# ON MASS CONCENTRATION FOR THE $L^{2}$-CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS 

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#### Abstract

We consider the mass concentration phenomenon for the $L^{2}$-critical nonlinear Schrödinger equations. We show the mass concentration of blow-up solutions contained in $L_{t}^{q} L_{x}^{r}$ space near the finite time. The new ingredient in this paper is a refinement of Strichartz's estimates with the mixed norm $L_{t}^{q} L_{x}^{r}$ for $2<q \leq r$.


## 1. INTRODUCTION

We consider the $L^{2}$-critical Cauchy problem in $\mathbb{R}^{d}, d \geq 2$,

$$
\left\{\begin{array}{l}
i u_{t}+\Delta u= \pm|u|^{\frac{4}{d}} u,(t, x) \in \mathbb{R}_{+} \times \mathbb{R}^{d}  \tag{1.1}\\
u(0, x)=u_{0}(x)
\end{array}\right.
$$

It is well known that (1.1) is locally well posed in the critical sense so that the time $T$ of existence depends not only the size of initial data but also on the profile of the data. For $u_{0} \in L^{2}\left(\mathbb{R}^{d}\right)$, there exists the unique solution $u(t, x)$,

$$
u \in C\left([0, T) ; L^{2}\left(\mathbb{R}^{d}\right)\right) \cap L^{\frac{2(d+2)}{d}}\left([0, T) ; L^{\frac{2(d+2)}{d}}\left(\mathbb{R}^{d}\right)\right)
$$

which conserves the mass;

$$
\|u(t, \cdot)\|_{L_{x}^{2}\left(\mathbb{R}^{d}\right)}=\|u(0, \cdot)\|_{L_{x}^{2}\left(\mathbb{R}^{d}\right)}, 0<t<T
$$

The existence time interval $[0, T]$ is extended as long as

$$
\|u\|_{L_{t, x}^{\frac{2(d+2)}{d}}}^{\left([0, T] \times \mathbb{R}^{d}\right)} \ll \infty .
$$

In the blow-up direction Bourgain [3] showed that if the $L^{2}$-well posed solution in $\mathbb{R}^{2}$ breaks down at some maximal time $0<T^{*}<\infty$ with

$$
\begin{equation*}
\|u\|_{L_{t, x^{d}}^{2(d+2)}}^{\left(\left[0, T^{*}\right) \times \mathbb{R}^{d}\right)}, ~=\infty \tag{1.2}
\end{equation*}
$$

then the blow-up solution has a mass concentration phenomenon when $d=2$ :

$$
\begin{equation*}
\limsup _{t \nearrow T^{*}} \sup _{x \in \mathbb{R}^{d}} \int_{B\left(x,\left(T^{*}-t\right)^{\frac{1}{2}}\right)}|u(t, x)|^{2} d x \geq \epsilon \tag{1.3}
\end{equation*}
$$

where $\epsilon=\left\|u_{0}\right\|_{2}^{-M}$ for some $M>0$. Later, this was extended to higher dimensions by Bégout and Vargas [1]. Both of the results were obtained by the use of refinement

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Figure 1. We are interested in the case where $(1 / r, 1 / q)$ is contained in the line segment $(A, B)$. The mass concentration for other admissible pairs follows from the earlier results (see $[1,3]$ ) dealing with non-mixed norm blow up.
of Strichartz's estimates which come from bilinear restriction estimate for the paraboloid $[14,17,18]$. On the other hand, in the focusing case the ground state mass concentration was studied with initial data in $H^{s}, s>0$. We also refer $[6,9,13,19]$ to the readers for further results in this direction.

In this paper we are concerned with the mass concentration of solution to the (1.1) when the initial datum $u_{0} \in L^{2}$ and its mixed $L_{t}^{q} L_{x}^{r}$-norm blows up in finite time. We basically rely on the argument due to Bourgain [3]. We also consider $L^{2}$-critical Hartree equation for which the similar approach works.

For $q, r \geq 2$, we say that a pair $(q, r)$ is admissible if

$$
\frac{2}{q}+\frac{d}{r}=\frac{d}{2}
$$

and $(q, r) \neq(2, \infty)$ when $d=2$. The following is our first result which is a natural generalization of results in $[1,3]$ to mixed norm spaces $L_{t}^{q} L_{x}^{r}$ with admissible ( $q, r$ ).

Theorem 1.1. Let $(q, r)$ be an admissible pair with $q>2$ when $d \geq 4$ and $q \geq(d+$ 4)/d when $d=2$, 3. Suppose that the solution of (1.1) satisfies $\|u\|_{L_{t}^{q} L_{x}^{r}\left([0, t) \times \mathbb{R}^{d}\right)}<\infty$ for $0<t<T^{*}<\infty$ and

$$
\begin{equation*}
\|u\|_{L_{t}^{q} L_{x}^{x}\left(\left[0, T^{*}\right) \times \mathbb{R}^{d}\right)}=\infty \tag{1.4}
\end{equation*}
$$

Then (1.3) holds.
The readers may try to establish the similar result for $d=1$ by following the argument (see also for related results [4, 10]). However, refinements of the usual

Strichatz's and restriction estimates (Proposition 2.3, 2.5) do not seem to be available in this case.
From the previously known results it is enough to consider the case $q \leq r$ in which $q \leq 2(d+2) / d$. From interpolation with the conserved mass it is clear that if $\|u\|_{L_{t}^{q_{0}} L_{x}^{r_{0}}\left(\left[0, T^{*}\right)\right)}=\infty$ for some admissible $\left(q_{0}, r_{0}\right)$ then $\|u\|_{L_{t}^{q} L_{x}^{r}\left(\left[0, T^{*}\right)\right)}=\infty$ for all admissible ( $q, r$ ) satisfying $q \leq q_{0}$ (see Figure 1). Hence from the results due to Bourgain [3], Bégout and Vargas [1] one can see there is a mass concentration if (1.4) holds and $q \geq 2(d+2) / d$.

We also remark that from the standard $L^{2}$-critical theory ([5]) based on contraction argument the local wellposedness of (1.1) can be established in the mixed norm space such that

$$
u \in C\left([0, T] ; L^{2}\left(\mathbb{R}^{d}\right)\right) \cap L^{q}\left([0, T] ; L^{r}\left(\mathbb{R}^{d}\right)\right)
$$

provided $(q, r)$ is admissible and $q \geq \max (2,(d+4) / d)$ and the blow-up criterion (1.2) is replaced with such admissible pair $(q, r)$. In fact, let us recall the homogeneous $\left\|e^{i t \Delta} f\right\|_{L_{t}^{q} L_{x}^{r}} \leq C\|f\|_{L^{2}}$ and inhomogeneous Strichartz's estimates

$$
\begin{equation*}
\left\|\int_{0}^{t} e^{i t \Delta(t-s)} F(s) d s\right\|_{L_{t}^{q} L_{x}^{L_{x}}} \leq C\|F\|_{L_{t}^{\tilde{q}^{\prime}} L_{x}^{\tilde{r}^{\prime}}} \tag{1.5}
\end{equation*}
$$

which are valid for admissible $(q, r)$ and $(\tilde{q}, \tilde{r})$ (see $[7,11,16])$. Applying the standard fixed point argument to (1.1) together with Duhamel's formula (see (3.2)), to make it sure that the nonlinear map is a contraction ${ }^{1}$ we need only to check that there are admissible pairs $(q, r),(\tilde{q}, \tilde{r})$ satisfying

$$
\begin{equation*}
\left(\frac{4}{d}+1\right) \frac{1}{q}=\frac{1}{\tilde{q}^{\prime}}, \quad\left(\frac{4}{d}+1\right) \frac{1}{r}=\frac{1}{\tilde{r}^{\prime}} . \tag{1.6}
\end{equation*}
$$

Since $\frac{2}{\widetilde{q}^{\prime}}+\frac{d}{\widetilde{r}^{\prime}}=2+\frac{d}{2}$ and the vectors $\left(\frac{1}{r}, \frac{1}{q}\right)$ and $\left(\frac{1}{\widetilde{r}^{\prime}}, \frac{1}{\widetilde{q}^{\prime}}\right)$ should be parallel, by a simple geometric consideration (see Figure 2) one can see that there are admissible pairs $(q, r),(\tilde{q}, \tilde{r})$ satisfying (1.6) long as $q \geq \max (2,(d+4) / d)$ when $q \leq r$.

