LEBESGUE SPACE ESTIMATES FOR A CLASS OF FOURIER INTEGRAL OPERATORS ASSOCIATED WITH WAVE PROPAGATION

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Dedicated to Professor Hans Triebel

ABSTRACT. We prove L^q estimates related to Sogge's conjecture for a class of Fourier integral operators associated with wave equations.

1. INTRODUCTION

In this note we prove a variable coefficient version of a recent result in [6] on the local L^q space-time regularity results for solutions of wave equations. The solution operators are Fourier integral operators satisfying the 'cinematic curvature' hypothesis introduced in [17] (see also [14]).

For the general setup let \mathcal{Y} and \mathcal{Z} be paracompact C^{∞} manifolds, dim $(\mathcal{Y}) = d$, dim $(\mathcal{Z}) = d + 1$; in the current paper we shall need to assume $d \geq 4$. We are interested in sharp local regularity estimates for Fourier integral operators $\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})$ (associated with the Fourier integral distributions defined in [8]). Here the canonical relation

$$\mathcal{C} \subset T^*\mathcal{Z} \setminus 0_L \times T^*\mathcal{Y} \setminus 0_R$$

is a conic manifold of dimension 2d + 1, which is Lagrangian with respect to the symplectic form $d\zeta \wedge dz - d\eta \wedge dy$. We denote by 0_L and 0_R the zero-sections in $T^*\mathcal{Z}$ and $T^*\mathcal{Y}$, respectively.

We formulate a curvature hypothesis which appeared in [3], [10] for classes of oscillatory integral operators (see [13], [2] for current results on these classes). We follow the exposition in [14] and impose conditions on the following projection maps.



We require that the projection $\pi_L : \mathcal{C} \to T^*\mathcal{Y}$ is a submersion (*i.e.* the differential has maximal rank 2d). We also require that the space projection $\Pi_{\mathcal{Z}} : \mathcal{C} \to \mathcal{Z}$ is a submersion (*i.e.* its differential has maximal rank d + 1). As discussed in §2 of [14] this implies that for fixed $z \in \Pi_Z \mathcal{C}$ the image of the projection to the fiber,

$$\Gamma_z = \{ \zeta :\in T_z^* \mathcal{Z} : \exists (y, \eta) \text{ such that } (z, \zeta, y, \eta) \in \mathfrak{C} \},\$$

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is an immersed conic (d-1)-dimensional hypersurface in $T_z^*\mathcal{Z} \setminus 0_L$. We then make an assumption on the curvature of the cones Γ_z :

Curvature hypothesis $\mathcal{H}(\ell)$: $\mathcal{C} \subset T^*\mathcal{Z} \setminus 0_L \times T^*\mathcal{Y} \setminus 0_R$, the projections π_R and $\Pi_{\mathcal{Z}}$ are submersions and for each z the cone Γ_z has at least ℓ nonvanishing principal curvatures at any $\zeta \in \Gamma_z$.

Theorem 1.1. Let $\ell \geq 3$, let \mathfrak{C} satisfy hypothesis $\mathfrak{H}(\ell)$ and let $\mathcal{F} \in I^{\mu-1/4}(\mathfrak{Z}, \mathfrak{Y}; \mathfrak{C})$. Suppose $\frac{2\ell}{\ell-2} < q < \infty$ and $\mu \leq \frac{d}{q} - \frac{d-1}{2}$. Then \mathcal{F} maps $L^q_{\text{comp}}(\mathfrak{Y})$ to $L^q_{\text{loc}}(\mathfrak{Z})$.

We may apply the theorem with $\ell = d - 1$ to solutions of the wave equation on a compact Riemannian manifold M, with initial data in L^q -Sobolev spaces $L^q_{\alpha}(M)$. Let Δ be the Laplace-Beltrami operator on M. If one combines Theorem 1.1 with the usual parametrix construction (cf. [4]) one obtains (arguing as in [14])

Corollary 1.2. Let $d \ge 4$, $\frac{2(d-1)}{d-3} < q < \infty$, and let I be a compact time interval. There is C > 0 such that

$$\left(\int_{I} \left\| e^{it\sqrt{-\Delta}} f \right\|_{L^{q}(M)}^{q} dt \right)^{1/q} \le C \|f\|_{L^{q}_{\alpha}(M)}, \quad \alpha = \frac{d-1}{2} - \frac{d}{q},$$

for all $f \in L^q_{\alpha}(M)$.

Note that the constant may strongly depend on the choice of I. There are further regularity improvements in the scale of Triebel-Lizorkin spaces (*cf.* §3 below); in particular $L^q_{\alpha}(M)$ can be replaced by the Besov space $B^q_{\alpha,q}(M)$.

In $\S2$ we prove a frequency localized version of Theorem 1.1 and combine the estimates corresponding to different frequencies in $\S3$. In $\S4$ we discuss some generalizations in the constant coefficient case.

Remarks. In the constant coefficient case one can recover from Theorem 1.1 the space time estimates of [6] which correspond to an endpoint version of Sogge's conjecture in the range given for $\ell = d - 1$, see also §4 for other generalizations. For previous partial results on Sogge's conjecture, also in lower dimensions, see the groundbreaking paper of Wolff [20] and the subsequent papers [12], [5].

