilon(d\theta) : \Lambda^{n-1}_{\mathbb{C}}H^* \to \Lambda^{n+1}_{\mathbb{C}}H^*$. Notice that $D_{R,n}$ is second order differential operator. Thus, if we let $\Lambda^k = \Lambda^k_1$ for $k = 0, \ldots, n-1$ and $\Lambda^k = \Lambda^k_2$ for $k = n+1, \ldots, 2n$ then we get the contact complex,

$$\begin{aligned} C^\infty(M) \stackrel{d_{R;0}}{\to} C^\infty(M, \Lambda^1) \stackrel{d_{R;1}}{\to} \ldots C^\infty(M, \Lambda^{n-1}) \stackrel{d_{R;n-1}}{\to} C^\infty(M, \Lambda^n_1) \stackrel{D_{R,n}}{\to} \\ C^\infty(M, \Lambda^n_2) \stackrel{d_{R;n}}{\to} C^\infty(M, \Lambda^{n+1}) \ldots \stackrel{d_{R;2n-1}}{\longrightarrow} C^\infty(M, \Lambda^{2n}), \end{aligned} \tag{6.3}$$

where $d_{R;k} = \pi_1 \circ d_{b;k}$ for $k = 0, \ldots, n-1$ (so that $d^*_{R;k+1} = d^*_{b;k+1}$) and $d_{R;k} = d_{b;k}$ for $k = n, \ldots, 2n-1$.

The contact Laplacian is defined as follows. In degree $k \neq n$ this is the differential operator $\Delta_{R,k} : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^k)$ such that

$$\Delta_{R,k} = \begin{cases} (n-k)d_{R,k-1}d^*_{R,k} + (n-k+1)d^*_{R,k+1}d_{R,k} & k = 0, \ldots, n-1, \\ (k-n-1)d_{R,k-1}d^*_{R,k} + (k-n)d^*_{R,k+1}d_{R,k} & k = n+1, \ldots, 2n. \end{cases} \tag{6.4}$$

For $k = n$ we have the differential operators $\Delta_{R,nj} : C^\infty(M, \Lambda^n_j) \to C^\infty(M, \Lambda^n_j)$, $j = 1, 2$, given by the formulas,

$$\Delta_{R,n1} = (d_{R,n-1}d^*_{R,n})^2 + D^*_{R,n}D_{R,n}, \quad \Delta_{R,n2} = D_{R,n}D^*_{R,n} + (d^*_{R,n+1}d_{R,n}). \tag{6.5}$$

Observe that $\Delta_{R,k}$, $k \neq n$, is a differential operator order 2, whereas $\Delta_{R,n1}$ and $\Delta_{R,n2}$ are differential operators of order 4. Moreover, Rumin [43] proved that in every degree the contact Laplacian is maximal hypoelliptic. In fact, in every degree the contact Laplacian has an invertible principal symbol, hence admits a parametrix in the Heisenberg calculus (see [34], [40], Sect. 3.5).

Let $\Pi_0(d_{R,k})$ and $\Pi_0(D_{R,n})$ be the orthogonal projections onto $\ker d_{R,k}$ and $\ker D_{R,n}$, and let $\Delta^{-1}_{R,k}$ and $\Delta^{-1}_{R,nj}$ be the partial inverses of $\Delta_{R,k}$ and $\Delta_{R,nj}$. Then as in (4.8) we have

$$\Pi_0(d_{R,k}) = \begin{cases} 1 - (n-k-1)^{-1} d^*_{R,k+1}\Delta^{-1}_{R,k+1}d_{R,k} & k = 0, \ldots, n-2, \\ 1 - d^*_{R,n}d_{R,n-1}d^*_{R,n}\Delta^{-1}_{R,n1}d_{R,n-1} & k = n-1, \\ 1 - (k-n)^{-1} d^*_{R,k+1}\Delta^{-1}_{R,k+1}d_{R,k} & k = n, \ldots, 2n-1, \end{cases} \tag{6.6}$$

$$\Pi_0(D_{R,n}) = 1 - D_{R,*}\Delta^{-1}_{R,n2}D_{R,n}. \tag{6.7}$$

As in each degree the principal symbol of the contact Laplacian is invertible, the operators $\Delta^{-1}_{R,k}$, $k \neq n$, and $\Delta^{-1}_{R,nj}$, $j = 1, 2$ are Ψ_HDOs of order -2 and order -4 respectively. Therefore, the above formulas for $\Pi_0(d_{R,k})$ and $\Pi_0(D_{R,n})$ show that these projections are zero'th order Ψ_HDOs.

Theorem 6.1. *The noncommutative residues* $\operatorname{Res}\Pi_0(d_{R,k})$*,* $k = 1, \ldots, 2n-1$ *and* $\operatorname{Res}\Pi_0(D_{R,n})$ *are Heisenberg diffeomorphism invariants of* M *and are invariant under deformation of the contact structure. In particular, their values depend neither on the contact form* θ*, nor on the almost complex structure* J*.*

Proof. Let us first show that the above noncommutative residues don't depend on θ or on J. Let θ' be a contact form on M and let J' be an almost complex structure on H calibrated with respect to θ'. We shall assign the superscript $'$ to objects associated to the pair (θ', J').

