RAAGs in Diffeos

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Abstract.

We survey embedability results related to RAAGs (right-angled Artin groups) and various automorphism groups of manifolds. We give two different methods of embedding a RAAG to into another, and deduce that every RAAG embeds into some braid group. This gives the unsolvability of the isomorphism problem for finitely presented subgroups of braid groups. Also, we prove that every RAAG is a quasi-isometrically embedded subgroup of the symplectomorphism groups of the disk and of the sphere, with suitable L^p metrics. Finally, we embed RAAGs in the smooth diffeomorphism group of the real line. These results show that many closed hyperbolic manifold groups embed as subgroups of diffeomorphism groups of manifolds.

§1. Right-angled Artin groups

Let Γ be a finite graph. We define the *RAAG (Right-Angled Artin Group)* on Γ as the group presentation

$$A(\Gamma) = \langle V(\Gamma) \mid [a, b] = 1 \text{ if } \{a, b\} \in E(\Gamma)\rangle.$$

We will also adopt an opposite notation; that is, we define

$$G(\Gamma) = \langle V(\Gamma) \mid [a, b] = 1 \text{ if } \{a, b\} \notin E(\Gamma)\rangle.$$

For example, $A(\bullet) \cong G(\bullet) \cong \mathbb{Z}$, $A(\Delta) \cong G(\{\text{three points}\}) \cong \mathbb{Z}^3$ and $A(\{\text{three points}\}) \cong G(\Delta) \cong F_3$. RAAGs are linear [24], residually torsion-free nilpotent [18] (hence bi-orderable), and have solvable word, conjugacy, and isomorphism problems. We refer the readers to [13] and bibliography therein for standard facts on RAAGs.

Charney pointed out that RAAGs "interpolate" between free groups and free abelian groups [13]. Note that subgroups of free and free abelian groups are again free and free abelian, respectively. In general, subgroups of RAAGs enjoy strong group theoretic restrictions.

Key words and phrases. right-angled Artin group, braid group, cancellation theory, hyperbolic manifold, quasi-isometry.

Theorem 1. (1) [23] *Every non-trivial subgroup of a RAAG surjects onto* $\mathbb{Z}$*.*

(2) [1] *If the fundamental group of a closed aspherical 3-manifold* M *embeds into a RAAG, then* M *virtually fibers over the circle.*

Despite the simplicity of presentations, RAAGs are known to have strikingly rich isomorphism types of subgroups. Haglund and Wise discovered an effective way of embedding groups into RAAGs, using codimension-one subgroups [22]. A group G *virtually* embeds into another group H, if a finite-index subgroup of G embeds into H. Many word-hyperbolic groups are known to virtually embed into RAAGs.

Theorem 2. (1) [36, 16] *The fundamental group of a (possibly non-orientable) closed hyperbolic surface with Euler characteristic not equal to -1 embeds into a RAAG.*

(2) [2] *If a word-hyperbolic group* G *is the fundamental group of a locally CAT(0) cube complex, then* G *virtually embeds into a RAAG.*

(3) [2, 38] *Every closed and ever finite-volume hyperbolic 3-manifold group virtually embeds into a RAAG, as shown by Agol and Wise respectively.*

(4) [6] *For each* $n \geq 2$*, there exists a closed hyperbolic* n*–manifold* M *such that* $\pi_1(M)$ *embeds into a RAAG.*

In particular, (3) in the above theorem relies on the Surface Subgroup Theorem by Kahn and Markovic [25]. Combining Theorems 1 and 2, one obtains a solution to the long-standing Virtual Haken conjecture by Waldhausen and Virtual Fiber Conjecture by Thurston [37].