The case $\ell = d - 1$ essentially corresponds to the assumption of cinematic curvature in [17]. We use Hörmander's convention for the definition of order, *i.e.*, in view of the different dimensions of \mathcal{Z} and \mathcal{Y} operators of class $I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})$ correspond to locally finite sums of operators with integral kernels in the standard representation (1) below, involving d frequency variables and standard symbols of order μ . One can use a partition of unity and finite decompositions in the fiber variable to reduce matters to the estimation of an integral operator with compactly supported kernel \mathcal{K} which is given as an oscillatory integral distribution in the sense of [8]. Namely if Z is an open set in \mathbb{R}^{d+1} and Y is an open set in \mathbb{R}^d we may assume that

(1)
$$\mathcal{K}(z,y) = \int a(z,y,\xi) e^{i(\phi(z,\xi) - \langle y,\xi \rangle)} d\xi$$

where a is a standard symbol of order μ , a is supported for z, y in compact subsets of Z and Y, resp., and ϕ is smooth away from the origin and homogeneous of order one with respect to the variable ξ , and supported in an open set which is conic in ξ . We then have $\nabla_z \phi(z,\xi) \neq 0$ for $\xi \neq 0$ and the mixed second derivative $(d+1) \times d$ matrix $\phi''_{z\xi}(z,\xi)$ has rank d. For fixed (z,ξ) , if the vector u is in the cokernel of $\phi''_{z\xi}(z,\xi)$ then the Hessian matrix $\nabla^2_{\xi\xi} \langle u, \nabla_z \phi \rangle(z,\xi)$ has rank at least ℓ , by our curvature assumption.

2. The frequency localized case

By making further localizations, changing variables in z and y, and ignoring error terms which are smoothing of high order we may assume that our kernel is given by

$$K(z,y) = \sum_{k=1}^{\infty} 2^{k\mu} K_k(z,y),$$

where

(2)
$$K_k(z,y) = \int \chi_k(z,y,2^{-k}\xi) e^{i(\varphi(z,\xi) - \langle y,\xi \rangle)} d\xi,$$

and the functions χ_k are smooth and supported in a compact subset of $Z \times Y \times \Xi$. Here Z is a small neighborhood of the origin in \mathbb{R}^{d+1} , Y is a small neighborhood of the origin in \mathbb{R}^d and Ξ is a small neighborhood of the vector $e_1 := (1, 0, \ldots, 0)$ in \mathbb{R}^d . Moreover

(3)
$$\varphi_{z\xi}''(0,e_1) = \begin{pmatrix} I_d \\ 0 \end{pmatrix}$$

(4)
$$\operatorname{rank} \nabla_{\xi\xi}^2 \varphi_{z_{d+1}}'(0, e_1) \ge \ell;$$

and in view of the small choice of Z, Y, Ξ we may assume that for all $(z,\xi) \in Z \times \Xi$ the gradient $\varphi'_z(z,\xi)$ is close to e_1 , and, with $z = (z', z_{d+1})$, we may assume that $\varphi''_{z'\xi}(z,\xi)$ is close to the identity matrix I_d and $\varphi''_{z_{d+1}\xi}(z,\xi)$ is small. We may further perform a rotation and assume that in coordinates $\xi = (\xi_1, \xi', \xi'')$ with $\xi' = (\xi_2, \ldots, \xi_{\ell+1})$ we have

(5)
$$\operatorname{rank} \nabla^2_{\xi'\xi'} \varphi'_{z_{d+1}}(0, e_1) = \ell.$$

Finally, $|\partial_{z,y,\xi}^{\alpha}\chi_k(z,y,\xi)| \leq C_{\alpha}$ for any multiindex α , uniformly in k, $(z,y,\xi) \in Z \times Y \times \Xi$. Let $T_k f(z) = \int K_k(z,y) f(y) dy$. Here we prove that the $L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^{d+1})$ operator

Let $T_k f(z) = \int K_k(z, y) f(y) dy$. Here we prove that the $L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^{d+1})$ operator norm of T_k is $O(2^{k(\frac{d-1}{2} - \frac{d}{q})})$ for $q > \frac{2\ell}{\ell-2}$, and in the next section we discuss how to put the estimates for T_k together. The L^{∞} estimate

$$|T_k f||_{L^{\infty}} \lesssim 2^{k\frac{d-1}{2}} ||f||_{\infty}$$

can be found in [16]. By interpolation it is enough to prove

Theorem 2.1. Let $\ell \geq 3$ and $q_{\ell} = \frac{2\ell}{\ell-2}$. The operator T_k is of restricted weak type (q_{ℓ}, q_{ℓ}) , with operator norm

$$||T_k||_{L^{q_\ell,1}(\mathbb{R}^{d+1})\to L^{q_\ell,\infty}(\mathbb{R}^d)} \lesssim 2^{k(d/\ell-1/2)}$$

By duality we need to prove the restricted weak type inequality for the adjoint operator T_k^* , given by

$$T_k^*F(y) = \int \int \chi_k(z, y, 2^{-k}\xi) e^{i(\langle y, \xi \rangle - \varphi(z, \xi))} d\xi F(z) dz;$$

i.e. for each measurable set E contained in $[-1/2, 1/2]^{d+1}$,

(6)
$$||T_k^*\chi_E||_{L^{p_\ell,\infty}}^{p_\ell} \lesssim 2^{k\frac{2d-\ell}{\ell+2}}|E|, \quad p_\ell = \frac{2\ell}{\ell+2}$$

The estimate (6) will be derived from the following Proposition 2.2, which is a discretized version of (6) and will be proved in §2.3.

