# The Schrödinger equation along curves and the quantum harmonic oscillator 

Sanghyuk Lee ${ }^{1, \mathrm{a}}$ Keith M. Rogers ${ }^{2, \mathrm{~b}}$<br>${ }^{\text {a }}$ School of Mathematical Sciences, Seoul National University, Seoul 151-742, Korea<br>'Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain

Key words: Schrödinger equation, harmonic oscillator, pointwise convergence

## 1. Introduction

We consider the free Schrödinger equation, $i \partial_{t} u=-\Delta u$, with initial data in $H^{s}\left(\mathbb{R}^{d}\right)$, the inhomogeneous Sobolev space with $s$ derivatives in $L^{2}\left(\mathbb{R}^{d}\right)$. A classical problem, originating in the work of Carleson [6], is to identify the exponents $s$ for which

$$
\lim _{t \rightarrow 0} e^{i t \Delta} f(x)=f(x) \quad \text { a.e. } x \in \mathbb{R}^{d}
$$

whenever $f \in H^{s}\left(\mathbb{R}^{d}\right)$.
In one spatial dimension, Carleson proved the convergence for data in $H^{s}(\mathbb{R})$ with $s \geqslant 1 / 4$, and Dahlberg and Kenig [9] proved that the convergence is not guaranteed when $s<1 / 4$. In two spatial dimensions, the first author [16] proved the convergence for data in $H^{s}\left(\mathbb{R}^{2}\right)$ with $s>3 / 8$, improving the work of a number of authors (see for example $[3,18,28,29])$. In higher dimensions, the best known result is that of Sjölin [26] and Vega [30] who proved the convergence for $H^{s}\left(\mathbb{R}^{d}\right)$ with $s>1 / 2$.

We also consider the Schrödinger equation for the harmonic oscillator, $i \partial_{t} u=\mathcal{H} u$, where $\mathcal{H}$ is the Hermite operator defined by

$$
\begin{equation*}
\mathcal{H}=\frac{1}{2}\left(-\Delta+|x|^{2}\right), \quad x \in \mathbb{R}^{d} . \tag{1}
\end{equation*}
$$

[^0]This is an important model in quantum mechanics, as it approximates any trapping Schrödinger equation with real potential at its point of equilibrium (see for example [11]). As for the free equation, there has been an effort to identify the exponents $s$ for which

$$
\lim _{t \rightarrow 0} e^{-i t \mathcal{H}} f(x)=f(x) \quad \text { a.e. } x \in \mathbb{R}^{d}
$$

whenever $f \in H^{s}(\mathbb{R})$.
The first nontrivial result which held in sufficient generality to include the harmonic oscillator is due to Cowling [8]. This was improved by Yajima [31] who proved convergence for data in $H^{s}\left(\mathbb{R}^{d}\right)$ with $s>1 / 2$. Recently, Sjögren and Torrea [25] proved the sharp result in one spatial dimension. That is to say, the convergence holds for data in $H^{s}(\mathbb{R})$ with $s \geqslant 1 / 4$, and the convergence is not guaranteed for data in $H^{s}(\mathbb{R})$ when $s<1 / 4$. For $d \geqslant 2$ it can be shown that the convergence fails for $s<1 / 4$ (see the paragraph below Theorem 3.1) but no result was known below $s=1 / 2$.

We improve Yajima's result in two spatial dimensions.
Theorem 1.1 Let $f \in H^{s}\left(\mathbb{R}^{2}\right)$ with $s>3 / 8$. Then

$$
\lim _{t \rightarrow 0} e^{-i t \mathcal{H}} f(x)=f(x) \quad \text { a.e. } \quad x \in \mathbb{R}^{2}
$$

Since the spectrum of $\mathcal{H}$ is discrete, recalling the free equation with periodic data (see for example [19]), one may expect that the usual analysis on Euclidean space does not work directly. However, by making use of a transformation (as in [25]) we are able to work with the free Schrödinger operator along curves $(\rho(x, t), t)=\left(\sqrt{1+t^{2}} x, t\right)$, and so we consider the problem in general. In the second section we will prove the following theorem in which $\mathbb{B}^{d}$ denotes the unit ball centred at the origin.

Theorem 1.2 Suppose that $\rho \in C^{1}\left(\mathbb{R}^{d} \times[0,1], \mathbb{R}^{d}\right)$ satisfies $\rho(x, 0)=x$, and that there exist constants $C_{s}>0$ such that

$$
\left\|\sup _{0<t<1}\left|e^{i t \Delta} f\right|\right\|_{L^{2}\left(\mathbb{B}^{d}\right)} \leqslant C_{s}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}, \quad s>s_{\mathrm{o}} .
$$

Then for all $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with $s>s_{\mathrm{o}}$,

$$
\lim _{t \rightarrow 0} e^{i t \Delta} f(\rho(x, t))=f(x) \quad \text { a.e. } x \in \mathbb{R}^{d}
$$

Combining this with the estimates of $[12,16,26,30]$ yields convergence along $C^{1}$ curves for all $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with $s>s_{d}$, where $s_{1}=1 / 4, s_{2}=3 / 8$ and $s_{d}=1 / 2$ if $d \geqslant 3$. In particular this yields Theorem 1.2. This also improves the result of Sjögren and Sjölin [24] who obtained the convergence for $\rho(x, t)=x+\alpha t$, with $\alpha \in \mathbb{R}^{d}$, for all $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with $s>1 / 2$.

In the third section we will prove the following equivalence between estimates for the free and Hermite Schrödinger operators. This yields Theorem 1.2 in the case $(\rho(x, t), t)=$ ( $\sqrt{1+t^{2}} x, t$ ). We denote by $B_{R} \equiv B(0, R)$ the ball of radius $R \geqslant 1$, centred at the origin.

Theorem 1.3 Let $q, r \geqslant 2$. If $r \neq \infty$, then there exist constants $c_{R}$ such that

$$
\begin{equation*}
\left\|e^{i t \Delta} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \leqslant c_{R}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)} \tag{2}
\end{equation*}
$$

if and only if there exist constants $C_{R}$ such that

$$
\begin{equation*}
\left\|e^{-i t \mathcal{H}} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \leqslant C_{R}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)} \tag{3}
\end{equation*}
$$

If $r=\infty$, then (2) holds for $s>s_{\mathrm{O}}$ if and only if (3) holds for $s>s_{\mathrm{O}}$.

In particular, taking $q=r=2$, Theorem 1.3 shows that the local smoothing estimate of Constantin and Saut [7], Sjölin [26] and Vega [30] for the free equation, is equivalent to that of Yajima [31] for the harmonic oscillator. Combining Theorem 1.3 with the work of Planchon, Tao and Vargas $[21,28,29]$ (see [17] for the endpoint) we also obtain the following corollary. It is not possible to bound the solution to the harmonic oscillator in $L_{x}^{q}\left(B_{R}, L_{t}^{r}(\mathbb{R})\right)$, with $r \neq \infty$, as the solution is periodic. Nor is it possible to bound the solution in $L_{x}^{q}\left(\mathbb{R}^{d}, L_{t}^{\infty}[0,1]\right)$ (see [25]). When $q=r$ however, estimates which are global in space are possible (see for example [14]).

Corollary 1.1 Let $\frac{2(d+3)}{d+1}<q<r<\infty, \frac{d+1}{q}+\frac{1}{r} \leqslant \frac{d}{2}$ and $s=\frac{d}{2}-\frac{d}{q}-\frac{2}{r}$. Then

$$
\left\|e^{-i t \mathcal{H}} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \leqslant C_{R}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}
$$

When $d=2$, the restriction on $q$ can be relaxed to $q>16 / 5$ by combining Theorem 1.3 with [17, 22].

Corollary 1.2 Let $\frac{16}{5}<q<r<\infty, \frac{3}{q}+\frac{1}{r} \leqslant 1$ and $s=1-\frac{2}{q}-\frac{2}{r}$. Then

$$
\left\|e^{-i t \mathcal{H}} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \leqslant C_{R}\|f\|_{H^{s}\left(\mathbb{R}^{2}\right)}
$$

In Section 2 we will prove an equivalence lemma which will be key to the proof of all of these results. This follows by developing in a Fourier series the exponential function evaluated at perturbations of the phase. However the Fourier coefficients are badly behaved when the time is localized at scale 1 . We get around this problem by proving a sharp temporal localization lemma which reduces estimates on time intervals of length 1 to estimates on intervals of length $\lambda^{-1}$ under the assumption that the frequency of the initial datum is localized at scale $\lambda$. We then combine these lemmas to prove Theorem 1.2. In Section 3, we will describe the aforementioned transformation for harmonic oscillator in more detail and see that the condition $\frac{d+1}{q}+\frac{1}{r} \leqslant \frac{d}{2}$ in Corollary 1.1 is sharp. We then prove a Littlewood-Paley style lemma, allowing us to prove equivalences without loss in regularity. This allows us to prove a somewhat more general version of Theorem 1.3. We also prove an equivalence of convergence along sequences for the free and Hermite Schrödinger equation. In the final section we discuss a refinement of almost everywhere convergence as in [1], and parts of the paper prior to that point are written in sufficient generality to be of use there.
Indeed, from now on $\mu, \nu$ will denote measures, and for an interval $I \subset \mathbb{R}$ we write

$$
\|F\|_{L_{\mu}^{q} L_{\nu}^{r}(I)}=\left(\int_{\mathbb{R}^{d}}\left(\int_{I}|F(x, t)|^{r} d \nu(t)\right)^{\frac{q}{r}} d \mu(x)\right)^{\frac{1}{q}}
$$

Also, $c$ and $C$ will denote positive constants that will depend on the dimension $d$. Their values may change from line to line.