It follows from [43] and [40], Chap. 2, that there are vector bundle isomorphisms $\phi_k : \Lambda^k_1 \to \Lambda^{k'}_1$, $k = 1, \ldots, n$, and $\psi_l : \Lambda^l_2 \to \Lambda^{l'}_2$, $l = n, \ldots, 2n-1$ such that $\phi_{k+1}d_{R,k} = d'_{R,k}\phi_k$ and $\psi_{l+1}d_{R,l} = d'_{R,l}\psi_l$ and also $\psi_n D_{R,n} = D'_{R,n}\phi_n$. In particular, we have $\ker d'_{R,k} = \phi_k(\ker d_{R,k})$. Since $\ker d_{R,k}$ is the range of $\Pi_0(d'_{R,k})$ and $\phi_k(\ker d_{R,k})$ that of $\phi_{k*}\Pi_0(d_{R,k})$, we see that $\Pi_0(d'_{R,k})$ and $\phi_{k*}\Pi_0(d_{R,k})$ have same range, so by Proposition 3.2 $\operatorname{Res}\Pi_0(d'_{R,k}) = \operatorname{Res}\phi_{k*}\Pi_0(d_{R,k}) = \operatorname{Res}\Pi_0(d_{R,k})$.

Similarly, $\operatorname{Res}\Pi_0(d'_{R,l}) = \operatorname{Res}\Pi_0(d_{R,l})$ and $\operatorname{Res}\Pi_0(D'_{R,n}) = \operatorname{Res}\Pi_0(D_{R,n})$. Hence $\operatorname{Res}\Pi_0(d_{R,k})$, $k = 1, \ldots, 2n-1$, and $\operatorname{Res}\Pi_0(D_{R,n})$ don't depend on θ or J.

Next, let (M', H') be an orientable contact manifold and let $\phi : M' \to M$ be a Heisenberg diffeomorphism. In addition, let θ' be a contact form on M' and let J' be a calibrated almost complex structure on H'. As before we shall assign the superscript $'$ to objects related to M'.

Since $\operatorname{Res}\Pi_0(d_{R,k})$, $k = 1, \ldots, 2n-1$, and $\operatorname{Res}\Pi_0(D_{R,n})$ don't depend on the contact form or on the almost complex structure, we may assume that $\theta' = \phi^*\theta$ and $J' = \phi^*J$. Then $\phi^*g_{\theta,J} = g_{\theta',J'}$ and $\phi^*d_{R,k} = d'_{R,k}$. Thus,

$$\phi^*\Pi_0(D_{R,n}) = \Pi_0(D'_{R,n}), \qquad \phi^*\Pi_0(d_{R,k}) = \Pi_0(d'_{R,k}). \tag{6.8}$$

Since $\operatorname{Res} \phi^*\Pi_0(d_{R,k}) = \operatorname{Res} \Pi_0(d_{R,k})$ it follows that $\operatorname{Res} \Pi_0(d'_{R,k}) = \operatorname{Res} \Pi_0(d_{R,k})$. Similarly, we have $\operatorname{Res} \phi^*\Pi_0(D'_{R,n}) = \operatorname{Res} \Pi_0(D_{R,n})$. Hence $\operatorname{Res} \Pi_0(D_{R,n})$ and $\operatorname{Res} \Pi_0(d_{R,k})$ are Heisenberg diffeomorphism invariants.

Finally, since the noncommutative residues $\operatorname{Res} \Pi_0(D_{R,n})$ and $\operatorname{Res} \Pi_0(d_{R,k})$ are invariant by Heisenberg diffeomorphism we may argue as in the proof of Proposition 4.6 to show that they are invariant under deformation of the contact structure. The proof is thus achieved. □

Remark 6.2. The residues $\operatorname{Res} \Pi_0(d^*_{R,k})$, $k = 1, \ldots, 2n-1$, and $\operatorname{Res} \Pi_0(D^*_{R,n})$ also yield invariants, but these are the same up to a sign factor as those from Theorem 6.1. Indeed, in (4.7) for $k \neq n$ we have $\Pi_0(\Delta_{R,k}) = \Pi_0(d_{R,k}) + \Pi_0(d^*_{R,k}) - 1$, so as $\Pi_0(\Delta_{R,k})$ is a smoothing operator we get $\operatorname{Res} \Pi_0(d^*_{R,k}) = -\operatorname{Res} \Pi_0(d_{R,k})$. Similarly, we have $\operatorname{Res} \Pi_0(d^*_{R,n}) = -\operatorname{Res} \Pi_0(D_{R,n})$ and $\operatorname{Res} \Pi_0(D^*_{R,n}) = -\operatorname{Res} \Pi_0(d_{R,n})$.

Remark 6.3. As with the invariants of the previous sections, we can make use of a Getzler's rescaling to compute the local densities associated to a supersymmtric version of the invariants of Theorem 6.1. However, it is not clear how efficient this would be to yield explicit formulas. More precisely, in the non-supersymetric setting there are no known explicit formulas for the fundamental solutions of the contact Laplacian and *a fortiori* for the inverse of its principal symbols. Therefore, it is all the more difficult to obtain explicit formulas in the supersymmetric setting.