Proposition 2.2. Let $p_{\ell} = \frac{2\ell}{\ell+2}$, $\ell \geq 3$. For k > 2 let $\mathcal{Z}_k = 2^{-k}\mathbb{Z}^{d+1} \cap [-\varepsilon^2, \varepsilon^2]^{d+1}$, for sufficiently small $\varepsilon > 0$. Suppose that for each $\mathfrak{z} \in \mathcal{Z}_k$ we are given a symbol $a_{k,\mathfrak{z}}$ supported in $\{\xi : 2^{k-1} < |\xi| < 2^{k+1}, |\frac{\xi}{|\xi|} - e_1| \leq \varepsilon^2\}$ so that

(7)
$$|\partial_{\xi}^{\alpha}a_{k,\mathfrak{z}}(\xi)| \le 2^{-k|\alpha|}, \quad |\alpha| \le 10d.$$

Define $S_{\mathfrak{z}} \equiv S_{\mathfrak{z}}^k$ by

(8)
$$S_{\mathfrak{z}}(y) = \int a_{k,\mathfrak{z}}(\xi) e^{i(\langle y,\xi\rangle - \varphi(\mathfrak{z},\xi))} d\xi.$$

Then for each $\mathcal{E} \subset \mathcal{Z}_k$ we have

(9)
$$\operatorname{meas}\left(\left\{y \in \mathbb{R}^d : \left|\sum_{\mathfrak{z} \in \mathcal{E}} S_{\mathfrak{z}}\right| > \alpha\right\}\right) \le C 2^{k(\frac{d+1}{2}p_{\ell}-1)} \alpha^{-p_{\ell}} \# \mathcal{E}.$$

In the following subsections we prove some preparatory L^1 and L^2 estimates, then prove Proposition 2.2, and that Proposition 2.2 implies (6). In §3 we combine the dyadic estimates in Theorem 2.1.

2.1. L^1 estimates. L^1 -estimates for the expressions $S_{\mathfrak{z}}$ can be found in [16]. In what follows we let Θ_k be a maximal $2^{-k/2}$ -separated set of unit vectors. Using a homogeneous extension of a partition of unity on the sphere one can split

$$a_{k,\mathfrak{z}}(\xi) = \sum_{\theta \in \Theta_k} a_{k,\mathfrak{z},\theta}(\xi)$$

where $a_{k,\mathfrak{z},\theta}$ is supported on the intersection of the cone $\{\xi : \left|\frac{\xi}{|\xi|} - \theta\right| \leq 2^{-k/2}\}$ with the support of $a_{k,\mathfrak{z}}$; moreover if u_i are unit vectors perpendicular to θ we have the estimates

$$\left|\langle\theta,\nabla_{\xi}\rangle^{M_{1}}\prod_{i=1}^{M_{2}}\langle u_{i},\nabla_{\xi}\rangle a_{k,\mathfrak{z},\theta}(\xi)\right| \leq C(M_{1},M_{2})2^{-kM_{1}}2^{-kM_{2}/2}$$

whenever $M_1 + M_2 \leq 10d$. Let

(10)
$$S_{\mathfrak{z},\theta}(y) = \int a_{k,\mathfrak{z},\theta}(\xi) e^{i(\langle y,\xi\rangle - \varphi(\mathfrak{z},\xi))} d\xi$$

By homogeneity we have

(11)
$$\phi_{\xi\xi}''(z,\theta)\theta = 0$$

Using this observation we get, as in [16], by an integration by parts

$$|S_{\mathfrak{z},\theta}(y)| \le C_d 2^{k\frac{d+1}{2}} \left(1 + 2^k |\langle \varphi'_{\xi}(\mathfrak{z},\theta) - y,\theta \rangle| + 2^{k/2} |\Pi_{\theta^{\perp}}(\varphi'_{\xi}(\mathfrak{z},\theta) - y)|\right)^{-10d};$$

here $\Pi_{\theta^{\perp}}$ denotes the projection to the orthogonal complement of θ . This estimate implies $\|S_{\mathfrak{z},\theta}\|_1 = O(1)$ and therefore

(12)
$$||S_{\mathfrak{z}}||_1 \lesssim 2^{k\frac{d-1}{2}}$$

Moreover we get for $1 \le R \le 2^k$,

$$\int_{\substack{|\Pi_{\theta^{\perp}}(\varphi'_{\xi}(\mathfrak{z},\theta)-y)|\\ \geq (2^{-k}R)^{1/2}}} |S_{\mathfrak{z},\theta}(y)| \, dy \lesssim \int_{\substack{w' \in \mathbb{R}^{d-1}\\ |w'| \geq (2^{-k}R)^{1/2}}} \frac{2^{k(d-1)/2}}{(1+2^{k/2}|w'|)^{10d-2}} dw' \lesssim R^{\frac{1-9d}{2}},$$

and similarly

$$\int_{\substack{|\langle \varphi'_{\xi}(\mathfrak{z},\theta)-y,\theta\rangle|\\\geq 2^{-k}R}} |S_{\mathfrak{z},\theta}(y)| \, dy \lesssim \int_{|w_d|\geq 2^{-k}R} \frac{2^k}{(1+2^k|w_d|)^{9d-1}} dw_d \lesssim R^{2-9d}.$$

Now clearly

$$\left|\varphi'_{\xi}(z,\theta)-\varphi'_{\xi}(\widetilde{z},\widetilde{\theta})\right| \lesssim |z-\widetilde{z}|+|\theta-\widetilde{\theta}|,$$

and by (11) also

$$\left|\left\langle \varphi_{\xi}'(z,\theta) - \varphi_{\xi}'(\widetilde{z},\widetilde{\theta}),\theta\right\rangle\right| \lesssim |z - \widetilde{z}| + |\theta - \widetilde{\theta}|^2.$$

Thus if

(13) $V_{\theta}^{k}(z,R) = \left\{ y : |\langle \varphi_{\xi}'(z,\theta) - y, \theta \rangle| \le R2^{-k}, \quad |\Pi_{\theta^{\perp}}(\varphi_{\xi}'(z,\theta) - y)| \le (R2^{-k})^{1/2} \right\}$ then the above calculations give

(14)
$$\|S_{\widetilde{\mathfrak{z}},\widetilde{\theta}}\|_{L^1(\mathbb{R}^d \setminus V^k_{\theta(\mathfrak{z},R)})} \le C(C_1)R^{-4d} \quad \text{if } |\widetilde{\mathfrak{z}} - \mathfrak{z}| \le C_1R2^{-k}, \quad |\widetilde{\theta} - \theta| \le C_1(R2^{-k})^{1/2}$$

for $C_1 \ge 1$.