## 2. Proof of Theorem 1.2

For $\rho: \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d}$, we define the operator $U_{\rho}$ by

$$
U_{\rho} f(x, t)=e^{i t \Delta} f(\rho(x, t))
$$

The following localization lemma extends and sharpens Lemma 2.3 of [16]. The proof makes use of the wave packet decomposition which has been used in the study of restriction and Bochner-Riesz problems (see for example [10,15,28]). In contrast with previous
arguments, we decompose $U_{\rho} f$ into pieces which are, in some sense, compactly supported in space instead of compactly supported in frequency. This enables us to exploit the localization property more effectively. Unfortunately this obscures the geometric reason behind why such a result should hold, and so we briefly describe the main idea. As the frequency is supported away from the origin, the wave packets have nonzero velocities, and so the space-time tubes to which they are adapted only interact with small pieces of the region of integration. The lemma is not true if the functions are Fourier supported in the ball $B_{\lambda}$ instead of the annulus $A_{\lambda}=\{\xi: \lambda / 2 \leqslant|\xi| \leqslant 2 \lambda\}$, as is easily seen by considering $\rho(x, t)=x$ with $q=r=2$, because then the tubes will interact with the whole region. On the other hand, the lemma continues to hold if the order of integration is interchanged.

Lemma 2.1 Let $q, r \in[2, \infty], \lambda \geqslant 1, \operatorname{supp}(\nu) \subset[-2,2], \lambda>\|1\|_{L_{\mu}^{q} L_{\nu}^{r}}^{1 / d}$, and suppose that

$$
\sup _{x \in \operatorname{supp}(\mu), t \in \operatorname{supp}(\nu)}|\rho(x, t)| \leqslant M
$$

where $M>1$. Suppose that, for a collection of boundedly overlapping intervals $I$ of length $\lambda^{-1}$, there exists a $C_{\mathrm{o}}>1$ such that

$$
\begin{equation*}
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)} \leqslant C_{\mathrm{o}}\|f\|_{2} \tag{4}
\end{equation*}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$. Then there is a constant $C_{d}>1$ such that

$$
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(\bigcup_{I)}\right.} \leqslant C_{d} M^{1 / 2} C_{\mathrm{o}}\|f\|_{2}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$.
We introduce two partitions of unity and decompose $U_{\rho} f$ into packets which are suited for our purpose. Fix a positive and smooth function $\psi$, supported in $B_{\sqrt{d}}$, such that

$$
\sum_{y \in \mathbb{Z}^{d}} \psi(x-y)=1
$$

We also fix a smooth $\phi_{\mathrm{o}}$, supported in $B_{2^{-4}}$, that satisfies $\phi_{\mathrm{o}} * \phi_{\mathrm{o}} * \phi_{\mathrm{o}} * \phi_{\mathrm{o}} * \phi_{\mathrm{o}} * \phi_{\mathrm{o}}(0)=1$, so that, by the Poison summation formula,

$$
\sum_{v \in \mathbb{Z}^{d}}\left(\widehat{\phi}_{o}\right)^{6}(\xi-v)=1
$$

We set $\phi=\phi_{0} * \phi_{0}$. Define $f_{y}$ and $f_{y v}$ by

$$
f_{y}(x)=\psi(x-y) f(x) \quad \text { and } \quad \widehat{f_{y v}}(\xi)=(\widehat{\phi})^{3}(\xi-v) \widehat{f}_{y}(\xi)
$$

respectively. It follows that

$$
\begin{equation*}
f=\sum_{y \in \mathbb{Z}^{d}} f_{y} \quad \text { and } \quad f=\sum_{y \in \mathbb{Z}^{d}} \sum_{v \in \mathbb{Z}^{d}} f_{y v} \tag{5}
\end{equation*}
$$

Note that $f_{y v}$ is supported in the ball of radius $(\sqrt{d}+1)$ with centre $y$. For the rest of this section $y$ and $v$ are reserved to denote elements in $\mathbb{Z}^{d}$.

Proof. Since the intervals are boundedly overlapping, by Minkowski's inequality we may assume that they are disjoint. We decompose $f$ as in (5) so that

$$
U_{\rho} f=\sum_{y, v: \lambda / 4<|v|<4 \lambda} U_{\rho} f_{y v}+\sum_{y, v:|v| \leqslant \lambda / 4,|v| \geqslant 4 \lambda} U_{\rho} f_{y v}
$$

where

$$
\begin{equation*}
U_{\rho} f_{y v}=\frac{1}{(2 \pi)^{d}} \int e^{i\left(\rho(x, t) \cdot \xi-t|\xi|^{2}\right)}(\widehat{\phi})^{3}(\xi-v) \widehat{f}_{y}(\xi) d \xi \tag{6}
\end{equation*}
$$

As $\widehat{f}$ is supported in $A_{\lambda}$, the second term is an error. Indeed, for any $N \geqslant 1$,

$$
\left|\widehat{f}_{y}\right|=\left|(f \psi(\cdot-y))^{\wedge}\right| \leqslant C_{N}|\widehat{f}| *(1+|\cdot|)^{-N}
$$

So, if $|\xi| \leqslant 3 \lambda / 8$ or $|\xi| \geqslant 5 \lambda / 2$, we have $\left|\widehat{f}_{y}(\xi)\right| \leqslant C_{N} \lambda^{-N}\|f\|_{2}$. Substituting this into (6), we see that $\left\|U_{\rho} f_{y v}\right\|_{\infty} \leqslant C_{N} \lambda^{-N}\|f\|_{2}$ when $|v| \leqslant \lambda / 4$ or $|v| \geqslant 4 \lambda$. As $\lambda>\|1\|_{L_{\mu}^{q} L_{\nu}^{r}}^{1 / d}$, this yields

$$
\left\|_{y, v:|v| \leqslant \lambda / 4,|v| \geqslant 4 \lambda} U_{\rho} f_{y v}\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(\bigcup_{I)}\right.} \leqslant C_{d}\|f\|_{2} .
$$

Thus, discarding this harmless error we can suppose that $\lambda / 4<|v|<4 \lambda$. For notational convenience we write simply

$$
U_{\rho} f=\sum_{y, v: \lambda / 4<|v|<4 \lambda} U_{\rho} f_{y v} .
$$

We now analyse the kernel of the $U_{\rho}$ combined with the projection operators. Note that

$$
\begin{equation*}
U_{\rho} f_{y v}=\int \mathcal{K}_{v}(x, z, t) f_{y}(z) d z \tag{7}
\end{equation*}
$$

where

$$
\mathcal{K}_{v}(x, z, t)=\int e^{i\left(\rho(x, t) \cdot \xi-t|\xi|^{2}-z \cdot \xi\right)}(\widehat{\phi})^{3}(\xi-v) d \xi
$$

By translation $\xi \rightarrow \xi+v$ the kernel $\mathcal{K}_{v}(x, z, t)$ is equal to

$$
e^{i\left((\rho(x, t)-z) \cdot v-t|v|^{2}\right)} \int e^{i(\rho(x, t)-2 t v-z) \cdot \xi}(\widehat{\phi})^{2}(\xi) e^{-i t|\xi|^{2}} \widehat{\phi}(\xi) d \xi
$$

Now, since $\widehat{\phi}$ is rapidly decaying and $|t| \leqslant 2$ on $\operatorname{supp}(\nu)$, we can write

$$
\begin{equation*}
e^{-i t|\xi|^{2}} \widehat{\phi}(\xi)=\int e^{i \eta \cdot \xi} \Phi(\eta, t) d \eta \tag{8}
\end{equation*}
$$

where $|\Phi(\eta, t)| \leqslant C_{N}(1+|\eta|)^{-N}$ uniformly in $t \in[-2,2]$. This decay is easily calculated by repeated integration by parts in the formula for the Fourier transform.

For notational simplicity let us set

$$
\begin{align*}
\mathcal{E}_{v}(x, z, t, \eta) & =e^{i\left((\rho(x, t)-z) \cdot v-t|v|^{2}\right)} \int e^{i(\rho(x, t)-2 t v-z) \cdot \xi}(\widehat{\phi})^{2}(\xi) e^{i \eta \xi} d \xi \\
& =e^{i\left((\rho(x, t)-z) \cdot v-t|v|^{2}\right)}(\phi * \phi)(\rho(x, t)-2 t v-z+\eta) \tag{9}
\end{align*}
$$

so that the kernel can be represented as the average

$$
\mathcal{K}_{v}(x, z, t)=\int \Phi(\eta, t) \mathcal{E}_{v}(x, z, t, \eta) d \eta
$$