Maybe the solution would be combine a Getzler's rescaling with diabatic limit techniques, since the contact complex appears naturally in the analysis of the asymptotic behavior of the de Rham complex under a diabatic limit (see [44], [10]). Let us also mention that another possible approach would be rely on the global K-theoretic techniques as alluded to in Appendix.

Appendix

The construction of the contact invariants from generalized Szegö projection in Section 5 is partly based on Lemma 3.7 stating that the noncommutative residue of a Ψ_HDO projection is a homotopy invariant of its principal symbol. In this Appendix we would like to explain that for a general Heisenberg manifold, not necessarily contact or CR, this leads us to a K-theoretic interpretation of the noncommutative residue of a Ψ_HDO projection.

First, as $S_0(\mathfrak{g}^*M)$ endowed with the product homogeneous symbols (2.10) is a Fréchet algebra, its K_0-group can be defined as follows.

Let $M_\infty(S_0(\mathfrak{g}^*M))$ be the algebra $\varinjlim M_q(S_0(\mathfrak{g}^*M))$, where the inductive limit is defined using the embedding $a \rightarrow \operatorname{diag}(a, 0)$ of $M_q(S_0(\mathfrak{g}^*M))$ into $M_{q+1}(S_0(\mathfrak{g}^*M))$ We say that idempotents e_1 and e_2 in $M_\infty(S_0(\mathfrak{g}^*M))$ are homotopic when there is a C^1-path of idempotents joining e_1 to e_2 is some space $M_q(S_0(\mathfrak{g}^*M))$ containing both e_1 and e_2.

The addition of idempotents is given by the direct sum $e_1 \oplus e_2 = \operatorname{diag}(e_1, e_2)$. This operation is compatible with the homotopy of idempotents, so turns the set of homotopy classes of idempotents in $M_\infty(S_0(\mathfrak{g}^*M))$ into a monoid. We then define $K_0(S_0(\mathfrak{g}^*M))$ as the associated Abelian group of this monoid, i.e., the group of formal differences of homotopy classes of idempotents in $M_\infty(S_0(\mathfrak{g}^*M))$.

In this sequel we will need to describe $K_0(S_0(\mathfrak{g}^*M))$ in terms of homotopy classes of symbols as follows.

Definition A.4. 1) An idempotent pair is a pair $(\pi, \mathcal{E})$ consisting of a (smooth) vector bundle $\mathcal{E}$ over M and an idempotent $\pi \in S_0(\mathfrak{g}^*M, \mathcal{E})$.

2) Two idempotent pairs $(\pi_1, \mathcal{E}_1)$ and $(\pi_2, \mathcal{E}_2)$ are equivalent when there exist smooth vector bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ over M, a vector bundle isomorphism ϕ from $\mathcal{E}_1 \oplus \mathcal{F}_1$ to $\mathcal{E}_2 \oplus \mathcal{F}_2$ and a homotopy of idempotents in $S_0(\mathfrak{g}^*M, \mathcal{E}_2 \oplus \mathcal{F}_2)$ from $\phi_*(\pi_1 \oplus 0_{\mathrm{pr}^* \mathcal{F}_1})$ to $\pi_2 \oplus 0_{\mathrm{pr}^* \mathcal{F}_2}$.

The set of equivalence classes of idempotent pairs becomes a monoid when we endow it with the addition given by the direct sum,

$$(\pi_1, \mathcal{E}_1) \oplus (\pi_2, \mathcal{E}_2) = (\pi_1 \oplus \pi_2, \mathcal{E}_1 \oplus \mathcal{E}_2). \tag{A.9}$$

We let $\mathcal{I}_0(\mathfrak{g}^*M)$ denote the Abelian group of formal differences of equivalence classes of idempotent pairs.

There is a natural map Θ from idempotents of $M_q(S_0(\mathfrak{g}^*M))$ to idempotent pairs obtained by assigning to any idempotent $e \in M_q(S_0(\mathfrak{g}_*M))$ the idempotent pair $\Theta(e) = (e, M \times \mathbb{C}^q)$. In fact, we have:

Lemma A.5. *Θ gives rise to an isomorphism from $K_0(S_0(\mathfrak{g}^*M))$ onto $\mathcal{I}_0(\mathfrak{g}^*M)$.*

Proof. First, let $(\pi, \mathcal{E})$ be an idempotent pair. There exists a (smooth) vector bundle $\mathcal{F}$ such that $\mathcal{E} \oplus \mathcal{F}$ is globally trivializable, i.e., there exists a vector bundle isomorphism $\phi : \mathcal{E} \oplus \mathcal{F} \simeq M \times \mathbb{C}^q$ (see, e.g., [3], Cor. 1.4.14). Then $(\pi, \mathcal{E})$ is equivalent to $\Theta(e) = (e, M \times \mathbb{C}^q)$, where e is the idempotent $\phi_*(\pi \oplus 0_{\mathrm{pr}\, \mathcal{F}}) \in M_q(S_0(\mathfrak{g}^*M))$. This shows that up to the equivalence of idempotent pairs the map Θ is surjective.