2.2. Estimates for scalar products. Based on standard calculations for oscillatory integrals ([10], [18], [1], [11], [14]) we prove some estimates for scalar products $\langle S_{\mathfrak{z}}, S_{\mathfrak{z}'} \rangle$; these results are closely related to the scalar product estimates in [6]. For the Fourier transforms we have

$$\widehat{S}_{\mathfrak{z}}(\xi) = a_{k,\mathfrak{z}}(\xi)e^{-i\varphi(\mathfrak{z},\xi)}$$

and

(15)
$$(2\pi)^d \langle S_{\mathfrak{z}}, S_{\widetilde{\mathfrak{z}}} \rangle = \langle \widehat{S}_{\mathfrak{z}}, \widehat{S}_{\widetilde{\mathfrak{z}}} \rangle = \int a_{k,\mathfrak{z}}(\eta) \overline{a_{k,\mathfrak{z}}(\eta)} e^{i(\varphi(\mathfrak{z},\eta) - \varphi(\mathfrak{z},\eta))} d\eta$$

(16)
$$= 2^{kd} \int b_{k,\mathfrak{z},\mathfrak{z}}(\xi) e^{i2^k (\varphi(\mathfrak{z},\xi) - \varphi(\mathfrak{z},\xi))} d\xi$$

where $b_{k,\mathfrak{z},\mathfrak{z}}$ is supported on a subset of diameter $O(\varepsilon^2)$ of the annulus $\{|\xi| \approx 1\}$, near e_1 , with ε sufficiently small. We may assume in what follows that $\mathfrak{z}, \mathfrak{z}$ are in a neighborhood of the origin in \mathbb{R}^{d+1} , of diameter $\lesssim \varepsilon^2$. We split coordinates $z = (z', z_{d+1})$, take advantage of (3) and get

$$|\varphi'_{\xi}(\mathfrak{z},\xi) - \varphi'_{\xi}(\mathfrak{z},\xi)| \ge c|\mathfrak{z}' - \mathfrak{z}'| - C\varepsilon|\mathfrak{z}_{d+1} - \mathfrak{z}_{d+1}|$$

and after an integration by parts we get

(17)
$$|\langle S_{\mathfrak{z}}, S_{\widetilde{\mathfrak{z}}}\rangle| \lesssim \frac{2^{kd}}{(1+2^{k}|\mathfrak{z}-\widetilde{\mathfrak{z}}|)^{9d}} \quad \text{if } |\mathfrak{z}'-\widetilde{\mathfrak{z}}'| \ge C_{1}\varepsilon|\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|.$$

For $s \in [0, 1]$ set $\mathfrak{z}_s = \mathfrak{z} + s(\mathfrak{z} - \mathfrak{z})$. If

$$|\mathfrak{z}' - \widetilde{\mathfrak{z}}'| \le C_2 \varepsilon |\mathfrak{z}_{d+1} - \widetilde{\mathfrak{z}}_{d+1}|$$

(with suitable $C_1 \ll C_2 \ll \varepsilon^{-1}$) we consider

$$\frac{\varphi(\widetilde{\mathfrak{z}},\xi) - \varphi(\mathfrak{z},\xi)}{\widetilde{\mathfrak{z}}_{d+1} - \mathfrak{z}_{d+1}} = \int_0^1 \left[\varphi_{z_{d+1}}'(\mathfrak{z}_s,\xi) + \left\langle \frac{\widetilde{\mathfrak{z}}' - \mathfrak{z}'}{\widetilde{\mathfrak{z}}_{d+1} - \mathfrak{z}_{d+1}}, \varphi_{z'}'(\mathfrak{z}_s,\xi) \right\rangle \right] ds$$

Note that $\frac{\varphi(\tilde{\mathfrak{z}},\xi)-\varphi(\mathfrak{z},\xi)}{\tilde{\mathfrak{z}}_{d+1}-\tilde{\mathfrak{z}}_{d+1}}$ is a small perturbation of $\varphi'_{z_{d+1}}(0,\xi)$ if ε is sufficiently small. We apply the method of stationary phase (with parameters, [8]) in the ξ' -variables, using (5). This yields

$$|\langle S_{\mathfrak{z}}, S_{\widetilde{\mathfrak{z}}}
angle| \lesssim rac{2^{kd}}{(1+2^k |\widetilde{\mathfrak{z}}_{d+1}-\mathfrak{z}_{d+1}|)^{\ell/2}}, \quad ext{ if } |\mathfrak{z}'-\widetilde{\mathfrak{z}}'| \leq C_2 arepsilon |\mathfrak{z}_{d+1}-\widetilde{\mathfrak{z}}_{d+1}|,$$

and combining this with (17) we get

(18)
$$\left| \langle S_{\mathfrak{z}}, S_{\tilde{\mathfrak{z}}} \rangle \right| \lesssim \frac{2^{kd}}{(1 + 2^k |\tilde{\mathfrak{z}} - \mathfrak{z}|)^{\ell/2}}$$

whenever $|\mathfrak{z} - \widetilde{\mathfrak{z}}| = O(\varepsilon^2)$.

2.3. **Proof of Proposition 2.2.** If $\alpha \leq 2^{k\frac{d+1}{2}}$ then the desired inequality follows from (12). Indeed by Tshebyshev's inequality the left hand side of (9) is $\lesssim \alpha^{-1} 2^{k(d-1)/2} \# \mathcal{E}$ which is dominated by the right hand side of (9) if $\alpha \leq 2^{k\frac{d+1}{2}}$.

In what follows we shall therefore assume that $\alpha > 2^{k\frac{d+1}{2}}$ and set

(19)
$$u_k(\alpha) := \left(\alpha 2^{-k\frac{d+1}{2}}\right)^{p_\ell} > 1.$$

The argument is a variant of one in [6]; it is based on a Calderón-Zygmund type decomposition at height $u_k(\alpha)$ where volume is replaced by diameter.