Secondly, we consider the $L^{2}$-critical Hartree equation, which is given by

$$
\left\{\begin{array}{l}
i u_{t}+\Delta u= \pm\left(|x|^{-2} *|u|^{2}\right) u  \tag{1.7}\\
u(0, x)=u_{0}(x) \in L^{2}\left(\mathbb{R}^{d}\right), \quad d \geq 3
\end{array}\right.
$$

The Hartree nonlinearity is the form of $\left(|x|^{-\gamma} *|u|^{2}\right) u$ for $0<\gamma<d$. The case $\gamma=2$ is mass critical and the equation (1.2) is scaling invariant under $u \rightarrow u^{\lambda}(t, x)=$

[^1]

Figure 2. The line segment $\left[\left(\frac{1}{2}, 0\right),\left(\frac{d-2}{2 d}, \frac{1}{2}\right)\right]$ stands for the reciprocal pair $\left(\frac{1}{r}, \frac{1}{q}\right)$ of admissible pair $(q, r)$, and $\left[\left(\frac{1}{2}, 1\right),\left(\frac{d+2}{2 d}, \frac{1}{2}\right)\right]$ stands for $\left(\frac{1}{\widetilde{r}^{\prime}}, \frac{1}{\tilde{q}^{\prime}}\right)$ of the dual exponents of admissible pairs $(\tilde{q}, \tilde{r})$. The points $A=(1 / r, 1 / q)$ and $B=\left(1 / \tilde{r}^{\prime}, 1 / \tilde{q}^{\prime}\right)$ satisfy the relation (1.6).
$\lambda^{-\frac{d}{2}} u\left(\frac{t}{\lambda^{2}}, \frac{x}{\lambda}\right)$. Even though the nonlinear term is different from (1.1), the local wellposedness theory and the blow-up criterion (1.2) for the equation (1.7) are completely the same as those of (1.1). One may be interested in a mass concentration for the finite time blow-up solutions for (1.7). It turns out that the mass concentration phenomenon is mostly involved with the homogeneous part of the solution. The method developed in $[1,3]$ can be applied to the solution of (1.7) without much modifications as long as one can control the nonlinear term effectively.

Theorem 1.2. Let $d \geq 3$. Let $(q, r)$ be an admissible pair with $2<q \leq r$. Suppose that the solution of (1.7) satisfies $\|u\|_{L_{t}^{q} L_{x}^{r}\left([0, t) \times \mathbb{R}^{d}\right)}<\infty$ for $0<t<T^{*}<\infty$ and (1.4). Then (1.3) holds.

It is also possible to obtain analogous results in mixed norm space $L_{t}^{q} L_{x}^{r}$ for the 2-dimensional non-elliptic Schrödinger equation which was considered in [15] as long as $(q, r)$ is admissible and $q \geq 3$.

The paper is organized as follows. In Section 2 we obtain preliminaries estimates which will be used in the proofs of Theorems. In Section 3 we prove Theorems 1.1 and 1.2.

## 2. PRELIMINARY

In this section we show several lemmas which will be used later for the proofs of the theorems. In order to deal with a mass concentration in the mixed norm space $L_{t}^{q} L_{x}^{r}$, we will use the mixed norm bilinear restriction estimates in the following.
2.1. The space $X_{p}^{q, r}(s)$. For each $j \in \mathbb{Z}$, we decompose $\mathbb{R}^{d}$ into disjoint dyadic (partly open) cubes of side length $2^{-j}$ which are given by

$$
Q_{k}^{j}=\prod_{i=1}^{d}\left[k_{i} 2^{-j},\left(k_{i}+1\right) 2^{-j}\right)
$$

with $k=\left(k_{1}, \ldots, k_{d}\right) \in \mathbb{Z}^{d}$. Let us set $f_{k}^{j}=f \chi_{Q_{k}^{j}}$. For $1 \leq p, r, s \leq \infty$ we define

$$
\|f\|_{X_{p}^{q, r}(s)}=\left(\sum_{j}\left[\sum_{k}\left(2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f_{k}^{j}\right\|_{L^{p}}\right)^{q}\right]^{\frac{r}{q}}\right)^{\frac{1}{r}} .
$$

The following is a slight generalization of Theorem 1.3 in [1].
Lemma 2.1. For $p<s<q \leq r$, there is a $\theta \in(0,1)$ such that

$$
\|f\|_{X_{p}^{q, r}(s)} \leq C\left(\sup _{j, k} 2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f_{k}^{j}\right\|_{L^{p}}\right)^{\theta}\|f\|_{L^{s}}^{1-\theta} .
$$

To show Lemma 2.1, it suffices to consider the case $q=r$ because $\ell^{q} \cap \ell^{\infty} \supset \ell^{r}$. Note that the $X_{p}^{q, r}(s)$-norm is actually a mixed norm. Hence in view of (complex) interpolation (see [2]) it is enough to show that

$$
\begin{equation*}
\|f\|_{X_{p}^{\infty, \infty}(s)} \leq \sup _{j, k} 2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f_{k}^{j}\right\|_{L^{p}} \tag{2.1}
\end{equation*}
$$

and for $p<s<q$

$$
\begin{equation*}
\|f\|_{X_{p}^{q, q}(s)} \leq C\|f\|_{L^{s}} \tag{2.2}
\end{equation*}
$$

Since (2.1) is obvious, we need only to show (2.2).
Proof of (2.2). To begin with, we may assume $\|f\|_{L^{s}}=1$ and it is enough to show $\|f\|_{X_{p}^{q, q}(s)} \leq C$. We break into $f=f^{j}+f_{j}$ where $f^{j}=\chi_{\left\{|f| \geq 2^{\left.j^{\frac{d}{s}}\right\}}\right.}$. Then,

$$
\|f\|_{X_{p}^{q, q}(s)}^{q} \leq 2^{q}\left(\sum_{j} \sum_{k}\left(2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f^{j} \chi_{Q_{k}^{j}}\right\|_{L^{p}}\right)^{q}+\sum_{j} \sum_{k}\left(2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f_{j} \chi_{Q_{k}^{j}}\right\|_{L^{p}}\right)^{q}\right)
$$

Since $p<s<q$, we have

$$
\sum_{j} \sum_{k}\left(2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f^{j} \chi_{Q_{k}^{j}}\right\|_{L^{p}}\right)^{q} \leq\left(\sum_{j} \sum_{k} 2^{j p\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f^{j} \chi_{Q_{k}^{j}}\right\|_{L^{p}}^{p}\right)^{\frac{q}{p}}
$$

Since $Q_{k}^{j}$ are disjoint for each fixed $j$, by taking summation along $k$ we obtain that

$$
\begin{aligned}
\sum_{j} \sum_{k}\left(2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f^{j} \chi_{Q_{k}^{j}}\right\|_{L^{p}}\right)^{q} & \leq\left(\int_{\mathbb{R}^{d}} \sum_{j} 2^{j p\left(\frac{d}{p}-\frac{d}{s}\right)}\left|f^{j}(x)\right|^{p} d x\right)^{\frac{q}{p}} \\
& =\left(\int_{\mathbb{R}^{d}} \sum_{|f(x)| \geq^{2^{j}}} 2^{j p\left(\frac{d}{p}-\frac{d}{s}\right)}|f(x)|^{p} d x\right)^{\frac{q}{p}} \\
& \leq C\|f\|_{L^{s}}^{\frac{q}{p} s} \leq C
\end{aligned}
$$

because $\|f\|_{s}=1$. On the other hand, by Hölder's inequality and the fact that $p<s<q$ we see that

$$
\sum_{j} \sum_{k}\left(2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f_{j} \chi_{Q_{k}^{j}}\right\|_{\left.L^{p}\right)^{q}} \leq \sum_{j} \sum_{k} 2^{j q\left(\frac{d}{q}-\frac{d}{s}\right)}\left\|f_{j} \chi_{Q_{k}^{j}}\right\|_{L^{q}}^{q} .\right.
$$

Again using the disjointness of $Q_{k}^{j}$ and taking summation along $k$, we have

$$
\begin{aligned}
\sum_{j} \sum_{k}\left(2^{j\left(\frac{d}{p}-\frac{d}{s}\right)}\left\|f_{j} \chi_{Q_{k}^{j}}\right\|_{\left.L^{p}\right)^{q}}\right. & \leq \int_{\mathbb{R}^{d}} \sum_{j} 2^{j q\left(\frac{d}{q}-\frac{d}{s}\right)}\left|f_{j}(x)\right|^{q} d x \\
& =\int_{\mathbb{R}^{d}} \sum_{|f(x)|<2^{j} \frac{d}{s}} 2^{j q\left(\frac{d}{q}-\frac{d}{s}\right)}|f(x)|^{q} d x \\
& \leq\|f\|_{L^{s}}^{s}=1
\end{aligned}
$$

Therefore, we get the desired.
2.2. Refinement of Strichartz's estimates. We recall the bilinear restriction estimates for the paraboloid Theorem 2.3 in [12]. It is a mixed norm generalization of bilinear restriction estimates due to Tao [17].

Theorem 2.2. Let $Q_{1}$ and $Q_{2}$ be cubes of side length 1. Suppose

$$
\operatorname{dist}\left(Q_{1}, Q_{2}\right) \sim 1
$$

and $\widehat{f}$ and $\widehat{g}$ are supported in $Q_{1}$ and $Q_{2}$, respectively. Then for $q, r>2$ satisfying $q>\min (2,8 /(d+1)), 2 / q<(d+1)\left(\frac{1}{2}-\frac{1}{r}\right)$, we have

$$
\begin{equation*}
\left\|e^{i t \Delta} f e^{i t \Delta} g\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C\|\widehat{f}\|_{L^{2}}\|\widehat{g}\|_{L^{2}} \tag{2.3}
\end{equation*}
$$

where $C$ is independent of $Q_{1}$ and $Q_{2}$.
From the interpolation between (2.3) and trivial $L^{\infty}$ - $L^{1}$ estimates, for any $2<$ $q \leq r$ and $2 / q+d / r=d / 2$ there is a $p<2$ such that

$$
\left\|e^{i t \Delta} f e^{i t \Delta} g\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C\|\widehat{f}\|_{L^{p}}\|\widehat{g}\|_{L^{p}}
$$

provided that supp $\widehat{f}$ and supp $\widehat{g}$ are contained in $B(0,1)$ and dist (supp $\widehat{f}$, supp $\widehat{g}$ ) $\sim$ 1. By the standard parabolic rescaling, there is a constant $C$ independent of $j \in \mathbb{Z}$
such that

$$
\begin{equation*}
\left\|e^{i t \Delta} f e^{i t \Delta} g\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C 2^{2 j\left(\frac{2}{q}+d\left(\frac{1}{r}+\frac{1}{p}-1\right)\right)}\|\widehat{f}\|_{L^{p}}\|\widehat{g}\|_{L^{p}} \tag{2.4}
\end{equation*}
$$

provided that supp $\widehat{f}$ and supp $\widehat{g}$ are contained in $B\left(x_{0}, 2^{1-j}\right)$ and dist (supp $\widehat{f}$, supp $\left.\widehat{g}\right) \sim$ $2^{-j}$.