Next, for $j = 1, 2$ let $e_j \in M_{q_j}(S_0(\mathfrak{g}^*M))$ be idempotent and suppose that $\Theta(e_1)$ and $\Theta(e_2)$ are equivalent idempotent pairs. Thus there exist vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ and a vector bundle isomorphism $\phi : (M \times \mathbb{C}^{q_1}) \oplus \mathcal{E}_1 \rightarrow (M \times \mathbb{C}^{q_2}) \oplus \mathcal{E}_2$ such that $\phi_*(e_1 \oplus 0_{\mathrm{pr}^* \mathcal{E}_1})$ is homotopic to $e_2 \oplus 0_{\mathrm{pr}^* \mathcal{E}_2}$ in $S_0(\mathfrak{g}^*M, (M \times \mathbb{C}^{q_2}) \oplus \mathcal{E}_2)$.

Let $\mathcal{F}$ be a vector bundle so that there exists a vector bundle isomorphism ψ_1 from $(M \times \mathbb{C}^{q_1}) \oplus \mathcal{E}_1 \oplus \mathcal{F}$ onto $M \times \mathbb{C}^q$. Composing ψ_1 with $\phi \oplus 1_{\mathcal{F}}$ we get a vector bundle isomorphism $\psi_2 : (M \times \mathbb{C}^{q_2}) \oplus \mathcal{E}_2 \oplus \mathcal{F} \rightarrow M \times \mathbb{C}^q$ such that there is a homotopy of idempotents in $S_0(\mathfrak{g}^*, M \times \mathbb{C}^q) = M_q(S_0(\mathfrak{g}^*))$ joining $\psi_{1*}(e_1 \oplus 0_{\mathrm{pr}^*(\mathcal{E}_1 \oplus \mathcal{F})})$ to $\psi_{2*}(e_2 \oplus 0_{\mathrm{pr}^*(\mathcal{E}_2 \oplus \mathcal{F})})$.

Using the identification $\mathbb{C}^q = \mathbb{C}^{q_1} \oplus \mathbb{C}^{q - q_1}$ let $\gamma = 1_{\mathbb{C}^q} \oplus 0$, where 0 is the zero vector bundle morphism from $\mathcal{E}_1 \oplus \mathcal{F}$ to $M \times \mathbb{C}^{q - q_1}$, and similarly define $\gamma^t = 1_{\mathbb{C}^q} \oplus 0$ using the the zero vector bundle morphism from $M \times \mathbb{C}^{q - q_1}$ to $\mathcal{E}_1 \oplus \mathcal{F}$. Let $\alpha = \psi_1 \circ \gamma$ and $\beta = \gamma^t \circ \psi_1^{-1}$. Then α and β are sections of $\mathrm{End}(M \times \mathbb{C}^q)$, i.e., are elements of $M_q(C^\infty(M))$, and we have $\alpha(e_1 \oplus 0_{\mathbb{C}^{q - q_1}})\beta = \psi_{1*}(e_1 \oplus 0_{\mathrm{pr}^*(\mathcal{E}_1 \oplus \mathcal{F})})$ and $(e_1 \oplus 0_{\mathbb{C}^{q - q_1}})\beta\alpha = e_1 \oplus 0_{\mathbb{C}^{q - q_1}}$. This shows that $e_1 \oplus 0_{\mathbb{C}^{q - q_1}}$ and $\psi_{1*}(e_1 \oplus 0_{\mathrm{pr}^*(\mathcal{E}_1 \oplus \mathcal{F})})$ are algebraically equivalent idempotents of $M_q(S_0(\mathfrak{g}^*M))$ in the sense of [11], Def. 4.2.1, hence are homotopic idempotents in $M_\infty(S_0(\mathfrak{g}^*M))$ (see [11], Props. 4.3.1, 4.4.1). Incidentally, e_1 and $\psi_{1*}(e_1 \oplus 0_{\mathrm{pr}^*(\mathcal{E}_1 \oplus \mathcal{F})})$ are homotopic to each other.

Similarly, e_2 and $\psi_{2*}(e_2 \oplus 0_{\mathrm{pr}^*(\mathcal{E}_2 \oplus \mathcal{F})})$ are homotopic idempotents. Since the latter is homotopic to $\psi_{1*}(e_1 \oplus 0_{\mathrm{pr}^*(\mathcal{E}_1 \oplus \mathcal{F})})$, it follows that e_1 is homotopic to e_2. This proves that Θ is injective up to the equivalence of idempotent pairs.

All this shows that Θ factorizes through a bijection from homotopy classes of idempotents in $M_\infty(S_0(\mathfrak{g}^*M))$ to equivalence classes of idempotent pairs. Furthermore, this map is additive. Indeed, if $e_j \in M_{q_j}(S_0(\mathfrak{g}^*M))$ then, up to the identification $\mathbb{C}^{q_1} \oplus \mathbb{C}^{q_2} = \mathbb{C}^{q_1 + q_2}$, we have $\Theta(e_1 \oplus e_2) = \Theta(e_1) \oplus \Theta(e_2)$. Thus Θ gives rise to isomorphism from $K_0(S_0(\mathfrak{g}^*M))$ onto $\mathcal{I}_0(\mathfrak{g}^*M)$. $\square$

We will also need the following lemma.