By the usual Vitali procedure there is a finite (possibly empty) family \mathfrak{B}^k of disjoint balls so that

(20)
$$u_k(\alpha)2^k \operatorname{diam}(B) \le \#(\mathcal{E} \cap B) \quad \text{for } B \in \mathfrak{B}^k;$$

moreover if we remove the balls in \mathfrak{B}^k and set

$$\mathcal{E}_* = \mathcal{E} \setminus \bigcup_{B \in \mathfrak{B}^k} B,$$

then

(21)
$$\#(\mathcal{E}_* \cap B) \le C_d u_k(\alpha) 2^k \operatorname{diam}(B)$$
 for every ball B .

Since $\mathcal{E} \subset \mathcal{Z}_k$ which is 2^{-k} -separated, we may assume that diam $(B) \ge 2^{-k}$ if $B \in \mathfrak{B}^k$. We need to establish the following two inequalities:

(22)
$$\operatorname{meas}\left(\left\{y \in \mathbb{R}^{d} : \left|\sum_{B \in \mathfrak{B}^{k}} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}}\right| > \alpha/2\right\}\right) \leq C2^{k(\frac{d+1}{2}p_{\ell}-1)} \alpha^{-p_{\ell}} \# \mathcal{E},$$

(23)
$$\operatorname{meas}\left(\left\{y \in \mathbb{R}^d : \left|\sum_{\mathfrak{z} \in \mathcal{E}_*} S_{\mathfrak{z}}\right| > \alpha/2\right\}\right) \le C 2^{k(\frac{d+1}{2}p_{\ell}-1)} \alpha^{-p_{\ell}} \# \mathcal{E}$$

Proof of (22). We first form an exceptional set as follows. Let z_B denote the center of a ball $B \in \mathfrak{B}^k$ and let $R_B = 10d2^k \operatorname{diam}(B) \gtrsim 1$. Let $\Theta(k, B)$ be a maximal $C_1(2^{-k}R_B)^{1/2}$ separated subset of S^{d-1} . Here C_1 is the constant in (14). Define (using the notation in (13))

(24)
$$\mathcal{V}^{k} = \bigcup_{B \in \mathfrak{B}^{k}} \bigcup_{\vartheta \in \Theta(k,B)} V^{k}_{\vartheta}(z_{B}, R_{B}).$$

Observe that meas($V_{\vartheta}^{k}(z_{B}, R_{B})$) is $O((R_{B}2^{-k})^{(d+1)/2})$ and $\#\Theta(k, B) = O((2^{k}R_{B}^{-1})^{(d-1)/2})$. Thus

$$\operatorname{meas}(\mathcal{V}^{k}) \lesssim \sum_{B \in \mathfrak{B}^{k}} \sum_{\vartheta \in \Theta(k,B)} \operatorname{meas}(V_{\vartheta}^{k}(z_{B}, R_{B})) \lesssim \sum_{B \in \mathfrak{B}^{k}} R_{B} 2^{-k}$$
$$\lesssim \sum_{B \in \mathfrak{B}^{k}} \operatorname{diam}(B) \lesssim \sum_{B \in \mathfrak{B}^{k}} 2^{-k} \frac{\#(\mathcal{E} \cap B)}{u_{k}(\alpha)} \lesssim 2^{k(-1 + \frac{d+1}{2}p_{\ell})} \alpha^{-p_{\ell}} \#\mathcal{E}$$

by the disjointness of the balls in \mathfrak{B}^k , (20), and the definition of $u_k(\alpha)$.

To conclude the proof of (22) we have to estimate the contribution in the complement of \mathcal{V}^k . For this we bound

$$\operatorname{meas}\left(\left\{y \in \mathbb{R}^{d} \setminus \mathcal{V}^{k} : \left|\sum_{B \in \mathfrak{B}^{k}} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}}\right| > \frac{\alpha}{2}\right\}\right) \lesssim \alpha^{-1} \left\|\sum_{B \in \mathfrak{B}^{k}} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}}\right\|_{L^{1}(\mathbb{R}^{d} \setminus \mathcal{V}^{k})}.$$

Now fix *B*. For every $\theta \in \Theta_k$ we may choose a $\vartheta = \vartheta_B(\theta) \in \Theta(k, B)$ so that $|\vartheta_B(\theta) - \theta| \le C_1(R_B 2^{-k})^{-1/2}$. Recalling $S_{\mathfrak{z}} = \sum_{\theta \in \Theta_k} S_{\mathfrak{z},\theta}$, we see

$$\begin{split} & \Big\| \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}} \Big\|_{L^1(\mathbb{R}^d \setminus \mathcal{V}^k)} \\ & \leq \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} \sum_{\theta \in \Theta_k} \| S_{\mathfrak{z}, \theta} \|_{L^1(\mathbb{R}^d \setminus V_{\vartheta_B(\theta)}^k(z_B, R_B))} \\ & \lesssim \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} \sum_{\theta \in \Theta_k} R_B^{-4d} \lesssim \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} 2^{k(d-1)/2} R_B^{-4d}. \end{split}$$

For the second inequality we use (14) and the last one follows from $\#\Theta_k = O(2^{k(d-1)/2})$. Now we note that $2^{k\frac{d-1}{2}}\alpha^{-1} = 2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}u_k(\alpha)^{1-\frac{1}{p_\ell}}$. Thus

$$\max\left(\left\{y \in \mathbb{R}^d \setminus \mathcal{V}^k : \left|\sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}}\right| > \frac{\alpha}{2}\right\}\right) \lesssim \alpha^{-1} \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} 2^{k(d-1)/2} R_B^{-4d}$$
$$\lesssim 2^{k(-1+\frac{d+1}{2}p_\ell)} \alpha^{-p_\ell} u_k(\alpha)^{1-\frac{1}{p_\ell}} \sum_{B \in \mathfrak{B}^k} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} R_B^{-4d}.$$