Proposition 2.3. Let $q, r$ be numbers satisfying $2<q \leq r$ and $2 / q+d / r=d / 2$. Then there are numbers $p_{*}, q_{*}$ and $r_{*}$ such that $p_{*}<2<q_{*} \leq r_{*}$ and

$$
\left\|e^{i t \Delta} f\right\|_{L_{L_{x}^{q} L_{x}^{r}}} \leq C\|\widehat{f}\|_{X_{p *}^{q *}, r_{*}(2)}
$$

This can be shown by interpolating the following two estimates:

$$
\left\|e^{i t \Delta} f\right\|_{L_{t}^{q} L_{x}^{r}} \leq C\|\widehat{f}\|_{X_{2}^{\infty, 2}(2)}
$$

and for some $p<2$,

$$
\begin{equation*}
\left\|e^{i t \Delta} f\right\|_{L_{t}^{q} L_{x}^{r}} \leq C\|\widehat{f}\|_{X_{p}^{2, q}(2)} \tag{2.5}
\end{equation*}
$$

In fact, the first estimate is actually the Strichartz's estimates and for the second inequality we use Theorem 2.2 . We also use the following simple lemma:

Lemma 2.4 ([18], Lemma 6.1). Let $\left\{R_{k}\right\}$ be a collection of rectangles in frequency space such that the dilates $\left\{2 R_{k}\right\}$ are boundedly overlapping, and suppose that $\left\{F_{k}\right\}$ are a collection of functions whose Fourier supports are contained in $\left\{R_{k}\right\}$. Then for $1 \leq p \leq \infty$ we have

$$
\left(\sum_{k}\left\|F_{k}\right\|_{L^{p}}^{p^{*}}\right)^{1 / p^{*}} \lesssim\left\|\sum_{k} F_{k}\right\|_{L^{p}} \lesssim\left(\sum_{k}\left\|F_{k}\right\|_{L^{p}}^{p_{*}}\right)^{1 / p_{*}}
$$

where $p_{*}=\min \left(p, p^{\prime}\right)$ and $p^{*}=\max \left(p, p^{\prime}\right)$.
Proof of (2.5). We say $Q_{k}^{j} \sim Q_{k^{\prime}}^{j}$ to mean that $Q_{k}^{j}$ and $Q_{k^{\prime}}^{j}$ are not adjacent but have adjacent parent cubes of diameter $2^{-j}$. So if $Q_{k}^{j} \sim Q_{k^{\prime}}^{j}$, then dist $\left(Q_{k}^{j}, Q_{k^{\prime}}^{j}\right) \sim 2^{-j}$. By a Whitney decomposition of $\mathbb{R}^{d} \times \mathbb{R}^{d}$ away from the diagonal $D$ of $\mathbb{R}^{d} \times \mathbb{R}^{d}$, ignoring some harmless measure zero set, we have

$$
\mathbb{R}^{d} \times \mathbb{R}^{d}=\bigcup_{j \in \mathbb{Z}} \bigcup_{Q_{k}^{\prime} \sim Q_{k^{\prime}}^{\prime}} Q_{k}^{j} \times Q_{k^{\prime}}^{j}
$$

(see also [18]). Hence $\sum_{j \geq 1} \sum_{Q_{k}^{j} \sim Q_{k^{\prime}}^{j}} \chi_{Q_{k}^{j}} \chi_{Q_{k^{\prime}}^{j}}=1$ almost everywhere. So we see that

$$
e^{i t \Delta} f e^{i t \Delta} f=\sum_{j} \sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}
$$

where $f_{k}^{j}=\left(\widehat{f} \chi_{Q_{k}^{j}}\right)^{\vee}$. Therefore, in order to get linear estimates it is enough to show that

$$
\begin{equation*}
\sum_{j}\left\|\sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C\|\widehat{f}\|_{X_{p}^{2, q}(2)}^{2} \tag{2.6}
\end{equation*}
$$

We observe that for each fixed $t$ the supports of $e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}$ are contained boundedly overlapping cubes when $k \sim k^{\prime}$. We also note that $r \leq 4$ when $d \geq 2$ because $2<q \leq r$. So by Lemma 2.4 we see that

$$
\left\|\sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C\left\|\left(\sum_{k \sim k^{\prime}}\left\|e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{x}^{r / 2}}^{r / 2}\right)^{2 / r}\right\|_{L_{t}^{q / 2}} .
$$

Since $q \leq r$, we get

$$
\left\|\sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C\left(\sum_{k \sim k^{\prime}}\left\|e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}}^{q / 2}\right)^{\frac{2}{q}} .
$$

Now using (2.4) and Schwarz's inequality, it follows that

$$
\begin{aligned}
\left\|\sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} & \leq C 2^{2 j\left(\frac{d}{p}-\frac{d}{2}\right)}\left(\sum_{k \sim k^{\prime}}\left\|\widehat{f}_{k}^{j}\right\|_{p}^{q / 2}\left\|\widehat{f}_{k^{\prime}}^{j}\right\|_{p}^{q / 2}\right)^{\frac{2}{q}} \\
& \leq\left(\sum_{k} 2^{j q\left(\frac{d}{p}-\frac{d}{2}\right)}\left\|\widehat{f}_{k}^{j}\right\|_{L^{p}}^{q}\right)^{\frac{2}{q}}
\end{aligned}
$$

Putting the above in the right hand side of (2.6), we get the required (2.5).
Proposition 2.5. Let $2<q \leq r \leq \infty$. Then if $\widehat{f}$ is supported in a ball of radius one, for $\frac{2}{q}+\frac{d}{r}<d\left(1-\frac{1}{q}\right)$ and $\frac{2}{q}+\frac{d+1}{r}<\frac{d+1}{2}$, there is a constant $C$ such that

$$
\left\|e^{i t \Delta} f\right\|_{L_{t}^{q} L_{x}^{r}} \leq C\|\widehat{f}\|_{L^{\infty}}
$$

Proof. Using the similar decomposition and notation which are used in the above, we need to show that for some $\epsilon>0$,

$$
\left\|\sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C 2^{-\epsilon j}\|\widehat{f}\|_{L^{p}}\|\widehat{g}\|_{L^{p}}
$$

because $\widehat{f}$ is supported in a ball of radius one.
Since the spatial Fourier supports of $e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}$ are boundedly overlapping, we can see that

$$
\begin{equation*}
\left\|\sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} \leq C\left\|\left(\sum_{k \sim k^{\prime}}\left\|e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{x}^{r / 2}}^{r^{*}}\right)^{1 / r^{*}}\right\|_{L_{t}^{q / 2}} \tag{2.7}
\end{equation*}
$$

where $r^{*}=\min \left(\left(r / 2,(r / 2)^{\prime}\right)\right.$. Since $q \leq r$ and $(q, r)$ is admissible, we have $r^{*} \geq q / 2$. Hence it is easy to see that the left hand side of (2.7) is bounded by

$$
C\left(\sum_{k \sim k^{\prime}}\left\|e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}}^{q / 2}\right)^{2 / q} .
$$

Using (2.4) with $p=\infty$, we have

$$
\begin{aligned}
\left\|\sum_{k \sim k^{\prime}} e^{i t \Delta} f_{k}^{j} e^{i t \Delta} f_{k^{\prime}}^{j}\right\|_{L_{t}^{q / 2} L_{x}^{r / 2}} & \leq C\left(\sum_{k \sim k^{\prime}} 2^{q j\left(\frac{2}{q}+d\left(\frac{1}{r}-1\right)\right)}\right)^{2 / q}\|\widehat{f}\|_{L^{\infty}}\|\widehat{f}\|_{L^{\infty}} \\
& \leq C 2^{2\left(\frac{2}{q}+d\left(\frac{1}{r}-1+\frac{1}{q}\right)\right)}\|\widehat{f}\|_{L^{\infty}}\|\widehat{f}\|_{L^{\infty}} .
\end{aligned}
$$

This completes the proof.
2.3. Two Lemmas. In this subsection we extend two lemmas in $[1,3]$ to mixed norm spaces. They were playing crucial roles in showing a mass concentration. The first one is concerned with decomposition of the initial datum into functions of which Fourier transform is spreading rather than concentrating. In view of uncertainty principle the spreading part of the initial datum will concentrate on some spatial region. The second one enables us to find regions where the linear Schrödinger wave concentrates in the mixed norm space $L_{t}^{q} L_{x}^{r}$ (here ( $q, r$ ) is admissible) if the Fourier transform of the initial data does not severely concentrate.