Lemma A.6. *Let $\pi_0 \in S_0(\mathfrak{g}^*M, \mathcal{E})$ be idempotent. Then we can always find a $\Psi_H DO$ projection $\Pi \in \Psi_H^0(M, \mathcal{E})$ with principal symbol π_0.*

Proof. Let $f_0 = \frac{1}{2}(1+\pi_0)$, so that $f_0 * f_0 = 1$. Since the principal symbol map $\sigma_0 : \Psi_H^0(M, \mathcal{E}) \to S_0(\mathfrak{g}^*M)$ is surjective (see [40], Prop. 3.2.6), there exists $P \in \Psi_H^0(M, \mathcal{E})$ with principal symbol f_0. Then P^2 has principal symbol $f_0 * f_0 = 1$, so $P^2 = 1 - R_1$ with $R_1 \in \Psi_H^{-1}(M, \mathcal{E})$.

Let $\sum_{k\geq 0} a_k z^k$ be the Taylor series at $z = 0$ of $(1-z)^{-\frac{1}{2}}$. Since R_1 has order ≤ -1 the symbolic calculus for Ψ_HDOs in [8] allows us to construct $Q \in \Psi_H^0(M, \mathcal{E})$ such that $Q = \sum_{k=0}^{N-1} a_k R_1^k \bmod \Psi_H^{-N}(M, \mathcal{E})$ for any integer N. Then we obtain $(1 - R_1)Q^2 = 1 \bmod \Psi^{-\infty}(M, \mathcal{E})$ and, letting $F = PQ$ and using the fact that $R_1 = 1 - P^2$ commutes with P, we see that $F^2 = P^2 Q = (1 - R_1)Q = 1 - R$ for some smoothing operator R.

Next, as for $\lambda \in \mathbb{C}$ we have $(F - \lambda)(F + \lambda) = F^2 - \lambda^2 = 1 - \lambda^2 - R$, we see that $\lambda \in \operatorname{Sp} F$ if, and only if, $\lambda^2 - 1 \in \operatorname{Sp} R$. Since R is smoothing this is a compact operator and so $\operatorname{Sp} R \setminus \{0\}$ is bounded and discrete. Incidentally $\operatorname{Sp} F \setminus \{\pm 1\}$ is a discrete set. Moreover, for $\lambda \notin (\operatorname{Sp} F \cup \{\pm 1\})$ we have

$$(F - \lambda)^{-1} = (F + \lambda)(1 - \lambda^2 - R)^2 = (\lambda^2 - 1)^{-1}(F + \lambda) - (F + \lambda)S(1 - \lambda^2), \tag{A.10}$$

where for $\mu \notin \operatorname{Sp} R \cup \{0\}$ we have let $S(\mu) = (\mu - R)^{-1} - \mu^{-1}$.

At first glance $(S(\mu))_{\mu \notin \operatorname{Sp} R \cup \{0\}}$ is a holomorphic family of bounded operators, but the equalities $S(\mu) = \mu^{-1} R(\mu - R)^{-1} = \mu^{-1}(\mu - R)^{-1} R$ imply that it actually is a holomorphic family of smoothing operators.

Now, since $\operatorname{Sp} F \setminus \{\pm 1\}$ is discrete we can find positive numbers $0 < r_1 < r_2 < 2$ such that $\operatorname{Sp} F \cap \{\lambda; r_1 < |\lambda - 1| < r_2\} = \emptyset$. Then $\{(F - \lambda)^{-1}\}_{r_1 < |\lambda - 1| < r_2}$ is a holomorphic family with values in $\Psi_H^0(M, \mathcal{E})$, so we define a projection in $L^2(M, \mathcal{E})$ by letting $\Pi = \frac{1}{2i\pi} \int_{|\lambda - 1| = r} (F - \lambda)^{-1} d\lambda$, with $r_1 < r < r_2$. In fact, as $(S(\mu))_{\mu \notin \operatorname{Sp} R \cup \{0\}}$ is a holomorphic family of smoothing operators, it follows from (A.10) that, up to a smoothing operator, Π agrees with $\frac{1}{2i\pi} \int_{|\lambda - 1| = r} (\lambda^2 - 1)^{-1}(F + \lambda) d\lambda = \frac{1}{2}(F + 1)$. Hence Π is a zero'th order Ψ_HDO projection with principal symbol $\frac{1}{2}(f_0 + 1) = \pi_0$. The lemma is thus proved. □

We can now give the topological interpretation of the noncommutative residue of a Ψ_HDO projection.

Proposition A.7. *There exists a unique additive map $\rho_R : K_0(S_0(M)) \to \mathbb{R}$ such that, for any vector bundle $\mathcal{E}$ over M and any projection $\Pi \in \Psi_H^0(M, \mathcal{E})$, we have*

$$\rho_R \circ \Theta^{-1}[\pi_0, \mathcal{E}] = \operatorname{Res} \Pi, \tag{A.11}$$

where π_0 denotes the principal symbol of Π.