By (20) we have for $B \in \mathfrak{B}^k$

$$u_k(\alpha) \lesssim \frac{\#(\mathcal{E} \cap B)}{R_B} \lesssim R_B^d$$

and therefore

$$\max\left(\left\{y \in \mathbb{R}^{d} \setminus \mathcal{V}^{k} : \left|\sum_{B \in \mathfrak{B}^{k}} \sum_{\mathfrak{z} \in \mathcal{E} \cap B} S_{\mathfrak{z}}\right| > \frac{\alpha}{2}\right\}\right)$$
$$\lesssim 2^{k(-1+\frac{d+1}{2}p_{\ell})} \alpha^{-p_{\ell}} u_{k}(\alpha)^{1-\frac{1}{p_{\ell}}-4} \sum_{B \in \mathfrak{B}^{k}} \#(\mathcal{E} \cap B) \lesssim 2^{k(-1+\frac{d+1}{2}p_{\ell})} \alpha^{-p_{\ell}} \#\mathcal{E}$$

since $u_k(\alpha) \ge 1$ and $R_B \gtrsim 1$.

Proof of (23). We check from (19) that

$$2^{kd}u_k(\alpha)^{2/\ell}\alpha^{-2} = 2^{k(-1+\frac{d+1}{2}p_\ell)}\alpha^{-p_\ell}.$$

Thus by Tshebyshev's inequality it suffices to prove

(25)
$$\left\|\sum_{\mathfrak{z}\in\mathcal{E}_*}S_{\mathfrak{z}}\right\|_2^2 \lesssim 2^{kd}u_k(\alpha)^{2/\ell}\#\mathcal{E}_*.$$

We set

$$\begin{split} L &:= u_k(\alpha)^{2/\ell},\\ I(n,L) &:= [n2^{-k}L, (n+1)2^{-k}L),\\ \mathcal{E}(n,L) &:= \{\mathfrak{z} \in \mathcal{E}_* : \mathfrak{z}_{d+1} \in I(n,L)\},\\ \mathfrak{S}_n &:= \sum_{\mathfrak{z} \in \mathcal{E}(n,L)} S_{\mathfrak{z}} \,. \end{split}$$

Now

$$\left\|\sum_{\mathfrak{z}\in\mathcal{E}_*}S_{\mathfrak{z}}\right\|_2^2 \lesssim \sum_n \sum_{\tilde{n}:|n-\tilde{n}|\leq 4} \langle\mathfrak{S}_n,\mathfrak{S}_{\tilde{n}}\rangle + \sum_n \sum_{\tilde{n}:|n-\tilde{n}|>4} \langle\mathfrak{S}_n,\mathfrak{S}_{\tilde{n}}\rangle =: I + II.$$

For I we use the Schwarz inequality and then (17) to get

$$\begin{split} |I| \lesssim &\sum_{n} \|\mathfrak{S}_{n}\|_{2}^{2} = \sum_{n} \left\|\sum_{\mathfrak{z}_{d+1} \in I(n,L)} \sum_{\mathfrak{z}'} S_{(\mathfrak{z}',\mathfrak{z}_{d+1})}\right\|_{2}^{2} \\ \lesssim &L \sum_{n} \sum_{\mathfrak{z}_{d+1} \in I(n,L)} \left\|\sum_{\substack{\mathfrak{z}':\\ (\mathfrak{z}',\mathfrak{z}_{d+1}) \in \mathcal{E}_{*}}} S_{(\mathfrak{z}',\mathfrak{z}_{d+1})}\right\|_{2}^{2} \\ \lesssim &L \sum_{n} \sum_{\mathfrak{z}_{d+1} \in I(n,L)} \sum_{\substack{\mathfrak{z}':\\ (\mathfrak{z}',\mathfrak{z}_{d+1}) \in \mathcal{E}}} \sum_{\widetilde{\mathfrak{z}'}} 2^{kd} (1+2^{k}|\mathfrak{z}'-\widetilde{\mathfrak{z}}'|)^{-8d} \lesssim L2^{kd} \# \mathcal{E} \,. \end{split}$$

For II we use (18) and estimate

$$|II| \lesssim \sum_{\mathfrak{z} \in \mathcal{E}_*} \sum_{\substack{\widetilde{\mathfrak{z}} \in \mathcal{E}_* \\ |\mathfrak{z}_{d+1} - \widetilde{\mathfrak{z}}_{d+1}| \ge 2^{-k}L}} |\langle S_{\mathfrak{z}}, S_{\widetilde{\mathfrak{z}}} \rangle| \lesssim \sum_{\mathfrak{z} \in \mathcal{E}_*} \sum_{\substack{\widetilde{\mathfrak{z}} \in \mathcal{E}_* : \\ |\mathfrak{z}_{d+1} - \widetilde{\mathfrak{z}}_{d+1}| \ge 2^{-k}L}} \frac{2^{kd}}{(1 + 2^k |\mathfrak{z} - \widetilde{\mathfrak{z}}|)^{\ell/2}} \, .$$

By (21) we have for $R \ge 1$ and fixed \mathfrak{z}

$$\sum_{\substack{\widetilde{\mathfrak{z}}\in\mathcal{E}_*\\2^{-k}R\leq|\mathfrak{z}-\widetilde{\mathfrak{z}}|\leq 2^{1-k}R}} (1+2^k|\mathfrak{z}-\widetilde{\mathfrak{z}}|)^{-\ell/2} \lesssim u_k(\alpha)R^{1-\frac{\ell}{2}}$$

and since $\ell/2 > 1$ we get (after setting $R = 2^m$ and summing over m with $2^m \ge L$)

$$|II| \lesssim 2^{kd} u_k(\alpha) L^{1-\frac{\ell}{2}} \# \mathcal{E}.$$

Hence

$$I + II \lesssim 2^{kd} \# \mathcal{E} \left(L + u_k(\alpha) L^{1 - \frac{\ell}{2}} \right)$$

and with the optimal choice of $L = [u_k(\alpha)]^{2/\ell}$ we obtain (25).