Substituting into (7), we see that

$$
\begin{equation*}
U_{\rho} f_{y v}(x, t)=\int \Phi(\eta, t) P_{y v}^{\eta}(x, t) d \eta \tag{10}
\end{equation*}
$$

where

$$
P_{y v}^{\eta}(x, t)=\int \mathcal{E}_{v}(x, z, t, \eta) f_{y}(z) d z
$$

Since $f_{y}$ is supported in a ball of radius $\sqrt{d}$ centred at $y$, and $\phi$ is supported in a ball of radius $2^{-3}$, from (9) we see that $P_{y v}^{\eta}$ is supported in the set

$$
T_{y v}^{\eta}=\{(x, t):|\rho(x, t)-2 t v-y+\eta| \leqslant 2 d\}
$$

Setting $Q_{I}=\operatorname{supp}(\mu) \times(I \cap \operatorname{supp}(\nu))$, when $r \geqslant q$, by concavity, (10) and Minkowski's inequality,

$$
\begin{align*}
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(\bigcup_{I)}\right.} & \leqslant\left(\sum_{I}\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)}^{q}\right)^{1 / q} \\
& \leqslant C \int(1+|\eta|)^{-(d+1)}\left(\sum_{I}\left\|\sum_{y, v} P_{y v}^{\eta}\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)}^{q}\right)^{1 / q} d \eta \\
& =C \int(1+|\eta|)^{-(d+1)}\left(\sum_{I}\left\|\sum_{y, v: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset} P_{y v}^{\eta}\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)}^{q}\right)^{1 / q} d \eta . \tag{11}
\end{align*}
$$

For the last equality we use the fact that $P_{y v}^{\eta}$ is supported on $T_{y v}^{\eta}$. On the other hand, when $r<q$, by the $L^{q / r}$-triangle inequality combined with similar arguments,

$$
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(\bigcup_{I)}\right.} \leqslant C \int(1+|\eta|)^{-(d+1)}\left(\sum_{I}\left\|\sum_{y, v: T_{y_{v} \cap Q_{I} \neq \emptyset}} P_{y v}^{\eta}\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(Q_{I}\right)}^{r}\right)^{1 / r} d \eta
$$

The arguments for each case are now essentially the same, so we only consider the case that $r \geqslant q$.

The strategy is to partially undo the decomposition and then apply the hypothesis. From (9) and the fact that $e^{i t|\xi|^{2}} \widehat{\phi}(\xi)=\int e^{i \zeta \cdot \xi} \cdot \Phi(\zeta,-t) d \zeta$, it follows that

$$
\begin{aligned}
\mathcal{E}_{v}(x, z, t, \eta)= & e^{i\left((\rho(x, t)-z) \cdot v-t|v|^{2}\right)} \\
& \times \iint e^{i(\rho(x, t)-2 t v-z) \cdot \xi} e^{-i t|\xi|^{2}} \widehat{\phi}(\xi) e^{i(\eta+\zeta) \cdot \xi} d \xi \Phi(\zeta,-t) d \zeta .
\end{aligned}
$$

By translation $\xi \rightarrow \xi-v$, this is equal to

$$
\iint e^{i\left(\left(\rho(x, t) \cdot \xi-t|\xi|^{2}-z \cdot \xi\right)\right.} \widehat{\phi}(\xi-v) e^{i(\eta+\zeta) \cdot(\xi-v)} d \xi \Phi(\zeta,-t) d \zeta
$$

So, we have that

$$
P_{y v}^{\eta}(x, t)=\int U_{\rho}\left[\widehat{\phi}(D-v) e^{i(\eta+\zeta) \cdot(D-v)} f_{y}\right] \Phi(\zeta,-t) d \zeta
$$

Here $m(D)$ is defined by $(m(D) f)^{\wedge}=m \widehat{f}$. Substituting this into (11) and applying Minkowski's inequality, we obtain

$$
\begin{aligned}
& \left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(\bigcup_{I)}\right.} \leqslant C \iint(1+|\eta|)^{-(d+1)}(1+|\zeta|)^{-(d+1)} \\
& \quad \times\left(\sum_{I}\left\|U_{\rho}\left[\sum_{y, v: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset} \widehat{\phi}(D-v) e^{i(\eta+\zeta) \cdot(D-v)} f_{y}\right]\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(Q_{I}\right)}^{q}\right)^{1 / q} d \eta d \zeta .
\end{aligned}
$$

By hypothesis, this yields

$$
\begin{aligned}
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(\bigcup I)} \leqslant C C_{\mathrm{o}} & \iint(1+|\eta|)^{-(d+1)}(1+|\zeta|)^{-(d+1)} \\
& \times\left(\sum_{I}\left\|\sum_{y, v: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset} \widehat{\phi}(D-v) e^{i(\eta+\zeta) \cdot(D-v)} f_{y}\right\|_{2}^{q}\right)^{1 / q} d \eta d \zeta .
\end{aligned}
$$

Now recall that $\widehat{\phi}=\left(\widehat{\phi}_{\mathrm{o}}\right)^{2}$. By Plancherel's theorem, the Cauchy-Schwarz inequality and making use of support properties of $\widehat{\phi}_{\mathrm{o}}$ and $\psi$ it is easy to see that

$$
\begin{aligned}
\left\|\sum_{y, v: T_{y v}^{n} \cap Q_{I} \neq \emptyset} \widehat{\phi}(D-v) e^{i(\eta+\zeta) \cdot(D-v)} f_{y}\right\|_{2}^{2} & \leqslant C \sum_{v}\left\|\sum_{y: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset} \widehat{\phi}_{\mathrm{o}}(D-v) f_{y}\right\|_{2}^{2} \\
& \leqslant C \sum_{y, v: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset}\left\|\widehat{\phi}_{\mathrm{o}}(D-v) f_{y}\right\|_{2}^{2} .
\end{aligned}
$$

Using the embedding $\ell^{2} \hookrightarrow \ell^{q}$ and then integrating in $\zeta$, we have that

$$
\begin{align*}
& \left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(\bigcup^{I)}\right.}  \tag{12}\\
& \quad \leqslant C C_{\mathrm{o}} \int(1+|\eta|)^{-(d+1)}\left(\sum_{I} \sum_{y, v: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset}\left\|\widehat{\phi}_{\mathrm{o}}(D-v) f_{y}\right\|_{2}^{2}\right)^{1 / 2} d \eta .
\end{align*}
$$

Now we claim that for $\rho$ satisfying $|\rho| \leqslant M$ on $\operatorname{supp}(\mu) \times \operatorname{supp}(\nu)$,

$$
\begin{equation*}
\sum_{I: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset} \leqslant C M \tag{13}
\end{equation*}
$$

uniformly in $\eta$. Assuming this for the moment, by changing the order of summation we see that

$$
\begin{aligned}
\sum_{I} \sum_{y, v: T_{y v}^{n} \cap Q_{I} \neq \emptyset}\left\|\widehat{\phi}_{\mathrm{o}}(D-v) f_{y}\right\|_{2}^{2} & =\sum_{y, v} \sum_{I: T_{y v}^{\eta} \cap Q_{I} \neq \emptyset}\left\|\widehat{\phi}_{\mathrm{o}}(D-v) f_{y}\right\|_{2}^{2} \\
& \leqslant C M \sum_{y, v}\left\|\widehat{\phi}_{\mathrm{o}}(D-v) f_{y}\right\|_{2}^{2}
\end{aligned}
$$

Substituting this into (12) and integrating in $\eta$, we see that

$$
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}\left(\bigcup_{I)}\right.} \leqslant C M^{1 / 2} C_{\mathrm{o}}\left(\sum_{y, v}\left\|\widehat{\phi}_{\mathrm{o}}(D-v) f_{y}\right\|_{2}^{2}\right)^{1 / 2}
$$

The result follows by showing $\sum_{y, v}\left\|\widehat{\phi}_{o}(D-v) f_{y}\right\|_{2}^{2} \leqslant C\|f\|_{2}^{2}$ which follows using the support properties of $\widehat{\phi}_{\mathrm{o}}$ and $\psi$.

It remains to prove (13). Since the intervals are disjoint and of length $\lambda^{-1}$, (13) will follow by proving that if $Q_{I_{\mathrm{o}}} \cap T_{y v}^{\eta} \neq \emptyset$ and $\operatorname{dist}\left(I, I_{\mathrm{o}}\right) \geqslant 50 d M \lambda^{-1}$, then $Q_{I} \cap T_{y v}^{\eta}=\emptyset$. Let $\left(x_{\mathrm{o}}, t_{\mathrm{o}}\right) \in Q_{I_{\mathrm{o}}} \cap T_{y v}^{\eta}$ and $(x, t) \in Q_{I}$, so that and $\left|\rho\left(x_{\mathrm{o}}, t_{\mathrm{o}}\right)-y-2 v t_{\mathrm{o}}+\eta\right| \leqslant 2 d$, and we are required to prove that $|\rho(x, t)-y-2 v t+\eta|>2 d$. Now

$$
\begin{aligned}
|\rho(x, t)-y-2 v t+\eta| & \geqslant\left|\rho(x, t)-y-2 v t+\eta-\left(\rho\left(x_{\mathrm{o}}, t_{\mathrm{o}}\right)-y-2 v t_{\mathrm{o}}+\eta\right)\right|-2 d \\
& =\left|2\left(t_{\mathrm{o}}-t\right) v+\left(\rho(x, t)-\rho\left(x_{\mathrm{o}}, t_{\mathrm{o}}\right)\right)\right|-2 d \\
& >\frac{\lambda}{2}\left|\left(t_{\mathrm{o}}-t\right)\right|-2 M-2 d>2 d
\end{aligned}
$$

where in the second inequality we use the fact that $|v| \geqslant \lambda / 4$ at the beginning. Thus $(x, t) \notin T_{y v}^{\eta}$, which proves (13), and we are done.

By taking $\rho(x, t)=x, q=r=2$ and $\mu$ and $\nu$ to be localized Lebesgue measure, (4) holds, with $\alpha=-1 / 2$, by Fubini's theorem and the conservation of the $L_{x}^{2}$ norm. Thus, this provides a new, somewhat geometric, proof of the local smoothing phenomena due to Constantin and Saut [7], Sjölin [26] and Vega [30].