Proof. Let $(\pi, \mathcal{E})$ be an idempotent pair. By Lemma A.6 there exists a projection $\Pi_{(\pi,\mathcal{E})}$ in $\Psi_H^0(M, \mathcal{E})$ whose principal symbol is π. The choice of $\Pi_{(\pi,\mathcal{E})}$ is not unique, but Lemma 3.7 insures us that the value of $\operatorname{Res} \Pi_{(\pi,\mathcal{E})}$ is independent of this choice. Furthermore, we know by Lemma 3.3 that $\operatorname{Res} \Pi_{(\pi,\mathcal{E})}$ is a real number. Therefore, we uniquely define a map from idempotent pairs to $\mathbb{R}$ by letting $\rho_R'(\pi, \mathcal{E}) = \operatorname{Res} \Pi_{(\pi,\mathcal{E})}$.

For $j = 1, 2$ let $(\pi_j, \mathcal{E}_j)$ be an idempotent pair and let $\Pi_j \in \Psi_H^0(M, \mathcal{E}_j)$ be a Ψ_HDO projection with principal symbol π_j. Then $\Pi_1 \oplus \Pi_2$ is a Ψ_HDO projection with principal symbol $\pi_1 \oplus \pi_2$, so $\rho'_R(\pi_1 \oplus \pi_2, \mathcal{E}_1 \oplus \mathcal{E}_2) = \operatorname{Res}(\Pi_1 \oplus \Pi_2)$. As we have $\operatorname{Res}(\Pi_1 \oplus \Pi_2) = \operatorname{Res}\Pi_1 + \operatorname{Res}\Pi_2$ we get $\rho'_R(\pi_1 \oplus \pi_2, \mathcal{E}_1 \oplus \mathcal{E}_2) = \rho'_R(\pi_1, \mathcal{E}_1) + \rho'_R(\pi_2, \mathcal{E}_2)$. Hence ρ'_R is an additive map.

Next, assume that $(\pi_1, \mathcal{E}_1)$ and $(\pi_2, \mathcal{E}_2)$ are equivalent idempotent pairs. Thus, there exist smooth vector bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ over M and a vector bundle isomorphism $\phi : \mathcal{E}_1 \oplus \mathcal{F}_1 \to \mathcal{E}_2 \oplus \mathcal{F}_2$ such that $\phi_*(\pi_1 \oplus 0_{\mathrm{pr}^* \mathcal{F}_1})$ and $\pi_2 \oplus 0_{\mathrm{pr}^* \mathcal{F}_2}$ are homotopic idempotents in $S_0(\mathfrak{g}^* M, \mathcal{E}_2 \oplus \mathcal{F}_2)$. Then by Lemma 3.7 we have $\rho'_R(\pi_1 \oplus 0_{\mathrm{pr}^* \mathcal{F}_1}, \mathcal{E}_1 \oplus \mathcal{F}_1) = \rho'_R(\pi_2 \oplus 0_{\mathrm{pr}^* \mathcal{F}_2}, \mathcal{E}_2 \oplus \mathcal{F}_2)$ and, as $\rho'_R(0_{\mathrm{pr}^* \mathcal{F}_j}, \mathcal{F}_j) = 0$, it follows from the additivity of ρ'_R that $\rho'_R(\pi_j \oplus 0_{\mathrm{pr}^* \mathcal{F}_j}, \mathcal{E}_j \oplus \mathcal{F}_j) = \rho'_R(\pi_j, \mathcal{E}_j)$. Hence we have $\rho'_R(\pi_1, \mathcal{E}_1) = \rho'_R(\pi_2, \mathcal{E}_2)$.

This shows that the value of $\rho'_R(\pi_1, \mathcal{E}_1)$ depends only on the equivalence class of $(\pi_1, \mathcal{E}_1)$. Since ρ'_R is additive it follows that it gives rise to an additive map from $\mathcal{I}_0(\mathfrak{g}^* M)$ to $\mathbb{R}$. Letting $\rho_R = \rho'_R \circ \Theta$ then defines the desired additive map from $K_0(S_0(\mathfrak{g}^* M))$ to $\mathbb{R}$ satisfying (A.11). $\square$

The above K-theoretic interpretation of the noncommutative residue of a Ψ_HDO projection is reminiscent of the K-theoretic interpretations of the residue at the origin of the eta function of a selfadjoint elliptic ΨDO by Atiyah-Patodi-Singer [5] and of the Fredholm index of an elliptic ΨDO by Atiyah-Singer [6]. Nevertheless, it differs from them on the fact that we have to use the K-theory of algebras rather than that of spaces as in [6] and [5]. Indeed, as the algebra of (scalar) zero'th order Heisenberg symbols is not commutative, it cannot be identified with the algebra of smooth functions on the cotangent unit sphere S^*M and we cannot make use of the Serre-Swan isomorphism to identify its K-theory with that of S^*M. Thus in order to give a K-theoretic interpretation of the noncommutative residue of a ΨDO projection we really have to rely on the K-theory of algebras.