2.4. Proof that Proposition 2.2 implies (6). We shall first assume that in (2)

(26)
$$\chi_k(z, y, 2^{-k}\xi) = \eta_k(z, 2^{-k}\xi)\chi_{\circ}(y)$$

where $\chi_{\circ} \in C_c^{\infty}(\mathbb{R}^d)$ is supported on a small neighborhood of the origin but so that $\chi_{\circ}(y) = 1$ for $y \in E$; moreover η_k is compactly supported in a set of diameter $O(\varepsilon^2)$ near $(z,\xi) = (0, e_1)$, and the derivatives of η_k up to order 10*d* are uniformly bounded.

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Let $Q_{\mathfrak{z}} = \prod_{i=1}^{d+1} [\mathfrak{z}_i, \mathfrak{z}_i + 2^{-k}]$. For $m \ge 0$ let $\mathcal{E}_m = \{\mathfrak{z} \in \mathcal{Z}_k : 2^{-k(d+1)-m-1} < |Q_{\mathfrak{z}} \cap E| \le 2^{-k(d+1)-m}\},$ (27) $E_m = \bigcup_{\mathfrak{z} \in \mathcal{E}_m} Q_{\mathfrak{z}} \cap E.$

And we also set

$$\begin{aligned} a_{k,\mathfrak{z},m}(\xi) &= 2^{m+(k+1)d} \int_{Q_{\mathfrak{z}}\cap E_m} \eta_k(z, 2^{-k}\xi) e^{i(\varphi(\mathfrak{z},\xi)-\varphi(z,\xi))} dz \,, \\ S_{\mathfrak{z},m}(y) &= \int a_{k,\mathfrak{z},m}(\xi) e^{i(\langle y,\xi\rangle-\varphi(\mathfrak{z},\xi))} d\xi \,. \end{aligned}$$

Then it follows that

$$T_k^* \chi_E(y) = \sum_{m=0}^{\infty} 2^{-m - (k+1)d} S_{\mathfrak{z},m}(y).$$

Since $\partial_{\xi}^{\alpha}(\varphi(\mathfrak{z},\xi)-\varphi(z,\xi)) = O(2^{-k})$ for any multiindex α it is easy to see that $a_{\cdot,\cdot,m}$ satisfies (7) uniformly in m. Hence, the result of Proposition 2.2 can be applied to $\sum_{\mathfrak{z}\in\mathcal{E}_m} S_{\mathfrak{z},m}(y)$ and we get uniform bounds. Thus

$$\left\|\sum_{\mathfrak{z}\in\mathcal{E}_m} S_{\mathfrak{z},m}\right\|_{L^{p_{\ell},\infty}} \lesssim 2^{k(\frac{d+1}{2}-\frac{1}{p_{\ell}})} (\#\mathcal{E}_m)^{1/p_{\ell}} \lesssim 2^{k(\frac{d+1}{2}-\frac{1}{p_{\ell}})} 2^{m/p_{\ell}} \left(2^{k(d+1)} \operatorname{meas}(E_m)\right)^{1/p_{\ell}}$$

with implicit constants independent of m. For the second inequality we use

$$\#\mathcal{E}_m \lesssim 2^m 2^{k(d+1)} \mathrm{meas}(E_m)$$

which follows from (27). Consequently we get

$$\begin{split} \|T_k^*\chi_E\|_{L^{p_{\ell},\infty}} &\lesssim \sum_{m=0}^{\infty} 2^{-m-(k+1)d} \|S_{\mathfrak{z},m}\|_{L^{p_{\ell},\infty}} \\ &\lesssim \sum_{m=0}^{\infty} 2^{-m(1-\frac{1}{p_{\ell}})} 2^{k(\frac{d}{p_{\ell}}-\frac{d+1}{2})} (\operatorname{meas}(E_m))^{1/p_{\ell}} \\ &\lesssim 2^{k(\frac{d}{p_{\ell}}-\frac{d+1}{2})} (\operatorname{meas}(E))^{1/p_{\ell}} \end{split}$$

which is the desired estimate.

Finally we have to remove the assumption (26). Here one uses Fourier series in y and expands $\chi_k(z, y, 2^{-k}\xi) = \sum_{\nu \in \mathbb{Z}^d} c_{k,\nu} \eta_{k,\nu}(z, 2^{-k}\xi) e^{i\langle y,\nu \rangle}$ where the functions $\eta_{k,\nu}(z, 2^{-k}\xi)$ are as before but now with a bound that decays fast in ν . We note that multiplication with $e^{i\langle y,\nu \rangle}$ does not affect the $L^{p_{\ell},1}$ norm, apply the previous bounds to the summands and sum in ν using the rapid decay in ν .

3. Combining the frequency localized pieces

We now combine the previous estimates on the operators T_k and prove the following result in Triebel-Lizorkin spaces $F_{a,s}^q$. Recall ([19]) that if $\{L_k\}_{k=0}^{\infty}$ is a standard inhomogeneous dyadic frequency decomposition then the norm $||f||_{F_{a,s}^q}$ can be defined as the $L^q(\ell^s)$ norm of the sequence $\{2^{ka}L_kf\}$. In view of the embeddings $L^q = F_{0,2}^q \subset F_{0,q}^q$ for $q \ge 2$, $L^q \supset F_{0,r}^q$ for $r \le 2$ the following result sharpens Theorem 1.1. For the case $r \ge 1$ one could argue by duality and follow [6] but we shall rely on a result in [15] which gives an estimate for all r > 0. **Theorem 3.1.** Let $\ell \geq 3$, let \mathcal{Z} , \mathcal{Y} be coordinate patches in \mathbb{R}^{d+1} , \mathbb{R}^d , resp., let $\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z},\mathcal{Y};\mathbb{C})$, with Schwartz kernel compactly supported in $\mathcal{Z} \times \mathcal{Y}$, and let \mathbb{C} satisfy hypothesis $\mathcal{H}(\ell)$. Suppose $\frac{2\ell}{\ell-2} < q < \infty$, $a = \mu + b + d(1/2 - 1/q) - 1/2$ and r > 0. Then \mathcal{F} maps $F_{a,q}^q(\mathbb{R}^d)$ boundedly to $F_{b,r}^q(\mathbb{R}^{d+1})$.