Having localized in time, we are now able to prove an equivalence between space-time estimates.

Lemma 2.2 Let $I=\left[t_{I}, t_{I}+\lambda^{-1}\right]$, and suppose that

$$
\sup _{x \in \operatorname{supp}(\mu), t_{1}, t_{2} \in \operatorname{supp}(\nu)} \frac{\left|\rho\left(x, t_{2}\right)-\rho\left(x, t_{1}\right)\right|}{\left|t_{2}-t_{1}\right|} \leqslant M
$$

Setting $\rho_{I}(x, t)=\rho\left(x, t_{I}\right)$, suppose that there exists $C_{\mathrm{o}}>1$ such that

$$
\begin{equation*}
\left\|U_{\rho_{I}} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)} \leqslant C_{\mathrm{o}}\|f\|_{2} \tag{14}
\end{equation*}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$. Then there exists $C_{d}>1$ such that

$$
\begin{equation*}
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)} \leqslant C_{d} M^{d+1} C_{\mathrm{o}}\|f\|_{2} \tag{15}
\end{equation*}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$. Conversely, if (14) holds with $\rho_{I}$ replaced by $\rho$, then (15) holds with $\rho$ replaced by $\rho_{I}$.

Proof. We show only the implication from (14) to (15), the converse being very similar. After scaling $\xi \rightarrow \lambda \xi$ we note that

$$
U_{\rho} f(x, t)=\frac{\lambda^{d}}{(2 \pi)^{d}} \int \psi(\xi) e^{i \lambda\left(\rho(x, t)-\rho\left(x, t_{I}\right)\right) \cdot \xi} e^{i\left(\lambda \rho\left(x, t_{I}\right) \cdot \xi-\lambda^{2} t|\xi|^{2}\right)} \widehat{f}(\lambda \xi) d \xi
$$

where $\psi$ is smooth and equal to 1 on $B_{2}$ and supported on $(-\pi, \pi)^{d}$. Now by hypothesis, we have

$$
\left|\lambda\left(\rho(x, t)-\rho\left(x, t_{I}\right)\right)\right| \leqslant M \lambda\left|t-t_{I}\right| \leqslant M, \quad t \in I .
$$

Thus, expanding in a Fourier series on $(-\pi, \pi)^{d}$,

$$
\psi(\xi) e^{i \lambda\left(\rho(x, t)-\rho\left(x, t_{I}\right)\right) \cdot \xi}=\frac{1}{(2 \pi)^{d}} \sum_{k \in \mathbb{Z}^{d}} a(x, t, k) e^{i \xi \cdot k},
$$

where the Fourier coefficients, which also depend on $\lambda$ and $t_{I}$, uniformly satisfy

$$
\begin{equation*}
|a(x, t, k)| \leqslant C M^{d+1}(1+|k|)^{-(d+1)}, \quad t \in I, x \in \operatorname{supp}(\mu) . \tag{16}
\end{equation*}
$$

This is easily calculated by integrating by parts the formula for the Fourier coefficients. Thus, we have that

$$
U_{\rho} f(x, t)=\frac{\lambda^{d}}{(2 \pi)^{2 d}} \sum_{k \in \mathbb{Z}^{d}} a(x, t, k) \int e^{i\left(\lambda \rho\left(x, t_{I}\right) \cdot \xi-\lambda^{2} t|\xi|^{2}\right)} e^{i \xi \cdot k} \widehat{f}(\lambda \xi) d \xi
$$

Now, by the triangle inequality, combined with (14) and (16), we see that

$$
\begin{aligned}
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)} & \leqslant C \sum_{k \in \mathbb{Z}^{d}}\|a(\cdot, \cdot, k)\|_{L_{\mu}^{\infty} L_{\nu}^{\infty}(I)}\left\|U_{\rho_{I}}\left(f\left(\lambda^{-1} k+\cdot\right)\right)\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)} \\
& \leqslant C \sum_{k \in \mathbb{Z}^{d}}(1+|k|)^{-(d+1)} M^{d+1} C_{\mathrm{o}}\|f\|_{2} \\
& \leqslant C M^{d+1} C_{\mathrm{o}}\|f\|_{2}
\end{aligned}
$$

and so we are done.

The following result yields Theorem 1.2 by standard arguments. Indeed, one can cover $\mathbb{R}^{d}$ by a countable number of the balls which are generated by Theorem 2.1, then extend the operator $U_{\rho}$ from the Schwartz functions to $H^{s}\left(\mathbb{R}^{d}\right)$ using the estimates (18). This yields a countable number of functions which are continuous in time for almost every $x \in \mathbb{R}^{d}$.

Theorem 2.1 Let $q, r \in[2, \infty], x_{\mathrm{o}} \in \mathbb{R}^{d}$, and let $\rho$, satisfying $\rho(x, 0)=x$, be continuously differentiable. Then there exist constants $C_{s}>0$ such that

$$
\begin{equation*}
\left\|e^{i t \Delta} f\right\|_{L_{x}^{q}\left(\mathbb{B}^{d}, L_{t}^{r}[0,1]\right)} \leqslant C_{s}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}, \quad s>s_{\mathrm{o}} \tag{17}
\end{equation*}
$$

if and only if there exist constants $\epsilon, c_{s}>0$ such that

$$
\begin{equation*}
\left\|U_{\rho} f\right\|_{L_{x}^{q}\left(B\left(x_{\mathrm{o}}, \epsilon\right), L_{t}^{r}[0, \epsilon]\right)} \leqslant c_{s}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}, \quad s>s_{\mathrm{o}} \tag{18}
\end{equation*}
$$

Proof. Since det $D_{x} \rho\left(x_{\mathrm{o}}, 0\right)=1$ and $D_{x} \rho$ is continuous, by the inverse function theorem there is an $\epsilon>0$ such that $\rho(\cdot, t): B\left(x_{\mathrm{o}}, 2 \epsilon\right) \rightarrow \mathbb{R}^{d}$ has its inverse $\rho^{-1}(\cdot, t)$ for all $t \in[-2 \epsilon, 2 \epsilon]$. The determinants of the Jacobians are uniformly bounded for all $t \in[0, \epsilon]$, and we set $E_{t}=\rho^{-1}\left(B\left(x_{\mathrm{o}}, \epsilon\right), t\right)$.
First we prove that (17) implies (18). By translation invariance and scaling if necessary, we have that

$$
\begin{equation*}
\left\|e^{i t \Delta} f\right\|_{L_{x}^{q}\left(E_{t^{\prime}}, L_{t}^{r}[0, \epsilon]\right)} \leqslant C_{s}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}, \quad s>s_{\mathrm{o}} \tag{19}
\end{equation*}
$$

for any fixed $t^{\prime} \in[0, \epsilon]$. We cover $[0, \epsilon]$ by a union of disjoint intervals $I=\left[t_{I}, t_{I}+\lambda^{-1}\right]$, where $\lambda>\epsilon^{-1}$. Then, by the change of variables $x \rightarrow \rho\left(x, t_{I}\right)$, the estimate (19) implies

$$
\left\|U_{\rho_{I}} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)} \leqslant C \lambda^{s}\|f\|_{2}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$, where $\rho_{I}=\rho\left(\cdot, t_{I}\right), d \mu(x)=\chi_{B\left(x_{o}, \epsilon\right)} d x$, and $d \nu(t)=$ $\chi_{[0, \epsilon]} d t$. By the boundedness of $\left|\partial_{t} \rho\right|$ on $B\left(x_{o}, \epsilon\right) \times[0, \epsilon]$ and the mean value theorem, we can apply Lemma 2.2 , so that

$$
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}(I)} \leqslant C \lambda^{s}\|f\|_{2}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$. By the boundedness of $|\rho|$ on $B\left(x_{\mathrm{o}}, \epsilon\right) \times[0, \epsilon]$, we can apply Lemma 2.1 to obtain

$$
\left\|U_{\rho} f\right\|_{L_{\mu}^{q} L_{\nu}^{r}[0, \epsilon]} \leqslant C \lambda^{s}\|f\|_{2}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$. Now the triangle inequality and summation along a geometric series gives the desired bound (18) for $s>s_{\mathrm{o}}$.

The converse direction is slightly easier. By the hypothesis (18), we have that

$$
\left\|U_{\rho} f\right\|_{L_{x}^{q}\left(B\left(x_{\circ}, \epsilon\right), L_{t}^{r}\left(I_{0}\right)\right)} \leqslant C \lambda^{s}\|f\|_{2}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$, where we take $I_{0}=\left[0, \lambda^{-1}\right]$. Then by Lemma 2.2 we may replace $\rho$ by $\rho_{0}=\rho(x, 0)=x$, obtaining

$$
\left\|e^{i t \Delta} f\right\|_{L_{x}^{q}\left(B\left(x_{o}, \epsilon\right), L_{t}^{r}\left(I_{0}\right)\right)} \leqslant C \lambda^{s}\|f\|_{2}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$. By time translation this is valid for any interval $I$ of length $\lambda^{-1}$, and we cover $[0, \epsilon]$ with such intervals. By Lemma 2.1 this yields

$$
\left\|e^{i t \Delta} f\right\|_{L_{x}^{q}\left(B\left(x_{0}, \epsilon\right), L_{t}^{r}[0, \epsilon]\right)} \leqslant C_{s} \lambda^{s}\|f\|_{2} .
$$

Summing a geometric series and scaling we get (17), and so we are done.