For a given cube $\tau$ let us denote the side length of $\tau$ by $\ell(\tau)$.
Lemma 2.6. Let $q$, $r$ be numbers satisfying $2<q \leq r$ and $2 / q+d / r=d / 2$. Suppose $f \in L^{2}\left(\mathbb{R}^{d}\right)$ and

$$
\begin{equation*}
\left\|e^{i t \Delta} f\right\|_{L_{t}^{q} L_{x}^{r}} \geq \epsilon \tag{2.8}
\end{equation*}
$$

for some $\epsilon>0$. Then there exists $f_{k} \in L^{2}\left(\mathbb{R}^{d}\right)$, cubes $\tau_{k}$ and $A_{k} \in(0, \infty)$ for $k=1,2, \cdots, N$ with $N=N\left(\|f\|_{L^{2}}, d, \epsilon\right)$ such that
(S1) For all $k=1, \cdots, N$, supp $\widehat{f}_{k} \subset \tau_{k}$ such that

$$
\ell\left(\tau_{k}\right) \leq C\|f\|_{L^{2} \epsilon^{-\nu}}^{c} A_{k} \text { and }\left|\widehat{f}_{k}\right|<A_{k}^{-d / 2}
$$

(S2) $\left\|e^{i t \Delta} f-\sum_{k=1}^{N} e^{i t \Delta} f_{k}\right\|_{L_{t}^{q} L_{x}^{r}}<\epsilon$,
(S3) $\|f\|_{L^{2}}^{2}=\sum_{k=1}^{N}\left\|f_{k}\right\|_{L^{2}}^{2}+\left\|f-\sum_{k=1}^{N} f_{k}\right\|_{L^{2}}^{2}$.
Here the constants $C, c, \nu$ depend only on $d$.
The proof of the above lemma is based on the following simpler one.
Lemma 2.7. Under the same assumption as in Lemma 2.6 there exists a function $h \in L^{2}\left(\mathbb{R}^{d}\right)$, a cube $\tau$ and a number $A>0$ satisfying the followings:
(1) supp $\widehat{h} \subset \tau$ such that

$$
\ell(\tau) \leq\|f\|_{L^{2} \epsilon^{-\nu}}^{c} A \text { and }|\widehat{h}|<A^{-d / 2}
$$

(2) $\|h\|_{L^{2}} \geq C\|f\|_{L^{2}}^{-a} \epsilon^{b}$ for some $a, b$ and $C$,
(3) $\|f\|_{L^{2}}^{2}=\|h\|_{L^{2}}^{2}+\|f-h\|_{L^{2}}$.

Proof. Using Lemma 2.3 and taking $s=2$ in Lemma 2.1, we see that there are $0<\theta<1$ and $p<2$ such that

$$
\epsilon \leq\left\|e^{i t \Delta} f\right\|_{L_{t}^{q} L_{x}^{r}} \leq\|f\|_{L^{2}}^{1-\theta}\left(\sup _{j, k} 2^{d j\left(\frac{1}{p}-\frac{1}{2}\right)}\left\|\widehat{f} \chi_{k}^{j}\right\|_{L^{p}}\right)^{\theta}
$$

From (2.8) and the above, there exists $\tau$ with $\ell(\tau)=2^{-j}$ such that

$$
\begin{equation*}
\|\widehat{f}\|_{L^{p}(\tau)}^{p} \geq\left(\epsilon^{\frac{1}{\theta}} 2^{d j\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_{L^{2}}^{1-1 / \theta}\right)^{p} . \tag{2.9}
\end{equation*}
$$

Since $p<2$, we have

$$
\int_{\tau \cap\{\widehat{f} \geq M\}}|\widehat{f}(\xi)|^{2}|\widehat{f}(\xi)|^{p-2} d \xi \leq M^{p-2} \int_{\tau \cap\{\hat{f} \geq M\}}|\widehat{f}(\xi)|^{2} d \xi
$$

Combining this with (2.9)

$$
\int_{\tau \cap\{\hat{f}<M\}}|\widehat{f}(\xi)|^{p} d \xi \geq\left(\epsilon^{\frac{1}{\bar{\theta}}} 2^{d j\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_{L^{2}}^{1-\frac{1}{\theta}}\right)^{p}-M^{p-2} \int_{\tau \cap\{\hat{f} \geq M\}}|\widehat{f}(\xi)|^{2} d \xi
$$

Now we choose $M$ which satisfies $M^{p-2}=\frac{1}{2} \epsilon^{\frac{p}{\theta}}\|f\|_{L^{2}}^{p-\frac{p}{\theta}-2} 2^{d j\left(\frac{1}{2}-\frac{1}{p}\right) p}$. Then it follows that

$$
\begin{equation*}
\ell(\tau)=2^{-j}=M^{-\frac{2}{d}}\left(\epsilon^{\frac{p}{\theta}}\|f\|_{L^{2}}^{p-\frac{p}{\partial}-2} / 2\right)^{\frac{2}{d(p-2)}} \tag{2.10}
\end{equation*}
$$

and

$$
\int_{\{\hat{f}<M\} \cap \tau}|\widehat{f}(\xi)|^{p} d \xi \geq \frac{1}{2}\left(\epsilon^{\frac{1}{\theta}} 2^{d j\left(\frac{1}{2}-\frac{1}{p}\right)}\|f\|_{L^{2}}^{1-\frac{1}{\theta}}\right)^{p} .
$$

By Hölder's inequality the left hand side of the above is bounded by

$$
\|\widehat{f}\|_{L^{2}(\{\hat{f}<M\} \cap \tau)}^{p} 2^{d j\left(\frac{p}{2}-1\right)}
$$

Hence we see that

$$
\begin{equation*}
\frac{1}{4} \epsilon^{\frac{2}{\theta}}\|f\|_{L^{2}}^{2\left(1-\frac{1}{\theta}\right)} \leq \int_{\{\widehat{f}<M\} \cap \tau}|\widehat{f}(\xi)|^{2} d \xi \tag{2.11}
\end{equation*}
$$

Now we take $\widehat{h}=\widehat{f} \chi_{\{\hat{f}<M\} \cap \tau}$ and $A=M^{-\frac{2}{d}} 2^{-\frac{2}{d(p-2)}}$. Then the property (1) follows from (2.10), the choices of $h$, and $A$. The inequality in (2) follows from (2.11), (3), and the choice of $h$.

Proof of Lemma 2.6. We apply Lemma 2.7 to $f$ repeatedly. We start with setting $f_{1}=h, \tau_{1}=\tau$, and $A_{1}=A$. Then by Lemma 2.7 we have
$(1)^{\prime} \operatorname{supp} \widehat{f}_{1} \subset \tau_{1}$ such that

$$
\ell\left(\tau_{1}\right) \leq\|f\|_{L^{2}}^{c} \epsilon^{-\nu} A_{1} \text { and }\left|\widehat{f}_{1}\right|<A_{1}^{-d / 2}
$$

(2) ${ }^{\prime}\left\|f_{1}\right\|_{L^{2}} \geq C\|f\|_{L^{2}}^{-a} \epsilon^{b}$ for some $a, b$ and $C$.
$(3)^{\prime}\|f\|_{L^{2}}^{2}=\left\|f_{1}\right\|_{L^{2}}^{2}+\left\|f-f_{1}\right\|_{L^{2}}$.
If we have

$$
\left\|e^{i t \Delta} f-e^{i t \Delta} f_{1}\right\|_{L_{t}^{q} L_{x}^{r}}<\epsilon
$$

our proof is finished with $N=1$. Otherwise, we apply Lemma 2.7 with $f$ replaced by $f-f_{1}$. We set $f_{2}=h, \tau_{2}=\tau$ and $A_{2}=A$. We apply Lemma 2.7 and (3)' to get the following :
$(1)^{\prime \prime} \operatorname{supp} \widehat{f}_{2} \subset \tau_{2}$ such that

$$
\ell\left(\tau_{2}\right) \leq\left\|f-f_{1}\right\|_{L^{2}}^{c} \epsilon^{-\nu} A_{2} \leq\|f\|_{L^{2}}^{c} \epsilon^{-\nu} A_{2} \text { and }\left|\widehat{f_{2}}\right|<A_{2}^{-d / 2}
$$

$(2)^{\prime \prime}\left\|f_{2}\right\|_{L^{2}} \geq C\left\|f-f_{1}\right\|_{L^{2}}^{-a} \epsilon^{b} \geq C\|f\|_{L^{2}}^{-a} \epsilon^{b}$ for some $a, b$, and $C$.
$(3)^{\prime \prime}\left\|f-f_{1}\right\|_{L^{2}}^{2}=\left\|f_{2}\right\|_{L^{2}}^{2}+\left\|\left(f-f_{1}\right)-f_{2}\right\|_{L^{2}}^{2}$.

We repeat the above process until we achieve

$$
\left\|e^{i t \triangle} f-\sum_{j=1}^{k} e^{i t \triangle} f_{j}\right\|_{L_{t}^{q} L_{x}^{r}}<\epsilon
$$

From Strichartz's estimates, we see that

$$
\begin{aligned}
\left\|e^{i t \Delta} f-\sum_{j=1}^{n} e^{i t \Delta} f_{j}\right\|_{L_{t}^{L_{x}^{r}}}^{2} & \leq\left\|f-\sum_{j=1}^{n} f_{j}\right\|_{L^{2}}^{2} \\
& =\|f\|_{L^{2}}^{2}-\sum_{j=1}^{n}\left\|f_{j}\right\|_{L^{2}}^{2} \\
& \leq C\left(\|f\|_{L^{2}}^{2}-C n\|f\|_{L^{2}}^{-a} \epsilon^{b}\right) .
\end{aligned}
$$

So this implies that our process ends in finite steps and it is also obvious that the number of steps only depends on $\epsilon$ and $\|f\|_{L^{2}}$.