The (full) index theorem of Atiyah-Singer [6] identifies in purely topological terms the K-theoretic analytical index map defined the Fredholm indices of elliptic ΨDOs. In turn, via a cohomological interpretation this provides us with a general topological formula to compute the index of an elliptic ΨDO in terms of the Chern character of its principal symbol.

Similarly, it would be interesting to have a topological formula for computing the noncommutative residue of a Ψ_HDO projection in terms of its principal symbol. As above-mentioned the algebra of zero'th order Heisenberg symbols is noncommutative, so we presumably have to rely on tools from Connes' noncommutative geometry to carry out this project. In particular, Connes [16] produced a fairly simple and general proof of the Atiyah-Singer index theorem by making use of the tangent groupoid of a manifold. The latter construction has been extended to Heisenberg manifolds in [39] (see also [46]). Therefore, the tangent groupoid of a Heisenberg manifold may well be a key tool to give a topological interpretation of the residue map ρ_R.

On the other hand, by a celebrated result of Atiyah-Patodi-Singer [5] and Gilkey [24] the eta function of a general selfadjoint elliptic ΨDO on a compact manifold is regular at the origin, so that the eta invariant of the operator is always well defined. This result was extended by Wodzicki [47] who established the vanishing of the noncommutative residue of a ΨDO projection. The original proof of Wodzicki is quite involved, but it was much simplified by Brüening-Lesch [15], Lem. 2.7, who showed that the result of Wodzicki is in fact equivalent to that of Atiyah-Patodi-Singer and Gilkey.

Similarly, in the framework of the Heisenberg calculus the vanishing of the noncommutative residue of a Ψ_HDO projection is equivalent to the regularity at the origin of the eta function of a selfadjoint hypoelliptic Ψ_HDO. Therefore, proving the vanishing of the map ρ_R would enable us to define the eta invariant of a selfadjoint hypoelliptic Ψ_HDO as the regular value at the origin of its eta function. Such a result would be interesting for dealing with hypoelliptic boundary values index problems on bounded strictly pseudoconvex complex domains, symplectic manifolds or even asymptotically complex hyperbolic spaces in the sense of [20]. This would also allow us to give a positive answer to a question left open in [10], Remark 9.3.

To summarize two interesting phenomena may occur:

- The map ρ_R may be non-trivial and understood in topological terms, which would allows us to compute the noncommutative residue of Ψ_HDO in terms of its principal symbol only;

- The noncommutative residue of a Ψ_HDO is always zero vanish, which would allow us to define the eta invariant of any hypoelliptic selfadjoint Ψ_HDOs.

Therefore, it is all the more important to further understand the noncommutative residue of a Ψ_HDO projection. We hope to go back to this in a future research.

References

[1] *S. Alexakis*, On the decomposition of global conformal invariants. I. E-print, arXiv, Oct. 05. To appear in Ann. of Math..

[2] *S. Alexakis*, On the decomposition of global conformal invariants. II. Adv. in Math. **206** (2006) 466–502.

[3] *M.F. Atiyah*, K-theory. 2nd edition, Addison-Wesley, Redwood City, CA, 1989.

[4] *M. Atiyah, R. Bott, V. Patodi,* On the heat equation and the index theorem. Invent. Math. **19** (1973) 279–330.

[5] *M. Atiyah, V. Patodi, I.M. Singer,* Spectral asymmetry and Riemannian geometry. III. Math. Proc. Camb. Philos. Soc. **79** (1976) 71–99 .

[6] *M. Atiyah, I.M. Singer,* The index of elliptic operators. I. Ann. of Math. (2) **87** (1968), 484–530.

[7] *T.N. Bailey, M.G. Eastwood, C.R. Graham,* Invariant theory for conformal and CR geometry. Ann. Math. **139** (1994) 491–552.

[8] *R. Beals, P.C. Greiner,* Calculus on Heisenberg manifolds. Annals of Mathematics Studies, vol. 119. Princeton University Press, Princeton, NJ, 1988.

[9] *R. Beals, P.C. Greiner, N.K. Stanton,* The heat equation on a CR manifold. J. Differential Geom. **20** (1984) 343–387.

[10] *O. Biquard, M. Herzlich, M. Rumin,* Diabatic limit, eta invariants and Cauchy-Riemann manifolds of dimension 3. E-print, arXiv, June 2005. To appear in Ann. Sc. Ec. Norm. Sup..

[11] *B. Blackadar,* K-theory for operator algebras. Second edition. MSRI Publications, 5. Cambridge University Press, 1998.

[12] *L. Boutet de Monvel,* Logarithmic trace of Toeplitz projectors. Math. Res. Lett. **12** (2005), 401–412.

[13] *L. Boutet de Monvel,* Vanishing of the logarithmic trace of generalized Szegö projectors. Algebraic Analysis of Differential Equations (Kyoto, Japan, July 2005), Springer 2007.

[14] *L. Boutet de Monvel, V. Guillemin,* The spectral theory of Toeplitz operators. Annals of Mathematics Studies, 99. Princeton University Press, Princeton, NJ, 1981.