## 3. The quantum harmonic oscillator

For $\mathbf{k}=\left(k_{1}, \ldots, k_{d}\right) \in \mathbb{N}_{0}^{d}$, let $h_{\mathbf{k}}$ be Hermite functions which are normalized in $L^{2}\left(\mathbb{R}^{d}\right)$. Then, the solution to the Schrödinger equation for the harmonic oscillator is given by

$$
e^{-i t \mathcal{H}} f=\sum_{\mathbf{k} \in \mathbb{N}_{0}^{d}} e^{-i t\left(|\mathbf{k}|+\frac{d}{2}\right)} a_{\mathbf{k}} h_{\mathbf{k}}
$$

where $a_{\mathbf{k}}$ are the Fourier-Hermite coefficients $a_{\mathbf{k}}=\int_{\mathbb{R}^{d}} f(x) h_{\mathbf{k}}(x) d x$. It follows that $\left\|e^{-i t \mathcal{H}} f\right\|_{L^{2}\left(\mathbb{R}^{d}\right)}=\|f\|_{L^{2}\left(\mathbb{R}^{d}\right)}$, for all time $t$. By the Mehler formula we also have the integral representation

$$
e^{-i t \mathcal{H}} f(x)=\int_{\mathbb{R}^{d}} K_{t}(x, y) f(y) d y, \quad t \in(0, \pi / 4)
$$

where

$$
K_{t}(x, y)=\frac{1}{[2 \pi i \sin t]^{d / 2}} \exp \left(\frac{i}{2}|x-y|^{2} \cot t-i x \cdot y \tan \frac{t}{2}\right) .
$$

By comparing this with the integral representation of the solution to the free equation,

$$
e^{i t \Delta} f(x)=\frac{1}{[4 \pi i t]^{d / 2}} \int_{\mathbb{R}^{d}} e^{\frac{i|x-y|^{2}}{4 t}} f(y) d y, \quad t \in(0, \infty)
$$

one can calculate (see [5], [20] or [25] for details) that for Schwartz functions, we have the transformation

$$
\begin{equation*}
e^{-i \tan ^{-1} t \mathcal{H}} f(x)=e^{-\frac{i t}{2}|x|^{2}}\left(1+t^{2}\right)^{d / 4} e^{i \frac{t}{2} \Delta} f\left(\sqrt{1+t^{2}} x\right), \quad t \in(0, \infty) \tag{20}
\end{equation*}
$$

By simple rescaling we see that $e^{i \frac{t}{2} \Delta} f(x)=e^{i t \Delta}\left[f\left(2^{-1 / 2} \cdot\right)\right]\left(2^{1 / 2} x\right)$, and $\left(1+\tan ^{2} t\right)^{d / 4} \leqslant$ $C$ for $t \in(0,1)$, so that

$$
\left|e^{-i t \mathcal{H}} f\left(2^{-1 / 2} x\right)\right| \sim\left|e^{i(\tan t) \Delta}\left[f\left(2^{-1 / 2} \cdot\right)\right]\left(\sqrt{1+\tan ^{2} t} x\right)\right|, \quad t \in(0,1) .
$$

Hence, the pointwise convergence problem for the harmonic oscillator can be thought of as the problem for $e^{i t \Delta} f$ along the curves $t \rightarrow\left(\sqrt{1+t^{2}} x, t\right)$. Of course this fits into the framework of the previous section, however slightly more can be said when $\rho$ takes this simple product structure.

For $\ell \geqslant 1$, define $\Gamma^{\ell}$ by

$$
\Gamma^{\ell}=\left\{\gamma \in C^{\ell}[-2,2]: 1 / 4<\gamma<4, \quad \sup _{t_{1}, t_{2} \in[-2,2]} \frac{\left|\gamma\left(t_{2}\right)-\gamma\left(t_{1}\right)\right|}{\left|t_{2}-t_{1}\right|} \leqslant 4\right\}
$$

and for $\gamma \in \Gamma^{\ell}$ we define the operator $S^{\gamma}$ by

$$
\begin{equation*}
S^{\gamma} f(x, t)=\frac{1}{(2 \pi)^{d}} \int e^{i\left(\gamma(t) x \cdot \xi-t|\xi|^{2}\right)} \widehat{f}(\xi) d \xi \tag{21}
\end{equation*}
$$

so that $S^{1} f=e^{i t \Delta} f$. Note that Lemmas 2.1 and 2.2 of the previous section hold in the Lipschitz case; for $\rho(x, t)=\gamma(t) x$ with $\gamma \in \Gamma^{0}$. The choice of 4 in the definition of $\Gamma^{\ell}$ is of no importance; it can be any positive number bigger than 1 . Similarly the $\gamma$ need not be defined in the whole interval $[-2,2]$.
For the proof of Theorem 1.3, we will require the following Littlewood-Paley-type lemma in order to sum estimates restricted to dyadic pieces without losing any regularity. Let $\chi$ be a smooth function such that $\operatorname{supp} \chi \subset A_{1}$ and

$$
\sum_{k \in \mathbb{Z}} \chi\left(2^{-k} \cdot\right)=1
$$

As usual, we define the projection operators $P_{k}$ by

$$
\widehat{P_{k} f}=\chi\left(2^{-k} \cdot\right) \widehat{f}
$$

Lemma 3.1 Let $1<r<\infty$ and $\gamma \in \Gamma^{\ell}$ with $\ell>1+\frac{d}{4}+\frac{N}{2}$. Then

$$
\left\|\sum_{2^{k}>8 R} S^{\gamma} P_{k} f(x, \cdot)\right\|_{L_{t}^{r}[0,1]} \leqslant C\left\|\left(\sum_{2^{k}>8 R}\left|S^{\gamma} P_{k} f(x, \cdot)\right|^{2}\right)^{\frac{1}{2}}\right\|_{L_{t}^{r}[-2,2]}+C_{N}\|f\|_{H^{-N}\left(\mathbb{R}^{d}\right)}
$$

whenever $x \in B_{R}$.

Proof. Let $\psi$ be smooth cutoff, equal to one on $[0,1]$, and supported in $(-2,2)$. Then for a fixed $x \in B_{R}$, we consider $\widetilde{S}^{\gamma} f$, defined by

$$
\widetilde{S}^{\gamma} f(t)=\psi(t) S^{\gamma} f(x, t)
$$

as a function of $t$ only. It will suffice to show that

$$
\left\|\sum_{2^{k}>8 R} \widetilde{S}^{\gamma} P_{k} f\right\|_{L^{r}[0,1]} \leqslant C\left\|\left(\sum_{2^{k}>8 R}\left|\widetilde{S}^{\gamma} P_{k} f\right|^{2}\right)^{\frac{1}{2}}\right\|_{L^{r}(\mathbb{R})}+C_{N, \ell}\|f\|_{H^{-N}\left(\mathbb{R}^{d}\right)}
$$

We define projection operators in time frequency. Let $\tilde{\chi}$ be a smooth function, equal to one on $\left\{c_{\mathrm{o}}^{-1} \leqslant|\tau| \leqslant c_{\mathrm{o}}\right\}$, and supported on $\left\{\left(2 c_{\mathrm{o}}\right)^{-1} \leqslant|\tau| \leqslant 2 c_{\mathrm{o}}\right\}$ for some large $c_{\mathrm{o}}>0$, and define $\widetilde{P}_{k}$ by

$$
\widehat{\widetilde{P}_{k} F}=\widetilde{\chi}\left(2^{-k} \tau\right) \widehat{F}(\tau), \quad k \geqslant 1 .
$$

Then by Minkowski's inequality,

$$
\left\|\sum_{2^{k}>8 R} \widetilde{S}^{\gamma} P_{k} f\right\|_{L^{r}[0,1]} \leqslant\left\|\sum_{2^{k}>8 R} \widetilde{P}_{2 k} \widetilde{S}^{\gamma} P_{k} f\right\|_{L^{r}(\mathbb{R})}+\sum_{2^{k}>8 R}\left\|\left(1-\widetilde{P}_{2 k}\right) \widetilde{S}^{\gamma} P_{k} f\right\|_{L^{r}[0,1]}
$$

The first term is majorized by a multiple of $\left\|\left(\sum_{2^{k}>8 R}\left|\widetilde{S}^{\gamma} P_{k} f\right|^{2}\right)^{\frac{1}{2}}\right\|_{L^{r}}$, by the usual Littlewood-Paley inequality, so it remains to show that for $N<2(\ell-1)-d / 2$,

$$
\sum_{2^{k}>8 R}\left\|\left(1-\widetilde{P}_{2 k}\right) \widetilde{S}^{\gamma} P_{k} f\right\|_{L^{r}[0,1]} \leqslant C\|f\|_{H^{-N}\left(\mathbb{R}^{d}\right)}
$$

which follows from

$$
\begin{equation*}
\left\|\left(1-\widetilde{P}_{2 k}\right) \widetilde{S}^{\gamma} P_{k} f\right\|_{L^{r}[0,1]} \leqslant C_{N} 2^{-N k}\left\|P_{k} f\right\|_{2} \tag{22}
\end{equation*}
$$

This can be shown by a routine integration by parts argument.
Indeed, write

$$
\begin{aligned}
& \left(1-\widetilde{P}_{2 k}\right) \widetilde{S}^{\gamma} P_{k} f\left(t^{\prime}\right) \\
= & \frac{1}{2 \pi} \int_{|\xi| \leqslant 2^{k+2}} \int \chi\left(2^{-k} \xi\right)\left(1-\widetilde{\chi}\left(2^{-2 k} \tau\right)\right) \mathcal{K}(x, \xi, \tau) \widehat{f}(\xi) e^{i \tau t^{\prime}} d \tau d \xi,
\end{aligned}
$$

where

$$
\mathcal{K}(x, \xi, \tau)=\int \psi(t) e^{i\left(\gamma(t) x \cdot \xi-t\left(|\xi|^{2}+\tau\right)\right)} d t
$$