As before, we can denote by $\tau(\xi, \rho)$ the cube centered at $\xi \in \mathbb{R}^{d}$ of side length $\rho$.
Lemma 2.8. Let $q$, $r$ be numbers satisfying $2<q \leq r$ and $2 / q+d / r=d / 2$. Suppose $g \in L^{2}\left(\mathbb{R}^{d}\right)$ and

$$
\operatorname{supp} \widehat{g} \subset \tau\left(\xi_{0}, C_{0} A\right) \text { and }|\widehat{g}|<A^{-\frac{d}{2}}
$$

for some $A>0$ and $C_{0}>0$. Then for any $\epsilon>0$, there exist $N_{1} \in \mathbb{N}, N_{1} \leq$ $C\left(d, C_{0}, \epsilon\right)$, and sets $\left(Q_{n}\right)_{1 \leq n \leq N_{1}} \subset \mathbb{R} \times \mathbb{R}^{d}$ which is given by

$$
\begin{equation*}
Q_{n}=\left\{(t, x) \in \mathbb{R} \times \mathbb{R}^{d} ; t \in I_{n} \text { and }\left(x-4 \pi t \xi_{0}\right) \in C_{n}\right\} \tag{2.12}
\end{equation*}
$$

where $I_{n} \subset \mathbb{R}$ is an interval with $\left|I_{n}\right|=A^{-2}$ and $C_{n}$ is a cube with $l\left(C_{n}\right)=A^{-1}$ such that

$$
\left\|e^{i t \Delta} g\right\|_{L_{t}^{q} L_{x}^{r}\left(\mathbb{R}^{d+1} \backslash \cup_{n=1}^{N_{1}} Q_{n}\right)}<\epsilon .
$$

Notation. Let $E$ be a measurable set in $R^{d+1}$ and $f: \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ is a measurable function. If $E_{t}=\{x:(t, x) \in E\}$ is measurable in $\mathbb{R}^{d}$ for all $t \in \mathbb{R}$, we define the mixed integral as

$$
\|f\|_{L_{t}^{q} L_{x}^{r}(E)}^{q}=\int_{\mathbb{R}}\left(\int_{E_{t}}|f(t, x)|^{r} d x\right)^{\frac{q}{r}} d t .
$$

Proof. We follow closely Lemma 3.3 in [1]. Let $g^{\prime} \in L^{2}\left(\mathbb{R}^{d}\right)$ be the normalized function of $g$ defined by $\widehat{g^{\prime}}\left(\xi^{\prime}\right)=A^{\frac{d}{2}} \widehat{g}\left(\xi_{0}+A \xi^{\prime}\right)$. Then supp $\widehat{g^{\prime}} \subset\left[-C_{0} / 2, C_{0} / 2\right]^{d}$, $\left\|g^{\prime}\right\|_{L^{2}}=\|g\|_{L^{2}}$ and $\left|\widehat{g^{\prime}}\right|<1$. We have the identity

$$
\left|e^{i A^{2} t \Delta} g^{\prime}\left(A\left(x-4 \pi t \xi_{0}\right)\right)\right|=A^{-\frac{d}{2}}\left|e^{i t \Delta} g(x)\right|,
$$

where $A^{-\frac{d}{2}}$ comes from the change of variables $\zeta=\xi_{0}+A \xi$. By the change of variables $(t, x) \rightarrow\left(t^{\prime}, x^{\prime}\right)=\left(A^{2} t, A\left(x-4 \pi t \xi_{0}\right)\right)$, we have

$$
\begin{equation*}
\left|e^{i t \Delta} g(x)\right|=A^{\frac{d}{2}}\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right| . \tag{2.13}
\end{equation*}
$$

We now note that the Fourier transform of $g^{\prime}$ is supported in a cube of side length one. Since $2<q \leq r$ and $(q, r)$ is admissible, from Proposition 2.5 we see that there are numbers $q^{*}<q, r^{*}<r$ such that $r^{*} / q^{*}=r / q$ and

$$
\begin{equation*}
\left\|e^{i t^{\prime} \Delta} g^{\prime}\right\|_{L_{t^{\prime}}^{q^{*}} L_{x^{\prime}}^{r^{*}}\left(\mathbb{R} \times \mathbb{R}^{d}\right)} \leq C(d)\left\|\widehat{g^{\prime}}\right\|_{L^{\infty}} \tag{2.14}
\end{equation*}
$$

Let $E \subset \mathbb{R} \times \mathbb{R}^{d}$ be the set $\left\{\left(t^{\prime}, x^{\prime}\right):\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right|<\lambda\right\}$ for a given $\lambda$. We write

$$
\left\|e^{i t \Delta} g^{\prime}\right\|_{L_{t^{\prime}}^{q} L_{x^{\prime}}^{r}(E)}^{q}=\int_{\mathbb{R}}\left(\int_{\left\{x^{\prime}:\left|e^{i t^{\prime}} \Delta g^{\prime}\left(x^{\prime}\right)\right|<\lambda\right\}}\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right|^{r^{*}+r-r^{*}} d x^{\prime}\right)^{\frac{q}{r}} d t^{\prime} .
$$

By using (2.14) and the choice of $q^{*}, r^{*}$ we see that the right hand side of the above is bounded by

$$
\left\|e^{i t \Delta} g^{\prime}\right\|_{L_{t^{\prime}}^{q} L_{x^{\prime}}^{r}(E)}^{q} \leq C\left(C_{0}, d\right) \lambda^{\left(r-r^{*}\right) \frac{q}{r}} \|{\widehat{g^{\prime}}}_{\|_{L^{\infty}}} \leq C\left(C_{0}, d\right) \lambda^{\left(r-r^{*}\right)^{\frac{q}{r}}}
$$

because $\|g\|_{L^{2}}=\left\|\widehat{g^{\prime}}\right\|_{L^{2}}$ and $\widehat{g^{\prime}}$ is supported in a cube of side length 1 . Since $r *<r$, due to the above estimate there exists a $\lambda_{0}\left(d, C_{0}, \epsilon\right) \in(0,1)$ small enough such that

$$
\left\|e^{i t^{\prime} \Delta} g^{\prime}\right\|_{L_{t^{\prime}}^{q} L_{x^{\prime}}^{r}(\tilde{E})}^{q} \leq \epsilon^{q}
$$

where $\widetilde{E}=\left\{\left(t^{\prime}, x^{\prime}\right):\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right|<2 \lambda_{0}\right\}$.
Since $\operatorname{supp} \widehat{g^{\prime}} \subset\left[-\frac{C_{0}}{2}, \frac{C_{0}}{2}\right]^{d}$ and $\left\|\widehat{g^{\prime}}\right\|_{L^{\infty}} \leq 1$, the map

$$
e^{i t^{\prime} \Delta} g^{\prime}(x)=\int_{\mathbb{R}^{d}} e^{2 \pi i\left(x \cdot \xi-2 \pi t|\xi|^{2}\right)} \widehat{g^{\prime}}(\xi) d \xi
$$

satisfies

$$
\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)-e^{i t^{\prime \prime} \Delta} g\left(x^{\prime \prime}\right)\right| \leq C\left(\left|t^{\prime}-t^{\prime \prime}\right|+\left|x^{\prime}-x^{\prime \prime}\right|\right)
$$

where $C=C\left(C_{0}, d\right) \geq 1$. So for such a constant, if $\left(t^{\prime}, x^{\prime}\right) \in E$ and if $\left(t^{\prime \prime}, x^{\prime \prime}\right) \in \mathbb{R} \times \mathbb{R}^{d}$ is such that $\left|t^{\prime}-t^{\prime \prime}\right| \leq \frac{\lambda_{0}}{2 C}<\frac{1}{2}$ and $\left|x^{\prime}-x^{\prime \prime}\right| \leq \frac{\lambda_{0}}{2 C}<\frac{1}{2}$, then $\left(t^{\prime \prime}, x^{\prime \prime}\right) \in \widetilde{E}$. So there exists a set $R$ and a family $\left(P_{r}\right)_{r \in R}=\left(J_{r}, K_{r}\right)_{r \in R} \subset \mathbb{R} \times \mathbb{R}^{d}$, where $J_{r} \subset \mathbb{R}$ is a closed interval of center $t^{\prime} \in \mathbb{R}$ with $\left|J_{r}\right|=\frac{\lambda_{0}}{C}$ and $K_{r} \in \mathcal{C}$ of center $x^{\prime} \in \mathbb{R}^{d}$ with $l\left(K_{r}\right)=\frac{\lambda_{0}}{C}$ and $\left(t^{\prime}, x^{\prime}\right) \in\left\{\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right| \geq 2 \lambda_{0}\right\}$. Thus we have that

$$
\begin{align*}
& \text { for all }(r, s) \in R \times R \text { with } r \neq s, \operatorname{Int}\left(P_{r}\right) \cap \operatorname{Int}\left(P_{s}\right)=\emptyset \text { and } \\
& \qquad\left\{\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right| \geq 2 \lambda_{0}\right\} \subset \bigcup_{r \in R} P_{r} \subset\left\{\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right| \geq \lambda_{0}\right\}, \tag{2.15}
\end{align*}
$$