Classifying isomorphism types of subgroups which occur in RAAGs can be quite a complicated task. For example, it is in general impossible to solve the isomorphism problem for finitely presented subgroups of a given RAAG [11]. So, it is reasonable to consider a smaller class of subgroups. A question we found interesting and useful for later uses is:

Question 1. *For given finite graphs* X *and* Y*, when does* $A(X)$ *embed into* $A(Y)$*?*

On the other hand, RAAGs arise quite ubiquitously as *subgroups* of homeomorphism groups. This is due to the *universal property of RAAGs*, defined as follows: Let M be a manifold. The *support* of $f \in \mathrm{Homeo}(M)$ is

$$\mathrm{supp}(f) = \overline{\{x \in M : \ f(x) \neq x\}}.$$

Suppose $f_1, f_2, \ldots, f_k \in \mathrm{Homeo}(M)$. We let Γ be the *intersection graph* of $\{\mathrm{supp}(f_1), \ldots, \mathrm{supp}(f_k)\}$; this means, $V(\Gamma) = \{f_1, \ldots, f_k\}$ and $\{f_i, f_j\} \in$

$E(\Gamma)$ if and only if $\mathrm{supp}(f_i) \cap \mathrm{supp}(f_j) \neq \varnothing$. Since the disjointness of supports implies commutativity of homeomorphisms, we have a natural group homomorphism $\phi:\ G(\Gamma) \to \mathrm{Homeo}(M)$ defined as $\phi(f_i) = f_i$ for each i. In general, it is a highly nontrivial question whether or not such ϕ is injective. One can also ask the same question in different categories such as $\mathrm{Diff}(M)$ or $\mathrm{Diff}^k(M)$ (meaning, C^k-diffeomorphism group).

We let $\mathrm{Mod}(S)$ denote the mapping class group of a surface S; see [20] for fundamental facts and notations regarding this group. We denote by $\mathrm{Symp}(S)$ the area-preserving diffeomorphism group of S, which plays an important role in fluid mechanics [3].

Question 2. *Given a surface S and a finite graph X, when does $A(X)$ embed into $\mathrm{Mod}(S)$ or into $\mathrm{Symp}(S)$?*

Finally, we address the question on 1-manifolds.

Question 3. *Given a one-manifold M, a finite graph X and a positive integer k, when does $A(X)$ embed into $\mathrm{Diff}^k(M)$?*

In this paper, we will survey our results on these three questions.

§2. RAAGs in RAAGs

A motivation for Question 1 comes from the question which RAAGs contain closed hyperbolic surface subgroups [21]. It is not still known whether or not a RAAG contains a closed hyperbolic surface subgroup if and only if it contains a one-ended word-hyperbolic group; see [28, 27] for related results.

Recall that an *induced subgraph* X of a graph Y is a subgraph X of Y that satisfies $E(X) = E(Y) \cap \binom{V(X)}{2}$. We denote $X \leq Y$ if X is an induced subgraph of Y. It was originally suspected that $A(\Gamma)$ contains a closed hyperbolic surface subgroup if and only if Γ contains an induced 5-cycle. This conjecture was disproved by showing that the RAAG on the complement graph of a cycle of length at least six contains a hyperbolic surface subgroup [29, 15, 5].

In [29], and by a different method in [5], it was proved that $G(X)$ embeds into $G(Y)$ if Y topologically contracts onto X. This can be much further generalized using the concept of *extension graph*. Let X be a finite graph. We define the *extension graph* X^e by the relations $V(X^e) = \{v^g :\ v \in V(\Gamma), g \in A(\Gamma)\}$ and $\{u^g, v^h\} \in E(X^e)$ iff $[u^g, v^h] = 1$ in $A(\Gamma)$. There is a natural right-action of $A(\Gamma)$ on X^e defined by $v^g.h = v^{gh}$. The *opposite graph* X^{opp} of a graph X is defined by $V(X^{\mathrm{opp}}) = V(X)$ and $E(X^{\mathrm{opp}}) = \binom{V(X)}{2} \setminus E(X)$. A graph X is *anti-connected* if X^{opp} is connected. Note that every graph is a join of anti-connected graphs.

Proposition 3. [31, 32] *Let X be a finite, connected and anti-connected graph, which is not a single vertex.*

(1) *X^e is a connected, locally infinite quasi-tree of infinite diamter.*
(2) *The action of $A(\Gamma)$ on X^e is* acylindrical.