Choosing $c_{\text {o }}$ sufficiently large, $\left|\frac{d}{d t}\left(\gamma(t) x \cdot \xi-t\left(|\xi|^{2}+\tau\right)\right)\right| \geqslant C \max \left(2^{2 k},|\tau|\right)$ on the region of integration because $2^{k}>8 R,\left|\gamma^{\prime}(t) x\right| \leqslant c R$ and $\tau \notin\left(2^{2 k} c_{\mathrm{o}}^{-1}, 2^{2 k} c_{\mathrm{o}}\right)$. By repeated integration by parts, we see that

$$
|\mathcal{K}(x, \xi, \tau)| \leqslant C_{\ell} 2^{-2(\ell-1-\mathrm{e}) k}(1+|\tau|)^{-(1+\mathrm{e})}, \quad \mathrm{e}>0
$$

whenever $\tau$ is in the region of integration. Hence, for $t \in[0,1]$, we obtain

$$
\begin{aligned}
\left|\left(1-\widetilde{P}_{2 k}\right) \widetilde{S}^{\gamma} P_{k} f\left(t^{\prime}\right)\right| & \leqslant C_{l} 2^{-2(\ell-1-\mathrm{e}) k} \int_{|\xi| \leqslant 2^{k+2}}\left|\chi\left(2^{-k} \xi\right) \widehat{f}(\xi)\right| d \xi \\
& \leqslant C_{l} 2^{-2(\ell-1-\mathrm{e}) k+\frac{d}{2} k}\left\|P_{k} f\right\|_{2}
\end{aligned}
$$

by Hölder's inequality and Plancherel's theorem, and this implies (22).
Now we show the necessary conditions for the space-time estimates (3), enabling us to take $N=1$ in the previous lemma. Consideration of $f$ given by $\widehat{f}=\psi\left(\lambda^{-1} \cdot\right)$, where $\psi$ is smooth and supported in $A_{1}$, reveals that the condition $s \geqslant \frac{d}{2}-\frac{d}{q}-\frac{2}{r}$ is a necessary for (3) to hold. In particular we may assume $s \geqslant-1$ when $q, r \geqslant 2$. To see that $\frac{d+1}{q}+\frac{1}{r} \leqslant \frac{d}{2}$ is necessary when $s=\frac{d}{2}-\frac{d}{q}-\frac{2}{r}$, we use $f$ given by $\widehat{f}=\phi\left(\lambda^{-1 / 2}\left(\xi-\lambda e_{1}\right)\right)$ with nontrivial $\phi \in C_{0}^{\infty}$. By a change of variables, it is easy to that $\left|S^{\gamma} f(x)\right| \geqslant C \lambda^{d / 2}$ the set defined by $\left|\gamma(t) x-2 \lambda e_{1} t\right|<c_{\mathrm{o}} \lambda^{-\frac{1}{2}}$ and $|t|<c_{\mathrm{o}} \lambda^{-1}$ for some small $c_{\mathrm{o}}>0$, so that

$$
\left\|S^{\gamma} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \geqslant C \lambda^{\frac{d}{2}-\frac{d-1}{2 q}-\frac{3}{2 r}} .
$$

Meanwhile $\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)} \leqslant C \lambda^{s+d / 4}$, so by letting $\lambda \rightarrow \infty$, we obtain the desired condition.
Thanks to the transformation (20), Theorem 1.3 is a consequence of the following proposition. There is no reason to believe that the conditions on $\ell$ or $\beta$ are sharp.

Proposition 3.1 Let $q, r \geqslant 2, r \neq \infty$, and suppose that

$$
\begin{equation*}
\left\|S^{\gamma} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \leqslant C R^{\alpha}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)} \tag{23}
\end{equation*}
$$

for some $\gamma \in \Gamma^{\ell}$ with $\ell>\frac{d+6}{4}$. Then for all $\gamma \in \Gamma^{\ell}$ and $\beta>2 d+3+\alpha$,

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{d}}\left(\int_{0}^{1}\left|S^{\gamma} f(x, t)\right|^{r} d t\right)^{q / r} \frac{d x}{(1+|x|)^{\beta q}}\right)^{1 / q} \leqslant C_{\beta}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)} \tag{24}
\end{equation*}
$$

If $r=\infty$ and $\ell=0$, then (23) for $s>s_{\mathrm{o}}$ implies that (24) holds for $s>s_{\mathrm{o}}$.

Proof. The proof is very similar to the proof of Theorem 2.1, the main difference being that $\rho$ is defined by $\rho(x, t)=\gamma(t) x$ and $\gamma$ is uniformly bounded above and away from zero, so we can invert $\rho$ for a fixed $t$ easily without need of the inverse function theorem. This means that the neighbourhoods of integration only change by mild dilation.

Consider first the case $r=\infty$ and $\ell=0$. Taking $d \mu(x)=\chi_{B_{R}} d x$ and $d \nu(t)=\chi_{[-2,2]} d t$, the constant $M$ which appears in the Lemmas 2.1 and 2.2 can be taken to be $4 R$. Combining the lemmas as before yields that

$$
\left\|S^{\gamma} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \leqslant C R^{\alpha}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}
$$

implies that

$$
\begin{equation*}
\left\|S^{1} f\right\|_{L_{x}^{q}\left(B_{R / 4}, L_{t}^{r}[0,1]\right)} \leqslant C R^{d+3 / 2+\alpha}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)} \tag{25}
\end{equation*}
$$

The domain of integration is smaller due to the change of variables, in order to convert $S^{\gamma\left(t_{I}\right)}$ to $S^{1}$. We can return to the original domain of integration by simply covering $B_{R}$ with balls of radius $R / 4$ and using the translation invariance of (25). Conversely, suppose that the roles of $S^{\gamma}$ and $S^{1}$ in the above are interchanged, then the same implication holds. Combining the two implications, then summing a geometric series in spatial dyadic annuli yields the result.

For the endpoint result when $r \neq \infty$, we apply the Littlewood-Paley Lemma 3.1. By using the symmetry and scaling invariance of (25), a very slight modification of the previous argument gives that (23) implies

$$
\begin{equation*}
\left\|S^{\gamma} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[-2,2]\right)} \leqslant C R^{2 d+3+\alpha} \lambda^{s}\|f\|_{2} \tag{26}
\end{equation*}
$$

whenever $\widehat{f}$ is supported in $A_{\lambda}$. It remains to combine the dyadic pieces without loss in regularity. By Lemma 3.1 with $N=1$,

$$
\left\|\sum_{2^{k}>8 R} S^{\gamma} P_{k} f(x, \cdot)\right\|_{L_{t}^{r}[0,1]} \leqslant C\left\|\left(\sum_{2^{k}>8 R}\left|S^{\gamma} P_{k} f(x, \cdot)\right|^{2}\right)^{\frac{1}{2}}\right\|_{L_{t}^{r}[-2,2]}+C\|f\|_{H^{-1}\left(\mathbb{R}^{d}\right)}
$$

whenever $x \in B_{R}$ and $\ell>\frac{d+6}{4}$, and trivially,

$$
\begin{aligned}
\left\|S^{\gamma} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} & \leqslant\|1\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)}\|\widehat{f}\|_{1} \\
& \leqslant C R^{\frac{d}{q}+\frac{d}{2}}\|f\|_{2} \leqslant C R^{d+1}\|f\|_{H^{-1}}
\end{aligned}
$$

whenever $\widehat{f}$ is supported in $B_{16 R}$. Hence by the triangle inequality we see that

$$
\begin{aligned}
\left\|S^{\gamma} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} & \leqslant\left\|\sum_{2^{k} \leqslant 8 R} S^{\gamma} P_{k} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)}+\left\|\sum_{2^{k}>8 R} S^{\gamma} P_{k} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} \\
& \leqslant C R^{d+1}\|f\|_{H^{-1}\left(\mathbb{R}^{d}\right)}+C\left(\sum_{2^{k}>2 C R}\left\|S^{\gamma} P_{k} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[-2,2]\right)}^{2}\right)^{1 / 2}
\end{aligned}
$$

From (26) it follows that

$$
\begin{aligned}
\left\|S^{\gamma} f\right\|_{L_{x}^{q}\left(B_{R}, L_{t}^{r}[0,1]\right)} & \leqslant C R^{d+1}\|f\|_{H^{-1}\left(\mathbb{R}^{d}\right)}+C R^{2 d+3+\alpha}\left(\sum_{2^{k}>8 R}\left\|2^{s k} P_{k} f\right\|_{2}^{2}\right)^{1 / 2} \\
& \leqslant C R^{2 d+3+\alpha}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)},
\end{aligned}
$$

which we can sum as before to obtain the desired estimate. In the final inequality we use the fact that $s$ is necessarily greater than -1 , as proven above.

Combining Proposition 3.1, the transformation (20) and the known bounds for the free Schrödinger operator one can obtain a global weighted estimate for the maximal operator $f \rightarrow \sup _{0<t<1}\left|e^{-i t \mathcal{H}} f\right|$. Failure of global unweighted estimates for the maximal operator was shown in [25].