where $\operatorname{Int}\left(P_{r}\right)$ denotes the interior of the set $P_{r}$. We set $N_{1}=\sharp R$. It follows from (2.15) and the Strichartz's estimate that

$$
\begin{aligned}
& N_{1}\left(\frac{\lambda_{0}}{C}\right)^{d+1}=\left|\bigcup_{r \in R} P_{r}\right| \leq\left|\left\{\left|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right| \geq \lambda_{0}\right\}\right| \\
& \leq \lambda_{0}^{-r}\left\|e^{i t^{\prime} \Delta} g^{\prime}\left(x^{\prime}\right)\right\|_{L_{t}^{q} L_{x}^{r}\left(\mathbb{R} \times \mathbb{R}^{d}\right)}^{q} \leq C \lambda_{0}^{-r}\|g\|_{L^{2}}^{q}
\end{aligned}
$$

from which we deduce that $N_{1}<\infty$ and $N_{1} \leq C\left(\|g\|_{L^{2}}, d, C_{0}, \epsilon\right)$. Actually, since our hypothesis implies that $\|g\|_{L^{2}} \leq C_{0}^{N / 2}$, we can also write $N_{1} \leq C\left(d, C_{0}, \epsilon\right)$. For any integer $1 \leq n \leq N_{1}$, let $\left(t_{n}, x_{n}\right)$ be the center of $P_{n}$, let $I_{n} \subset \mathbb{R}$ be the interval of
center $\frac{t_{n}}{A^{2}}$ with $\left|I_{n}\right|=\frac{1}{A^{2}}$, let $I_{n}^{\prime}=A^{2} I_{n}$, let $C_{n} \in \mathcal{C}$ of center $\frac{1}{A} x_{n}$ with $\ell\left(C_{n}\right)=\frac{1}{A}$, let $C_{n}^{\prime}=A C_{n}$ and let $Q_{n}$ be defined by (2.12). We obtain

$$
\left\|e^{i t^{\prime} \Delta} g^{\prime}\right\|_{L_{t^{\prime}}^{q} L_{x^{\prime}}^{r}\left(\mathbb{R}^{d+1} \backslash \cup_{n=1}^{N_{1}} I_{n}^{\prime} \times C_{n}^{\prime}\right)}<\epsilon^{q}
$$

By (2.13) and reversing the change of variables $\left(t^{\prime}, x^{\prime}\right) \rightarrow(t, x)$, we have

$$
\left\|e^{i t \Delta} g\right\|_{L_{t}^{q} L_{x}^{r}\left(\mathbb{R}^{d+1} \backslash \cup_{n=1}^{N_{1}} Q_{n}\right)}^{q}=A^{d\left(\frac{q}{2}-\frac{q}{r}\right)-2}\left\|e^{i t \Delta} g^{\prime}\right\|_{L_{t^{\prime}}^{q} L_{x^{\prime}}^{r}\left(\mathbb{R}^{d+1} \backslash \cup_{n=1}^{\left.N_{1} I_{n}^{\prime} \times C_{n}^{\prime}\right)}\right.}^{q}<\epsilon^{q}
$$

since $(q, r)$ is admissible. This concludes the proof of the lemma.

## 3. Mass concentration for the Schrödinger operator

3.1. Proof of Theorem 1.1. Let $u$ be the maximal solution to (1.1) over the maximal forward existence time interval $\left[0, T^{*}\right)$ so that (1.4) is satisfied for some Strichartz admissible pairs $(q, r), 2<q \leq r$ and $\|u\|_{L_{t}^{q} L_{x}^{r}\left([0, t) \times \mathbb{R}^{d}\right)}<\infty$ for $0<t<$ $T^{*}<\infty$.

Let $\eta$ be a small positive number. Then we can find a strictly increasing sequence $t_{1}<t_{2}<\cdots<t_{n}<\cdots$ contained in $\left[0, T^{*}\right)$ such that

$$
\lim _{n \rightarrow \infty} t_{n}=T^{*}
$$

and for every $n \in \mathbb{N}$

$$
\begin{equation*}
\|u\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)}=\eta . \tag{3.1}
\end{equation*}
$$

By Duhamel's formula we have for $t \in\left(0, T^{*}\right)$

$$
\begin{equation*}
u(t)=e^{i \Delta\left(t-t_{n}\right)} u\left(t_{n}\right)+i \int_{t_{n}}^{t} e^{i \Delta(t-s)}|u(s)|^{\frac{4}{d}} u(s) d s \tag{3.2}
\end{equation*}
$$

3.1.1. Controlling the inhomogeneous part. Since the power $1+4 / d$ is bigger than one, we can throw away the inhomogeneous part just comparing the size of norm with the homogeneous part. It can be done by using the Strichatz's estimates.

For any $t \in\left(t_{n}, t_{n+1}\right)$ let us set

$$
F_{n}(u)=i \int_{t_{n}}^{t} e^{i \Delta(t-s)}|u|^{\frac{4}{d}} u(s) d s
$$

Applying Strichartz's estimate together with (3.1), we see

$$
\begin{equation*}
\left\|F_{n}(u)\right\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)} \leq C\|u(s)\|_{L_{t}^{\underline{a}} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)}^{\frac{4+d}{\frac{1}{d}}}=C \eta^{\frac{4+d}{d}} \tag{3.3}
\end{equation*}
$$

as long as $\frac{d+4}{d} \leq q \leq \frac{4(d+1)}{d}$ for $d=2,3$ and $2<q \leq \frac{4(d+1)}{d}$ for $d \geq 4$ in view of (1.6) and the argument around it. Hence from (3.1), (3.3) and time translation we get

$$
\left\|e^{i\left(t-t_{n}\right) \Delta} u\left(t_{n}\right)\right\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)} \geq \frac{\eta}{2},
$$

for $\eta$ sufficiently small. Hence it is enough to deal with the linear propagator in the last of the proof.
3.1.2. Decomposition to the initial datum with non-concentration Fourier transforms. Now we decompose $u\left(t_{n}\right)$ into functions with non-concentration Fourier transforms using Lemma 2.6.

Fix $n \in \mathbb{N}$ and the time interval $\left(t_{n}, t_{n+1}\right)$. We denote $f=u\left(t_{n}\right)$. Then by the mass conservation we have

$$
\|f\|_{L^{2}\left(\mathbb{R}^{d}\right)}=\left\|u_{0}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)} .
$$

Thus, applying Lemma 2.6 to $f$ with $\epsilon=\eta^{\frac{4+d}{d}}$, there exists $\left\{f_{\sigma}\right\}_{1 \leq \sigma \leq N_{0}}$ such that $\widehat{f_{\sigma}}$ is supported in a cube $\tau_{\sigma}$,

$$
\begin{equation*}
\left|\widehat{f}_{\sigma}\right| \leq C_{0} A_{\sigma}^{-\frac{d}{2}} \tag{3.4}
\end{equation*}
$$

such that

$$
\begin{equation*}
\left\|e^{i t \Delta} f-\sum_{\sigma=1}^{N_{0}} e^{i t \Delta} f_{\sigma}\right\|_{L_{t}^{q} L_{x}^{r}\left(\mathbb{R} \times \mathbb{R}^{d}\right)}<\eta^{\frac{4+d}{d}} \tag{3.5}
\end{equation*}
$$

where $N_{0}=N_{0}\left(\|f\|_{L^{2}}, d, \eta\right)$.
Now we decompose (3.1) into three parts;

$$
\begin{align*}
\eta^{q} & =\int_{t_{n}}^{t_{n+1}}\left(\int_{\mathbb{R}^{d}}|u|^{r} d x\right)^{\frac{q}{r}} d t  \tag{3.6}\\
& \leq I+\mathrm{II}+\int_{t_{n}}^{t_{n+1}}\left(\int_{\mathbb{R}^{d}}|u|^{2}\left|\sum_{\sigma=1}^{N_{0}} e^{i\left(t-t_{n}\right) \Delta} f_{\sigma}\right|^{r-2} d x\right)^{\frac{q}{r}} d t
\end{align*}
$$

where

$$
\begin{aligned}
\mathrm{I} & =\int_{t_{n}}^{t_{n+1}}\left(\int_{\mathbb{R}^{d}}|u|^{2}\left|u(t, x)-e^{i\left(t-t_{n}\right) \Delta} u\left(t_{n}\right)\right|^{r-2} d x\right)^{\frac{q}{r}} d t, \quad \text { and } \\
\text { II } & =\int_{t_{n}}^{t_{n+1}}\left(\int_{\mathbb{R}^{d}}|u|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} u\left(t_{n}\right)-\sum_{\sigma=1}^{N_{0}} e^{i\left(t-t_{n}\right) \Delta} f_{\sigma}\right|^{r-2} d x\right)^{\frac{q}{r}} d t .
\end{aligned}
$$

Recalling the definition of $F_{n}(u)$ and using Hölder's inequality with $\frac{2}{r}+\frac{r-2}{r}=1$ successively,

$$
\begin{aligned}
\mathrm{I} & =\int_{t_{n}}^{t_{n+1}}\left(\int_{\mathbb{R}^{d}}|u|^{2}\left|F_{n}(u)\right|^{r-2} d x\right)^{\frac{q}{r}} d t \\
& \leq\|u\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)}^{\frac{2 q}{q}}\left\|F_{n}(u)\right\|_{L_{t}^{q} L_{x}^{q}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)}^{\frac{q(r-2)}{q}} .
\end{aligned}
$$