[15] *J. Brüning, M. Lesch,* On the eta-invariant of certain nonlocal boundary value problems. Duke Math. J. **96** (1999), no. 2, 425–468.

[16] *A. Connes,* Noncommutative geometry. Academic Press, Inc., San Diego, CA, 1994.

[17] *S. Deser, A. Schwimmer,* Geometric classification of conformal anomalies in arbitrary dimensions. Phys. Lett. B **309** (1993), no. 3-4, 279–284.

[18] *Y. Eliashberg, W. Thurston,* Confoliations. University Lecture Series, 13, AMS, Providence, RI, 1998.

[19] *C.L. Epstein, R.B. Melrose,* The Heisenberg algebra, index theory and homology. Preprint, 2000. Available online at `http://www-math.mit.edu/rbm/book.html`.

[20] *C.L. Epstein, R.B. Melrose, G. Mendoza*, Resolvent of the Laplacian on strictly pseudoconvex domains. Acta Math. **167** (1991), no. 1-2, 1–106.

[21] *C.L. Fefferman,* Parabolic invariant theory in complex analysis. Adv. in Math. **31** (1979), no. 2, 131–262.

[22] *G. Folland, E. Stein*, Estimates for the $\overline{\partial}_b$-complex and analysis on the Heisenberg group. Comm. Pure Appl. Math. **27** (1974) 429–522.

[23] *E. Getzler,* A short proof of the local Atiyah-Singer index theorem. Topology **25**, 111–117 (1986).

[24] *P.B. Gilkey,* The residue of the global η function at the origin. Adv. in Math. **40** (1981), no. 3, 290–307.

[25] *P.B. Gilkey,* Invariance theory, the heat equation, and the Atiyah-Singer index theorem. 2nd edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[26] Gray, J.W.: Some global properties of contact structures. Ann. of Math. **69** (1959) 421–450.

[27] *P.C. Greiner,* Personal communication, December 2005.

[28] *M. Gromov,* Carnot-Carathéodory spaces seen from within. Sub-Riemannian geometry, 79–323, Progr. Math., 144, Birkhäuser, Basel, 1996.

[29] *V.W. Guillemin,* A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. in Math. **55** (1985), no. 2, 131–160.

[30] *V.W. Guillemin,* Residue traces for certain algebras of Fourier integral operators. J. Funct. Anal. **115** (1993), no. 2, 391–417.

[31] *K. Hirachi,* Construction of boundary invariants and the logarithmic singularity of the Bergman kernel. Ann. of Math. **151** (2000) 151–191.

[32] *K. Hirachi,* Logarithmic singularity of the Szegö kernel and a global invariant of strictly pseudoconvex domains. Ann. of Math. **63** (2006), 499–515.

[33] *K. Hirachi,* Personal communication.

[34] *P. Julg, G. Kasparov,* Operator K-theory for the group $\mathrm{SU}(n,1)$. J. Reine Angew. Math. **463** (1995), 99–152.

[35] *J.J. Kohn,* Boundaries of complex manifolds. 1965 Proc. Conf. Complex Analysis (Minneapolis, 1964) pp. 81–94. Springer, Berlin.

[36] *J.J. Kohn, H. Rossi,* On the extension of holomorphic functions from the boundary of a complex manifold. Ann. of Math. **81** (1965) 451–472.

[37] *R. Ponge,* Calcul fonctionnel sous-elliptique et résidu non commutatif sur les variétés de Heisenberg. C. R. Acad. Sci. Paris, Série I, **332** (2001) 611–614.

[38] *R. Ponge,* A new short proof of the local index formula and some of its applications. Comm. Math. Phys. **241** (2003) 215–234.

[39] *R. Ponge,* The tangent groupoid of a Heisenberg manifold. Pacific Math. J. **227** (2006) 151–175.

[40] *R. Ponge,* Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds. E-print, arXiv, September 2005, 138 pages. To appear in Mem. Amer. Math. Soc..

[41] *R. Ponge,* Noncommutative residue for Heisenberg manifolds and applications in CR and contact geometry. E-print, arXiv, July 2006. To appear in J. Funct. Anal..

[42] *C. Rockland,* Intrinsic nilpotent approximation. Acta Appl. Math. **8** (1987), no. 3, 213–270.

[43] *M. Rumin,* Formes différentielles sur les variétés de contact. J. Differential Geom. **39** (1994), no.2, 281–330.

[44] *M. Rumin,* Sub-Riemannian limit of the differential form spectrum of contact manifolds. Geom. Funct. Anal. **10** (2000) 407–452 .

[45] *M.E. Taylor*, Noncommutative microlocal analysis. I. Mem. Amer. Math. Soc. 52 (1984), no. 313.

[46] *E. Van Erp*, PhD dissertation. Pennsylvania State University, 2005.

[47] *M. Wodzicki,* Local invariants of spectral asymmetry, Invent. Math. **75** (1984) 143–177.

[48] *M. Wodzicki,* Noncommutative residue. I. Fundamentals. K-theory, arithmetic and geometry (Moscow, 1984–1986), 320–399, Lecture Notes in Math., 1289, Springer, 1987.

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