Remark 3.1 Let $e^{i t P(D)} f$ be the solution of the equation

$$
i \partial_{t} u+P(D) u=0, \quad u(\cdot, 0)=f
$$

where $P$ is a polynomial of degree $m \geqslant 2$ with $\nabla P(\xi) \neq 0$ if $|\xi|$ is sufficiently large. Lemmas 2.1 and 2.2 can be extended to this case, however Lemma 3.1 is unavailable. Thus Theorem 1.2 and the nonendpoint part of Proposition 3.1 hold for the nonelliptic Schrödinger operator, which, combined with the estimates of [23], yields convergence along curves in this case. We note that the nonendpoint part of the previous proposition also holds with Lebesgue measure in time replaced by a general measure $\nu$ such that $\|1\|_{L_{x}^{q}\left(B_{R}, L_{\nu}^{r}[-2,2]\right)} \leqslant C R^{d}$. Again this is because Lemma 3.1 is not used.
From this we obtain the following equivalence. We remark that an analogous equivalence for the Bochner-Riesz and Hermite Bochner-Riesz problems was proven by Kenig, Stanton and Tomas [13].

Theorem 3.1 Let $\left(t_{k}\right)$ be a real sequence that converges to zero. Then

$$
\lim _{k \rightarrow \infty} e^{i \tan t_{k} \Delta} f(x)=f(x) \text { a.e. } x \in \mathbb{R}^{d}
$$

whenever $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with $s>s_{\mathrm{o}}$ if and only if

$$
\lim _{k \rightarrow \infty} e^{-i t_{k} \mathcal{H}} f(x)=f(x) \text { a.e. } x \in \mathbb{R}^{d}
$$

whenever $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with $s>s_{\mathrm{o}}$.
Proof. Thanks to the transformation (20) and scaling we have

$$
\left|e^{-i t \mathcal{H}} f\left(2^{-1 / 2} x\right)\right| \sim\left|e^{i(\tan t) \Delta}\left[f\left(2^{-1 / 2} \cdot\right)\right]\left(\sqrt{1+\tan ^{2} t} x\right)\right|, \quad t \in(0,1)
$$

By Proposition 3.1 and Remark 3.1 we see that

$$
\left\|S^{1} f\right\|_{L_{x}^{2}\left(B_{R}, L_{\nu}^{\infty}[0,1]\right)} \leqslant C_{s}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}, \quad s>s_{\mathrm{o}}
$$

if and only if

$$
\left\|S^{\gamma} f\right\|_{L_{x}^{2}\left(B_{R}, L_{\nu}^{\infty}[0,1]\right)} \leqslant c_{s}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}, \quad s>s_{\mathrm{O}}
$$

where $\gamma(t)=\sqrt{1+(\tan 1)^{2} t^{2}}$ and $\nu$ is the discrete measure which has mass at $\tan t_{k}$. Thus, the result follows by applying the Nikisin-Stein maximal principle [27], with the
weak $(2,2)$ estimates converted into strong estimates by interpolation with the trivial $H^{s} \rightarrow L^{\infty}, s>n / 2$, estimate followed by Hölder's inequality.

From this one can easily deduce the failure of $\lim _{t \rightarrow 0} e^{-i t \mathcal{H}} f=f$ a.e. for certain $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with $s<1 / 4$. This was shown in [25] when $d=1$. Indeed, if the convergence held for all $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with some $s<1 / 4$, then in particular $\lim _{k \rightarrow \infty} e^{-i \tan ^{-1} \frac{1}{k} \mathcal{H}} f=f$ a.e., so that $\lim _{k \rightarrow \infty} e^{\frac{i}{k} \Delta} f=f$ a.e. by Theorem 3.1. We remark that this is the sequence of time along which Carleson originally considered the convergence [6]. By the NikisinStein maximal principle [27] followed by interpolation and Hölder's inequality as before, we would get

$$
\left\|\sup _{k \geqslant 1}\left|e^{\frac{i}{k} \Delta} f\right|\right\|_{L^{2}\left(B_{R}\right)} \leqslant C_{s}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}
$$

for all $f \in H^{s}\left(\mathbb{R}^{d}\right)$ with some $s<1 / 4$, and this is well-known to be false via the DahlbergKenig counterexample [9]. Indeed to see this one can consider $\widehat{f}=\phi\left(\lambda^{-1 / 2}\left(\xi-\lambda e_{1}\right)\right)$ with nontrivial $\phi \in C_{0}^{\infty}$. Note that $\left|e^{i t \Delta} f(x)\right| \geqslant C \lambda^{d / 2}$ if $(x, t) \in A=\left\{(x, t):\left|x-2 \lambda e_{1} t\right|<\right.$ $\left.c_{\mathrm{o}} \lambda^{-\frac{1}{2}},|t|<c_{\mathrm{o}} \lambda^{-1}\right\}$ for some small $c_{\mathrm{o}}>0$. If $\left(x_{\mathrm{o}}, t_{\mathrm{o}}\right) \in A$, there is an interval $I$ of length $\sim \lambda^{-\frac{3}{2}}$ such that $t_{\mathrm{o}} \in I \subset\left(0, c_{\mathrm{o}} \lambda^{-1}\right)$ and $\left\{x_{\mathrm{o}}\right\} \times I \subset A$. Since $\frac{1}{k}-\frac{1}{k+1}<$ $c^{2} \lambda^{-2}$ if $\frac{1}{k}<c \lambda^{-1}$, there is a $k_{\mathrm{o}}$ such that $\frac{1}{k_{\mathrm{o}}} \in I$. Thus, $\left|e^{\frac{i}{k_{\mathrm{o}}} \Delta} f\left(x_{\mathrm{o}}\right)\right| \geqslant C \lambda^{d / 2}$, and it follows that $\sup _{k \geqslant 1}\left|e^{\frac{i}{k} \Delta} f(x)\right| \geqslant C \lambda^{d / 2}$ if $\left|x_{1}\right|<c_{\mathrm{o}}$ and $\left|\left(x_{2}, x_{3}, \ldots, x_{d}\right)\right| \leqslant c_{o} \lambda^{-1 / 2}$. Since $\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)} \leqslant C \lambda^{s+d / 4}$, the maximal bound implies $\lambda^{(d+1) / 4} \leqslant C \lambda^{s+d / 4}$. Letting $\lambda \rightarrow \infty$ this gives $s \geqslant 1 / 4$ which is a contradiction.

## 4. Fractal dimension of the divergence set

We have proven that, under various conditions, the set of points where convergence fails is null with respect to Lebesgue measure. In this section we attempt to bound the the Hausdorff dimension of this set. This makes no sense while considering Sobolev spaces, as the functions are only defined up to a set of full Hausdorff dimension. Instead, we consider the potential spaces, which we also call $H^{s}\left(\mathbb{R}^{d}\right)$, defined by

$$
H^{s}\left(\mathbb{R}^{d}\right)=\left\{G_{s} * f: f \in L^{2}\left(\mathbb{R}^{d}\right)\right\}
$$

Here $\widehat{G}_{s}(\xi)=\left(1+|\xi|^{2}\right)^{-s / 2}$, so that each equivalence class of the Sobolev space has a representative in the potential space.
Similarly we have to take more care with the definition of $S^{\gamma}$. We may define $S^{\gamma} f$ as the pointwise limit

$$
\begin{equation*}
S^{\gamma} f(\cdot, t)=\lim _{N \rightarrow \infty} S_{N}^{\gamma} f(\cdot, t) \tag{27}
\end{equation*}
$$

whenever the limit exists, where the operator $S_{N}^{\gamma}$ is defined by

$$
S_{N}^{\gamma} f(x, t)=\frac{1}{(2 \pi)^{d}} \int_{\mathbb{R}^{d}} \psi\left(N^{-1}|\xi|\right) \widehat{f}(\xi) e^{i\left(\gamma(t) x \cdot \xi-t|\xi|^{2}\right)} d \xi
$$

Here, for convenience, we take $\psi$ to be the Gaussian $\psi(r)=e^{-r^{2}}$. By standard arguments, this coincides with the traditional $L^{2}$-limit, almost everywhere with respect to Lebesgue measure, however it is also well defined with respect to fractal measures when $s>0$ (see [1]).