Hence using (3.1) and (3.3), we see that

$$
\mathrm{I} \leq C \eta^{\frac{2 q}{r}} \eta^{\frac{(d+4)(r-2) q}{d r}}<\frac{\eta^{q}}{4}
$$

since $\frac{(d+4) r-8}{d}>r$ for $r \geq q>2$. Similarly, using (3.5) we get

$$
\mathrm{II} \leq\|u\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)}^{\frac{2 q}{r}}\left\|e^{i\left(t-t_{n}\right) \Delta} u\left(t_{n}\right)-\sum_{\sigma=1}^{N_{0}} e^{i\left(t-t_{n}\right) \Delta} f_{\sigma}\right\|_{L_{t}^{q} L_{x}^{r}}^{\frac{q(r-2)}{r}}<\frac{\eta^{q}}{4} .
$$

Thus from (3.6), $I$, and $I I$ we find that

$$
\int_{t_{n}}^{t_{n+1}}\left(\int_{\mathbb{R}^{d}}|u|^{2}\left|\sum_{\sigma=1}^{N_{0}} e^{i\left(t-t_{n}\right) \Delta} f_{\sigma}\right|^{r-2} d x\right)^{\frac{q}{r}} d t \geq \frac{\eta^{q}}{2}
$$

Since $N_{0}=N_{0}\left(\left\|u_{0}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}, \eta\right)$, there exists an $n_{0}$ and an $f_{0}=f_{n_{0}}$ supported on a cube $\tau_{0}$ such that

$$
\begin{equation*}
\int_{t_{n}}^{t_{n+1}}\left(\int_{\mathbb{R}^{d}}|u(t, x)|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f_{0}(x)\right|^{r-2} d x\right)^{\frac{q}{r}} d t \geq \epsilon_{0} \tag{3.7}
\end{equation*}
$$

where we denote by $\epsilon_{0}=\frac{1}{2} \frac{\eta^{q}}{N_{0}^{(r-2) q / r}}$ and $f_{0}=f_{0}(n)$. Let us set

$$
\ell\left(\tau_{0}\right)=A
$$

Then from (3.4) it follows that $\left|\widehat{f}_{0}\right| \leq C A^{-d / 2}$.
3.1.3. Figuring out the concentrating region. By Lemma 2.8, there is $N_{1}=N_{1}\left(\left\|f_{0}\right\|_{L^{2}}, \eta\right)$ and a set of regions $\left\{Q_{n}\right\}_{1 \leq n \leq N_{1}}$ defined by

$$
Q_{n}=\left\{(t, x) \in \mathbb{R} \times \mathbb{R}^{d} ; t \in I_{n} \text { and }\left(x-4 \pi t \xi_{0}\right) \in C_{n}\right\},
$$

where $C_{n}$ is a cube of measure $l\left(C_{n}\right)=A^{-1}$, and $I_{n}$ is an interval of length $l\left(I_{n}\right)=$ $A^{-2}$ such that

$$
\left\|e^{i\left(t-t_{n}\right) \Delta} f_{0}\right\|_{L_{t}^{q} L_{x}^{r}\left(\mathbb{R} \times \mathbb{R}^{d} \backslash \cup_{n=1}^{N_{1}} Q_{n}\right)}<\left(\frac{\epsilon_{0}}{2 \eta^{2 q / r}}\right)^{r / q(r-2)} .
$$

Then by Hölder's inequality with $\frac{2}{r}+\frac{r-2}{r}=1$ twice, we have

$$
\begin{aligned}
& \left\||u|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f_{0}\right|^{r-2}\right\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d} / \cup_{n=1}^{N_{1}} Q_{n}\right)}^{q} \\
\leq & \|u\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d}\right)}^{\frac{2 q}{q}}\left\|e^{i\left(t-t_{n}\right) \Delta} f_{0}\right\|_{\left.L_{t}^{q} L_{x}^{r}\left(\mathbb{R} \times \mathbb{R}^{d}\right) \backslash \cup_{n=1}^{N_{1}} Q_{n}\right)}^{(r-} \\
< & \eta^{2 q / r} \frac{\epsilon_{0}}{2 \eta^{2 q / r}}=\frac{\epsilon_{0}}{2} .
\end{aligned}
$$

Thus from (3.7) it follows that

$$
\left\||u|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f_{0}\right|^{r-2}\right\|_{L_{t}^{q} L_{x}^{r}\left(\left(t_{n}, t_{n+1}\right) \times \mathbb{R}^{d} \cap\left(\cup_{n=1}^{N_{1}} Q_{n}\right)\right)}^{q} \geq \frac{\epsilon_{0}}{2} .
$$

It implies there is a region $Q_{0} \in\left\{Q_{n}\right\}_{n=1}^{n=N_{1}}$ such that

$$
\begin{equation*}
\int_{\left(t_{n}, t_{n+1}\right) \cap I_{0}}\left(\int_{Q_{0}^{t}}|u(t, x)|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f_{0}(x)\right|^{r-2} d x\right)^{\frac{q}{r}} d t \geq \frac{1}{2 N_{1}} \epsilon_{0}:=\epsilon_{1} \tag{3.8}
\end{equation*}
$$

where $Q_{0}^{t}=\left\{x:(x, t) \in Q_{0}\right\}$.
3.1.4. Determining the size of window. Since $\left|\widehat{f}_{0}\right| \leq C A^{-d / 2}$ and $\widehat{f}_{0}$ is supported in a cube of measure $A^{d}$, we see that

$$
\begin{equation*}
\left|e^{i\left(t-t_{n}\right) \Delta} f_{0}(x)\right| \leq \int_{\tau}\left|\widehat{f}_{0}(\xi)\right| d \xi \leq C A^{\frac{d}{2}} \tag{3.9}
\end{equation*}
$$

Thus from $d q(r-2) / r=4$, it follows that

$$
\begin{aligned}
\epsilon_{1} & \leq \int_{\left(t_{n}, t_{n+1}\right) \cap I_{0}}\left(\int_{Q_{0}^{t}}|u(t, x)|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f_{0}(x)\right|^{r-2} d x\right)^{\frac{q}{r}} d t \\
& \leq C A^{\frac{d q(r-2)}{2 r}} \int_{\left(t_{n}, t_{n+1}\right) \cap I_{0}}\left(\int_{Q_{0}^{t}}|u(t, x)|^{2} d x\right)^{\frac{q}{r}} d t \\
& \leq C A^{2}\|u\|_{L_{x}^{2}\left(\mathbb{R}^{d}\right)}^{\frac{2 q}{q}}\left(t_{n+1}-t_{n}\right) .
\end{aligned}
$$

We thus have

$$
t_{n+1}-t_{n} \geq C A^{-2} \epsilon_{1}
$$

By (3.9), similarly we can choose $\kappa$ small enough so that

$$
\begin{aligned}
\int_{t_{n+1}-C \kappa A^{-2} \epsilon_{1}}^{t_{n+1}} & \left(\int_{Q_{0}^{t}}|u(t, x)|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f(x)\right|^{r-2} d x\right)^{\frac{q}{r}} d t \\
& \leq C \kappa \epsilon_{1}\left\|u_{0}\right\|_{L^{2}}^{\frac{2 q}{r}} \leq \frac{\epsilon_{1}}{2}
\end{aligned}
$$

In view of this and (3.8), we obtain that

$$
\int_{\left(t_{n}, t_{n+1}-C \kappa A^{-2} \epsilon_{1}\right) \cap I_{0}}\left(\int_{Q_{0}^{t}}|u(t, x)|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f(x)\right|^{r-2} d x\right)^{\frac{q}{r}} d t \geq \frac{\epsilon_{1}}{2} .
$$

Again by (3.9)

$$
\begin{aligned}
\frac{\epsilon_{1}}{2} & \leq C\left|I_{0}\right| \sup _{t \in\left(t_{n}, t_{n+1}-C \kappa A^{-2} \epsilon_{1}\right)}\left(\int_{Q_{0}^{t}}|u|^{2}\left|e^{i\left(t-t_{n}\right) \Delta} f\right|^{r-2} d x\right)^{\frac{q}{r}} \\
& \leq C\left|I_{0}\right| A^{2}\left(\sup _{t \in\left(t_{n}, t_{n+1}-C \kappa A^{-2} \epsilon_{1}\right)} \int_{Q_{0}^{t}}|u|^{2} d x\right)^{\frac{q}{r}} .
\end{aligned}
$$

The length $\left|I_{0}\right|=A^{-2}$ implies

$$
\sup _{t \in\left(t_{n}, t_{n+1}-C \kappa A^{-2} \epsilon_{1}\right)} \int_{Q_{0}^{t}}|u|^{2} d x \geq C\left(\frac{\epsilon_{1}}{2}\right)^{\frac{r}{q}} .
$$

Therefore, for each $t_{n}$ there are $t_{0} \in\left(t_{n}, t_{n+1}-C \kappa A^{-2}\right]$ and a cube $Q_{0}^{t_{0}}$ such that

$$
\int_{Q_{0}^{t_{0}}}\left|u\left(t_{0}, x\right)\right|^{2} d x \geq \frac{C}{4}\left(\frac{\epsilon_{1}}{2}\right)^{\frac{r}{q}} .
$$

Since $l\left(Q_{0}^{t_{0}}\right)=1 / A$, then $Q_{0}^{t_{0}}$ is contained in a ball of radius $C_{d} / A$. Since $t_{n+1}-t_{0} \geq$ $C \kappa A^{-2} \epsilon_{1}$,

$$
\frac{1}{A} \leq C\left(t_{n+1}-t_{0}\right)^{\frac{1}{2}} \leq C\left(T_{*}-t_{0}\right)^{\frac{1}{2}}
$$

Hence $Q_{0}^{t_{0}}$ can be covered by a finite number (depending only on $\eta, d$ and $\left\|u_{0}\right\|_{2}$ ) of balls of radius $R=\left(T^{*}-t_{0}\right)^{\frac{1}{2}}$. Therefore, there exists $c \in \mathbb{R}^{d}$ such that

$$
\int_{B(c, R)}\left|u\left(t_{0}, x\right)\right|^{2} d x \geq \epsilon\left(\left\|u_{0}\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}, d, \eta\right)
$$

3.2. Hartree type nonlinearity. In this section we prove Theorem 1.2. As it was seen in the proof of Theorem 1.1, it is enough to deal with homogeneous part of the solution once the inhomogeneous part is controlled properly.