For the definition of *acylindricity* and related results for mapping class group actions curve complexes, see [8]. The above proposition illustrates some analogies between extension graphs of RAAGs and curve complexes of surfaces. Coming back to our original motivation of Question 1, let us define another concept, the *clique graph*. A *clique* in a graph is a collection of vertices which are pairwise adjacent. For a possibly infinite graph Y, we define the clique graph Y_k of Y by declaring that $V(Y_k)$ is the set of cliques in Y and we join $K, L \in V(Y_k)$ if and only if $K \cup L \in V(Y_k)$.

Let us write $H \hookrightarrow G$ for two groups H and G if there is an injective group homomorphism from H into G.

Theorem 4. [31] *Let X and Y be finite graphs.*

(1) *If X embeds into Y^e as an induced subgraph, then $A(X)$ embeds into $A(Y)$.*
(2) *If $A(X)$ embeds into $A(Y)$, then X embeds into $(Y^e)_k$ as an induced subgraph.*
(3) *If Y is triangle-free, then $X \leq Y^e$ and $A(X) \hookrightarrow A(Y)$ are equivalent.*

For two groups G and H with given metrics, we say G is a *q.i. embedded subgroup* of H if there exist a group embedding $f : G \to H$ and a constant $C \geq 1$ such that every $x, y \in G$ satisfies

$$\frac{1}{C} d(x,y) - C \leq d(f(x), f(y)) \leq C d(x,y) + C.$$

We will need another type of embeddings between RAAGs. Let X be a finite graph. Consider the universal covering map $p \colon \tilde{X} \to X$, and an arbitrary finite subtree $T \leq \tilde{X}$. We can define a map $\phi(X,T) \colon G(X) \to G(T)$ by

$$\phi(X,T)(v) = \prod_{u \in p^{-1}(v) \cap T} u.$$

Note that $\phi(X,T)$ is well-defined, but not necessarily injective. An important observation is that it becomes injective for sufficiently large T.

Theorem 5 ([34])**.** *For each X, there exists a finite tree T such that $G(X)$ is a q.i. embedded subgroup of $G(T)$.*

§3. RAAGs in Mods

The embedability of RAAGs in mapping class groups has surprising similarity with Theorem 4. For a surface S, we let $\mathcal{C}(S)$ be the curve graph of S, namely the one-skeleton of the curve complex.

Theorem 6. *Let X be a finite graph and S be a connected orientable surface with finite negative Euler characteristic.*

(1) [35] *If X embeds into $\mathcal{C}(S)$ as an induced subgraph, then $A(X)$ embeds into* $\mathrm{Mod}(S)$.

(2) [33] *If $A(X)$ embeds into* $\mathrm{Mod}(S)$, *then X embeds into $\mathcal{C}(S)_k$ as an induced subgraph.*

(3) [30] *If $\mathcal{C}(S)$ is triangle-free, then $X \le \mathcal{C}(S)$ and $A(X) \hookrightarrow \mathrm{Mod}(S)$ are equivalent. If $\mathcal{C}(S)$ contains a complete graph on four vertices, then there exists a subgroup $A(X) \hookrightarrow \mathrm{Mod}(S)$ for which X is not a subgraph of $\mathcal{C}(S)$.*

In particular, if $A(X)$ embeds into $\mathrm{Mod}(S)$, then the chromatic number of X is at most 2^N, where N is the chromatic number of $\mathcal{C}(S)$. This gives a new obstruction for a RAAG to embedability into a mapping class group; the reader may compare this with the abelian rank obstruction given in [7].

Each RAAG embeds into some mapping class group. This was first seen by Crisp and Farb (unpublished), and can be deduced again from Theorem 6 (1). Regarding genus zero case, note first that we can embed $G(T)$ for each finite tree T into some planar braid group B_n [17]. Combining with Theorem 5, we have the following.