We denote by $\alpha_{d}(s, \gamma)$ the supremum of

$$
\begin{equation*}
\operatorname{dim}_{H}\left\{x \in \mathbb{R}^{d}: S^{\gamma} f\left(x, t_{k}\right) \nrightarrow f(x) \text { as } k \rightarrow \infty\right\} \tag{28}
\end{equation*}
$$

over all $f \in H^{s}\left(\mathbb{R}^{d}\right)$ and all sequences $\left(t_{k}\right)$ that converge to zero. Here, as usual, $\operatorname{dim}_{H}$ denotes the Hausdorff dimension. In [1], a sharp result was proven,

$$
\alpha_{1}(s, 1)=\left\{\begin{array}{lr}
1, & s<1 / 4 \\
1-2 s, & 1 / 4 \leqslant s<1 / 2 \\
0, & 1 / 2 \leqslant s
\end{array}\right.
$$

which improved upon previous upper bounds due to Sjögren and Sjölin [24]. The lower bound is a consequence of the fact that $G_{s} * f$, with $f \in L^{2}\left(\mathbb{R}^{d}\right)$, can be singular on sets of dimension $\alpha$ when $\alpha<d-2 s$ (see [32]), combined with the Dahlberg-Kenig counterexample [9]. Restricting attention to radial data, in [2] it was proven that

$$
\alpha_{d}(s, 1)=\left\{\begin{array}{lr}
d, & s<1 / 4 \\
d-2 s, & 1 / 4 \leqslant s<1 / 2 \\
0, & 1 / 2<s
\end{array}\right.
$$

which is again sharp.
We say that a positive Borel measure $\mu$ is $\alpha$-dimensional if

$$
c_{\alpha}(\mu):=\sup _{x \in \mathbb{R}^{d}, r>0} \frac{\mu(B(x, r))}{r^{\alpha}}<\infty, \quad 0 \leqslant \alpha \leqslant n
$$

and denote by $\mathcal{M}^{\alpha}\left(B_{R}\right)$ the $\alpha$-dimensional measures which are supported in $B_{R}$. Upper bounds for $\alpha_{d}$ follow from appropriate maximal estimates. Indeed, if for all $R>1$, we have

$$
\begin{equation*}
\left\|\sup _{k \geqslant 1} \sup _{N \geqslant 1}\left|S_{N}^{\gamma} f\left(\cdot, t_{k}\right)\right|\right\|_{L^{2}(d \mu)} \leqslant C_{R} \sqrt{c_{\alpha}(\mu)}\|f\|_{H^{s}\left(\mathbb{R}^{d}\right)}, \quad \alpha>\alpha_{\mathrm{o}}, \tag{29}
\end{equation*}
$$

whenever $\mu \in \mathcal{M}^{\alpha}\left(B_{R}\right), f \in H^{s}\left(\mathbb{R}^{d}\right)$ and $\left(t_{k}\right) \in(0,1)^{\mathbb{N}}$, then $\alpha_{d}(s, \gamma) \leqslant \alpha_{o}$. This is can be proven by standard arguments including an application of Frostman's lemma (see [2, Appendix B] for details).

Using the results of Section 2 we are able to extend these results (losing the endpoint $s=1 / 4$ ), so that they hold for

$$
\alpha_{d}^{*}(s):=\sup _{\gamma \in \Gamma^{0}} \alpha_{d}(s, \gamma) .
$$

In particular we extend the refinement to the quantum harmonic oscillator. In the following theorem we consider general data; the radial data extension is proven similarly.

Theorem 4.1 Let $\frac{d}{4}<s \leqslant \frac{d}{2}$. Then $\alpha_{d}^{*}(s)=d-2 s$.

Proof. Writing $f=G_{s} * g$, we are required to prove

$$
\begin{equation*}
\left\|\sup _{k \geqslant 1} \sup _{N \geqslant 1}\left|S_{N}^{\gamma}\left(G_{s} * g\right)\left(\cdot, t_{k}\right)\right|\right\|_{L^{2}(d \mu)} \leqslant C_{R} \sqrt{c_{\alpha}(\mu)}\|g\|_{2}, \quad \alpha>d-2 s, \tag{30}
\end{equation*}
$$

whenever $\gamma \in \Gamma^{0}, \mu \in M^{\alpha}\left(B_{R}\right)$ and $g \in L^{2}\left(\mathbb{R}^{d}\right)$. First we reduce this to proving

$$
\begin{equation*}
\left\|\sup _{0<t<1}\left|S^{\gamma}\left(G_{s} * g\right)(\cdot, t)\right|\right\|_{L^{2}(d \mu)} \leqslant C_{R} \sqrt{c_{\alpha}(\mu)}\|g\|_{2}, \quad \alpha>d-2 s \tag{31}
\end{equation*}
$$

whenever $\gamma \in \Gamma^{0}, \mu \in M^{\alpha}\left(B_{R}\right)$ and $g \in L^{2}\left(\mathbb{R}^{d}\right)$ with compact Fourier support. To see this, we note that by the Fundamental Theorem of Calculus,

$$
\begin{equation*}
\sup _{N \geqslant 1}\left|S_{N}^{\gamma}\left(G_{s} * g\right)\right| \leqslant\left|S_{1}^{\gamma}\left(G_{s} * g\right)\right|+\int_{1}^{\infty}\left|\frac{d}{d N} S_{N}^{\gamma}\left(G_{s} * g\right)\right| d N \tag{32}
\end{equation*}
$$

and we can calculate $\left|\frac{d}{d N} S_{N}^{\gamma}\left(G_{s} * g\right)\right|=N^{-2}\left|S^{\gamma}\left(\psi^{\prime}\left(N^{-1}|\cdot|\right)|\cdot| \widehat{G_{s}} \widehat{g}\right)^{\vee}\right|$. Substituting this into (32), and (32) into (30), by Minkowski's integral inequality, it will suffice to prove

$$
\left\|\sup _{0<t<1}\left|S^{\gamma}\left(\widehat{G}_{s} \psi(|\cdot|) \widehat{g}\right)^{\vee}(\cdot, t)\right|\right\|_{L^{2}(d \mu)} \leqslant C_{R} \sqrt{c_{\alpha}(\mu)}\|g\|_{2}
$$

which follows from $(31)$ as $\left\|(\psi(|\cdot|) \widehat{g})^{\vee}\right\|_{2} \leqslant\|g\|_{2}$, and for $0<\epsilon<1 / 100$,

$$
\int_{1}^{\infty}\left\|\sup _{0<t<1}\left|S^{\gamma}\left(\widehat{G}_{s}\left[\frac{\psi^{\prime}\left(N^{-1}|\cdot|\right)|\cdot| \widehat{g}}{\left(1+|\xi|^{2}\right)^{\mathrm{e} / 2}}\right]\right)^{\vee}(\cdot, t)\right|\right\|_{L^{2}(d \mu)} \frac{d N}{N^{2}} \leqslant C_{R} \sqrt{c_{\alpha}(\mu)}\|g\|_{2}
$$

where we lose an e of regularity which is permissable. Now this would follow from (31) as

$$
\left\|\left(\frac{\psi^{\prime}\left(N^{-1}|\cdot|\right)|\cdot| \widehat{g}}{\left(1+|\xi|^{2}\right)^{\mathrm{e} / 2}}\right)^{\vee}\right\|_{2} \leqslant C N^{1-\mathrm{e}}\|g\|_{2}
$$

so it remains to prove (31).
We will apply Lemmas 2.1 and 2.2 as before; the difference here is that the measures $\mu$ are not necessarily translation invariant, and so the changes of variables in order to pass from $S^{1}$ to $S^{\gamma\left(t_{I}\right)}$ change the supports of the measures. However we can argue in a very similar way. Indeed, in [1, Proposition 3.1 and Lemma C.1], it was proven that

$$
\begin{equation*}
\left\|\sup _{0<t<1}\left|S^{1} f(\cdot, t)\right|\right\|_{L^{2}(d \mu)} \leqslant C \sqrt{c_{\alpha}(\mu)} \lambda^{s}\|f\|_{2}, \quad \alpha>d-2 s \tag{33}
\end{equation*}
$$

whenever $\mu \in M^{\alpha}\left(\mathbb{B}^{d}\right)$ and $\widehat{f}$ is supported in $A_{\lambda}$. Now any measure in the class $\mathcal{M}^{\alpha}\left(B_{4 R}\right)$ can be represented as the sum of a finite number of translated $\alpha$-dimensional measures supported in $\mathbb{B}^{d}$, so that (33) yields

$$
\begin{equation*}
\left\|\sup _{0<t<1}\left|S^{1} f(\cdot, t)\right|\right\|_{L^{2}(d \mu)} \leqslant C_{R} \sqrt{c_{\alpha}(\mu)} \lambda^{s}\|f\|_{2}, \quad \alpha>d-2 s \tag{34}
\end{equation*}
$$

whenever $\mu \in \mathcal{M}^{\alpha}\left(B_{4 R}\right)$ and $\widehat{f}$ is supported in $A_{\lambda}$. Applying Lemmas 2.1 and 2.2 as before, from this we can conclude that

$$
\left\|\sup _{0<t<1}\left|S^{\gamma} f(\cdot, t)\right|\right\|_{L^{2}(d \mu)} \leqslant C_{R} \sqrt{c_{\alpha}(\mu)} \lambda^{s}\|f\|_{2}, \quad \alpha>d-2 s
$$

whenever $\gamma \in \Gamma^{0}, \mu \in \mathcal{M}^{\alpha}\left(B_{R}\right)$ and $\widehat{f}$ is supported in $A_{\lambda}$ with $\lambda>\|\mu\|^{1 / d}$. Supposing for a moment that $\mu$ is a probability measure, trivially we also have that

$$
\left\|\sup _{0<t<1}\left|S^{\gamma} f(\cdot, t)\right|\right\|_{L^{2}(d \mu)} \leqslant\|\widehat{f}\|_{1} \leqslant \sqrt{c_{\alpha}(\mu)}\|f\|_{2}, \quad \alpha>d-2 s
$$

whenever $\gamma \in \Gamma^{0}, \mu \in \mathcal{M}^{\alpha}\left(B_{R}\right)$, and $\widehat{f}$ is supported in $A_{\lambda}$ with $\lambda \leqslant\|\mu\|^{1 / d}$. The argument is completed by summing a geometric series, and then considering $\|\mu\|^{-1} \mu$ in order to remove the condition that $\|\mu\|=1$.

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[^0]:    Email addresses: shlee@math.snu.ac.kr (Sanghyuk Lee), keith.rogers@icmat.es (Keith M. Rogers).
    ${ }^{1}$ Supported in part by the NRF grant 2009-0072531 (Korea)
    ${ }^{2}$ Supported in part by the ERC grant 277778 and the MICINN grant MTM2010-16518 (Spain)