Indeed, let $u$ be the maximal solution to (1.7) over the maximal forward existence time interval $\left[0, T^{*}\right)$ so that (1.4) is satisfied for some Strichartz admissible pairs $(q, r), 2<q \leq r$ and $\|u\|_{L_{t}^{q} L_{x}^{r}\left([0, t) \times \mathbb{R}^{d}\right)}<\infty$ for $0<t<T^{*}<\infty$. Let $\eta$ and sequence $t_{1}, \ldots, t_{n}, \ldots$ be given as before such that $t_{n} \nearrow T^{*}$ and (3.1) is satisfied for every $n \in \mathbb{N}$. By the Duhamel's formula we may write for $t \in\left(0, T^{*}\right)$

$$
u(t)=e^{i \Delta\left(t-t_{n}\right)} u\left(t_{n}\right) \pm i \int_{t_{n}}^{t} e^{i(t-s) \Delta}\left[\left(|x|^{-2} *|u(s)|^{2}\right) u(s)\right] d s
$$

We need to show the similar estimate as (3.3) for the solution of Hartree equation.
Lemma 3.1. Let $(q, r)$ be an admissible pair, $2<q \leq r$ and let $t_{n}, \eta$ be the same as in (3.1). Then for the solution $u$ of (1.7) there is a constant $C>0$ such that

$$
\begin{equation*}
\left\|\int_{t_{n}}^{t} e^{i(t-s) \Delta}\left[\left(|x|^{-2} *|u(s)|^{2}\right) u(s)\right] d s\right\|_{L_{t}^{q} L_{x}^{r}\left[\left(t_{n}, t_{n+1}\right] \times \mathbb{R}^{d}\right)} \leq C \eta^{1+\theta} \tag{3.10}
\end{equation*}
$$

for some $0<\theta<1$.
After achieving this we only need to deal with the homogeneous part of the solution to show the mass concentration. That is to say, if we have (3.10), the remaining arguments are the same as those of Sections 3.1.2 through 3.1.4. So we omit the details.

Proof of Lemma 3.1. By the inhomogeneous Strichartz's estimate (1.5), the left hand side of (3.10) is bounded by

$$
\left\|\left(|x|^{-2} *|u|^{2}\right) u\right\|_{L_{t}^{\tilde{q}^{\prime}} L_{x}^{\tilde{\prime}^{\prime}}\left(\left[t_{n}, t_{n+1}\right] \times \mathbb{R}^{d}\right)}
$$

for any $\widetilde{q}$ and $\widetilde{r}$ satisfying $\widetilde{q}, \widetilde{r} \geq 2$ and $2 / \widetilde{q}+d / \widetilde{r}=d / 2$. Here $\widetilde{r^{\prime}}$ denotes the conjugate exponent of $\widetilde{r}$. By Hölder's inequality, it follows that

$$
\begin{equation*}
\left\|\int_{t_{n}}^{t} e^{i(t-s) \Delta}\left[\left(|x|^{-2} *|u(s)|^{2}\right) u(s)\right] d s\right\|_{L_{t}^{q} L_{x}^{r}\left(\left[t_{n}, t_{n+1}\right] \times \mathbb{R}^{d}\right)} \leq C I_{1} \cdot I_{2} \tag{3.11}
\end{equation*}
$$

where for all $\mu, s>1$

$$
\begin{aligned}
& I_{1}=\left\||x|^{-2} *|u|^{2}\right\|_{L_{t}^{\tilde{q}^{\prime} \mu^{\prime}}} \quad L_{x}^{\tilde{\tau}_{x}^{\prime} s^{\prime}}\left(\left[t_{n}, t_{n+1}\right] \times \mathbb{R}^{d}\right) \\
& I_{2}=\|u\|_{L_{t}^{\tilde{q^{\prime}} \mu} L_{x}^{\tilde{T_{s}^{\prime}} s}\left(\left[t_{n}, t_{n+1}\right] \times \mathbb{R}^{d}\right)^{\prime}}, \text { and }
\end{aligned}
$$

We make choices of $\mu$ and $s$ such that $1 / \mu=\widetilde{q^{\prime}} / q<1$ and $1 / s=\widetilde{r^{\prime}} / r<1$. It is possible because $2<q \leq r$. (We shall determine the exact $\widetilde{r}$ and $\widetilde{q}$ later.) Then

$$
I_{2}=\|u\|_{L_{t}^{q} L_{x}^{r}\left(\left[t_{n}, t_{n+1}\right] \times \mathbb{R}^{d}\right)}=\eta .
$$

By applying the Hardy-Sobolev inequality on the $x$-space,

$$
\begin{equation*}
I_{1} \leq C\|u\|_{L_{t}^{q_{2}} L_{x}^{r_{2}}\left(\left[t_{n}, t_{n+1}\right] \times \mathbb{R}^{d}\right)}^{2} \tag{3.12}
\end{equation*}
$$

where $\left(1 / q_{2}, 1 / r_{2}\right)$ is given by

$$
\begin{align*}
\frac{1}{q_{2}} & =\frac{1}{2 \widetilde{q}^{\prime} \mu^{\prime}}=\frac{1}{2}\left[1-\left(\frac{1}{\widetilde{q}}+\frac{1}{q}\right)\right], \text { and } \\
\frac{1}{r_{2}} & =\frac{1}{2}\left[\frac{1}{\widetilde{r}^{\prime} s^{\prime}}+\frac{d-2}{d}\right]=\frac{1}{2}\left[1-\left(\frac{1}{\widetilde{r}}+\frac{1}{r}\right)+\frac{d-2}{d}\right] . \tag{3.13}
\end{align*}
$$

Note that $\left(q_{2}, r_{2}\right)$ is an admissible pair, that is, $2 / q_{2}+d / r_{2}=d / 2$. We now choose an admissible $(\widetilde{q}, \widetilde{r})$ such that

$$
\left(\frac{1}{q_{2}}, \frac{1}{r_{2}}\right) \in\left(\left(\frac{1}{q}, \frac{1}{r}\right),\left(0, \frac{1}{2}\right)\right) .
$$

Since $\left(q_{2}, r_{2}\right)$ is an admissible pair, from (3.13) we obtain that such a choice is always possible as long as $0<(1-1 / \widetilde{q}-1 / q) / 2<1 / q$. This is obviously satisfied because $2<q \leq \frac{2(d+2)}{d}$. Hence $\left(1 / q_{2}, 1 / r_{2}\right)=\theta(1 / q, 1 / r)+(1-\theta)(1 / \infty, 1 / 2)$ for some $\theta \in(0,1)$. Therefore, via interpolation between two space $L_{t}^{\infty} L_{x}^{2}$ and $L_{t}^{q} L_{x}^{r}$ we get that

$$
\|u\|_{L_{t}^{q_{2}} L_{x}^{r_{2}}} \leq\|u\|_{L_{t}^{q} L_{x}^{r}}^{\theta}\left\|u_{0}\right\|_{L_{x}^{2}}^{1-\theta} .
$$

Combining this with the conserved mass and (3.12), we obtain $I_{1} \leq C \eta^{\theta}$. Finally, from (3.11), (3.12) and the above we get the desired estimate (3.10).

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[^1]:    ${ }^{1}$ This amounts to showing $\left\|\left.\int_{0}^{T} e^{i t \Delta(t-s)}\left[|u(s)|^{\frac{4}{d}} u(s)-|v(s)|^{\frac{4}{d}} v(s)\right] d s \right\rvert\,\right\|_{L_{t}^{q} L_{x}^{r}} \leq C\left(\|u\|_{L_{t}^{\tilde{q}^{\prime}} L_{x}^{\tilde{r}^{\prime}}}^{\frac{4}{d}}+\right.$ $\left.\|u\|_{L_{t}^{\tilde{q}^{\prime}} L_{x}^{\tilde{\tau}^{\prime}}}^{\frac{4}{4}}\right)\|u-v\|_{L_{t}^{\tilde{q}^{\prime}} L_{x}^{\tilde{r}^{\prime}}}$ which follows from the inhomogeneous Strichartz's estimate and Hölder's inequality.