Theorem 7. *Each RAAG embeds into some braid group.*

Bridson found a RAAG which has an unsolvable isomorphism problem for finitely presented subgroups [11]. Using the above mentioned result of Crisp and Farb, Bridson deduced that a sufficiently large genus mapping class group does not have a solvable isomorphism problem for finitely presented subgroups. Theorem 7 immediately implies the same for braid groups:

Corollary 8 ([34])**.** *The isomorphism problem for f.p. subgroups is not solvable for B_n if n is sufficiently large.*

§4. RAAGs in Diffeos

Let S be a surface with a fixed Riemannian metric. We denote by $\mathrm{Symp}(S)$ the group of orientation- and area-preserving smooth diffeomorphisms on S. Suppose $\alpha : [0,1] \to \mathrm{Symp}(S)$ be a smooth isotopy on

S. For $p \geq 1$, we define the L^p*-length* of α as the integral

$$\ell_p(\alpha) = \int_0^1 \left(\int_{x \in S} \left\| \frac{\partial \alpha_t}{\partial t} \right\|^p dx \right)^{1/p} dt.$$

This defines a right-invariant length metric d_p and a norm $\|\cdot\|_p$ on $\mathrm{Symp}(S)$.

In [17], it was shown that if X is a finite *planar* graph then $G(X)$ is a q.i. embedded subgroup of $\mathrm{Symp}(D^2)$ equipped with L^2 metric; see [9] for a generalization for L^p metric for $p \geq 1$ in the case of finitely generated free abelian groups. M. Kapovich proved that every RAAG embeds into $\mathrm{Symp}(S^2)$, asking whether or not it could be q.i. embedded with respect to the L^2 metric of the latter group [26]. A remarkable inequality relating word length of a spherical braid group and the L^p metric on $\mathrm{Symp}(S^2)$ for $p > 2$ is given in [10]. Building on these ideas and using the embedding (which happens to be quasi-isometric) found in Theorem 5, we have the following results.

Theorem 9 ([34]). (1) *Every RAAG is a q.i. embedded subgroup of* $\mathrm{Symp}(D^2)$ *with the* L^p *metric, for* $p \geq 1$.
(2) *Every RAAG is a q.i. embedded subgroup of* $\mathrm{Symp}(S^2)$ *with the* L^p *metric, for* $p > 2$.

Let us describe an argument for (2) above. By Theorem 5, we have only to consider $G(T)$ for a tree T. Let $V(T) = \{v_1, v_2, \ldots, v_k\}$. There exists a collection of simply connected subsurfaces $S_1, \ldots, S_k$ in S^2 whose intersection graph is T. Let us consider a set $P = \{m_1, m_2, \ldots, m_n\}$ of marked points in S^2, which contains at least two points from each of the regions in $S^2 \setminus \cup_i \partial S_i$.

Denote by $X_n \subseteq S^2 \times \cdots \times S^2$ the configuration space of n distinct points. We let $\{D_1, D_2, \ldots, D_n\}$ be a fixed collection of disjoint neighbourhoods of m_i, and put $D_0 = D_1 \times \cdots \times D_n \subseteq X_n$. We define $\mathcal{P}_n \leq \mathrm{Symp}(S^2)$ as the subgroup of diffeomorphisms that restrict to the identity on each D_i. We can choose a pseudo-Anosov diffeomorphism f_i on $S_i \setminus \cup_j D_j$ and extend f_i as the identity outside S_i and within $\cup_j D_j$. Recall we have a commutative diagram

$$\begin{array}{ccc} G(T) & \xrightarrow{\ \psi\ } & \mathcal{P}_n \\ & \searrow^{\phi} & \downarrow{\scriptstyle f \mapsto [f]} \\ & & \mathrm{Mod}(S^2 \setminus P) \end{array}$$

where $\phi: v_i \mapsto [f_i^N]$. By a theorem of Clay, Leininger and Mangahas in [14], the map ϕ embeds $G(T)$ into $\mathrm{Mod}(S^2 \setminus P)$ as a q.i. embedded subgroup for each sufficiently large N.

It is more convenient to lift the map $\mathcal{P}_n \to \mathrm{Mod}(S^2 \setminus P)$ with respect to the two-sheeted universal covering map $p : \widetilde{\mathrm{Symp}(S^2)} \to \mathrm{Symp}(S^2)$. Namely, let us consider the quotient map from $\widetilde{\mathcal{P}_n} = p^{-1}(\mathcal{P}_n)$ to the pure braid group $PB_n(S^2)$. For each isotopy α in $\mathrm{Symp}(S^2)$ joining the identity to an element in $\mathcal{P}_n$ and an n-tuple $x \in D_0$, we denote by $[\alpha(x)]$ the n-strand braid defined as the trace of x over the isotopy α. We let $\|[\alpha(x)]\|$ as the word-length in $PB_n(S^2)$. It now suffices to show the following claim:

Claim. *Let $p > 2$. There exists C such that for each isotopy $\alpha : [0,1] \to \mathrm{Symp}(S^2)$ with $\alpha(0) = \mathrm{Id}$ and $\alpha(1) \in \mathcal{P}_n$, we have*

$$\|[\alpha(P)]\| \leq C\ell_p(\alpha) + C.$$

The proof of the claim is essentially given in [10], and can be summarised as follows. We may first assume that $\alpha(D_0) \subseteq \mathbb{C}^n \subseteq (S^2)^n$ possibly after disregarding a measure zero set. So we can consider the stereographic projection of $\alpha(x)$ onto $\mathbb{C}$ for $x \in D_0$. Let $1 \leq i \neq j \leq n$. For each $x = (x_1, x_2, \ldots, x_n) \in D_0$, let us define the *crossing number* $c_\alpha(x_i, x_j, \omega)$ of $\alpha(x)$ for $\omega \in S^1$ as the sum of the counts of t such that $\alpha(t)(x_i) - \alpha(t)(x_j)$ is parallel to ω. We let $c_\alpha(x_i, x_j)$ be the average of $c_\alpha(x_i, x_j, \omega)$ over $\omega \in S^1$. By the minimality of the word-length, $\sum_{i,j} c_\alpha(x_i, x_j)$ is asymptotically bounded below by $\|[\alpha(P)]\| = \|[\alpha(x)]\|$ for $x \in D_0$.

For each $x \in D_0$ and $i \neq j$, let us consider the closed curve $\gamma_\alpha(x_i, x_j; t)$ in S^1 defined by the unit vector in the direction of $\alpha(t)(x_i) - \alpha(t)(x_j)$ at each time t. By the co-area formula, $c_\alpha(x_i, x_j)$ is equal to the length of the curve $\gamma_\alpha(x_i, x_j; t)$.

The final step of the proof is a clever application of Hölder inequality [10], to show that there is C' such that

$$\int_0^1 \int_{D_i} \int_{D_j} \left| \frac{\partial \gamma_\alpha(x_i, x_j; t)}{\partial t} \right| dx_j\, dx_i\, dt \leq C'\ell_p(\alpha) + C'.$$

We finally remark that Theorem 5 is also used in the proof of the following:

Theorem 10 ([12]). *Every RAAG embeds into the group of area-preserving, boundary-fixing piecewise linear homeomorphism group of the square I^2.*

§5. One-dimensional manifolds

The mapping class group of a puctured surface embeds into $\mathrm{Homeo}(S^1)$ but not in $\mathrm{Diff}^2(S^1)$ [19]. It is a famous open question whether or not mapping class groups virtually embed into $\mathrm{Diff}^2(S^1)$ or $\mathrm{Homeo}(\mathbb{R})$. Every RAAG A embeds into some mapping class group G, and once there is such an embedding, A embeds into every finite-index subgroup of G. With Baik, the authors proved that RAAGs can be given C^∞ regularity in the following sense:

Theorem 11. [4] *Every RAAG embeds into* $\mathrm{Diff}^\infty(\mathbb{R})$.

Related to the aforementioned question on mapping class groups, we currently do not know an answer to the following:

Question 4. *Does every RAAG embed into* $\mathrm{Diff}^\infty(S^1)$*?*

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