We state (a slight variant of) the result from [15]. In this setting one is given operators \mathcal{T}_k defined on the Schwartz space $\mathcal{S}(\mathbb{R}^{d_1})$,

$$\mathcal{T}_k f(z) = \int \mathcal{K}_k(z, y) f(y) \, dy, \quad z \in \mathbb{R}^{d_2}$$

and each \mathcal{K}_k is continuous and bounded. Let $\zeta \in \mathcal{S}(\mathbb{R}^{d_2})$ and $\zeta_k(z) = 2^{kd_2}\zeta(2^k z)$, and define $P_k g = \zeta_k * g$.

Theorem 3.2 ([15]). Let $d_1 \le d_2$, $0 < \gamma < d_2$, $\varepsilon > 0$, $1 < q_0 < q < \infty$, and assume

(28)
$$\sup_{k>0} 2^{k\gamma/q_0} \|\mathcal{T}_k\|_{L^{q_0}(\mathbb{R}^{d_1}) \to L^{q_0}(\mathbb{R}^{d_2})} < \infty$$

(29)
$$\sup_{k>0} \|\mathcal{T}_k\|_{L^{\infty}(\mathbb{R}^{d_1}) \to L^{\infty}(\mathbb{R}^{d_2})} < \infty.$$

Furthermore assume that for each cube Q there is a measurable set $\mathcal{W}_Q \subset \mathbb{R}^{d_2}$ so that

(30)
$$\operatorname{meas}(\mathcal{W}_Q) \le C \max\{|Q|^{1-\gamma/d_2}, |Q|\}$$

and there is $\delta > 0$ such that for every $k \in \mathbb{N}$ and every cube Q with $2^k \operatorname{diam}(Q) \geq 1$

(31)
$$\sup_{x \in Q} \int_{\mathbb{R}^{d_2} \setminus \mathcal{W}_Q} |\mathcal{K}_k(x, y)| dy \le C \max\{(2^k \operatorname{diam}(Q))^{-\delta}, 2^{-k\delta}\}.$$

Then for $q_0 < q < \infty$, r > 0

$$\left\| \left(\sum_{k} 2^{k\gamma r/q} |P_k \mathcal{T}_k f_k|^r \right)^{1/r} \right\|_q \lesssim \left\| \left(\sum_{k} \|f_k\|_q^q \right)^{1/q} \right\|_q$$

This (or a slightly sharper version) was formulated in [15] only for the case $d_1 = d_2$, but the result there implies the version cited above. Indeed if $d_1 < d_2$ we can define an operator $\widetilde{\mathcal{T}}_k$ on functions F on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2-d_1}$ by

$$\widetilde{\mathcal{T}}_k F(z) = \iint K_k(z, y) \chi(w) F(y, w) dy dw$$

where χ is a nontrivial $C_c^{\infty}(\mathbb{R}^{d_2-d_1})$ function. The assumptions on \mathcal{T}_k imply the corresponding assumptions on $\tilde{\mathcal{T}}_k$, by Minkowski's and Hölder's inequalities. Thus the equidimensional case in [15] may be applied and if in the conclusion we specialize to tensor products, $F(y, w) = f(y)\chi_1(w)$ we get the above generalization.

In order to prepare for our application of Theorem 3.2 we let $T_k f(z) = \int K_k(z, y) f(y) dy$ with K_k as in (2). Let β_0 be a C^{∞} -function supported in $\{\eta \in \mathbb{R}^{d+1} : |\eta| \leq 3/2\}$ so that $\beta_0(\eta) = 1$ for $|\eta| \leq 1$; and, for $k \geq 1$ let $\beta_k(\eta) = \beta_0(2^{-k}(\eta)) - \beta_0(2^{1-k}(\eta))$. Define L_k on functions on \mathbb{R}^{d+1} by $\widehat{L_kg} = \beta_k(\eta)\widehat{g}(\eta)$. We use calculations in [8] and first observe that there is a constant $A_0 > 1$ so that

(32)
$$||L_{\widetilde{k}}T_k||_{L^q \to L^q} \le C_N \min\{2^{-kN}, 2^{-\widetilde{k}N}\} \quad \text{if } |k - \widetilde{k}| \ge A_0.$$

This follows from the assumption that C does not meet the zero sections, which, by homogeneity implies that $c_1|\xi| \leq |\varphi'_z(z,\xi)| \leq C_1|\xi|$. The kernel $\mathcal{K}_{k\tilde{k}}$ of $L_{\tilde{k}}T_k$ is given by

$$\mathcal{K}_{k\widetilde{k}}(z,y) = \iiint \beta_{\widetilde{k}}\eta)\chi_k(z,y,2^{-k}\xi)e^{i(\langle z-w,\eta\rangle + \varphi(w,\xi) - \langle y,\xi\rangle)}dw\,d\eta\,d\xi$$

and if $k - \tilde{k}$ are sufficiently large then $|\eta + \varphi'_w(w,\xi)| \approx \max\{|\xi|, |\eta|\}$ on the support of the amplitude. Thus in this case we may use an integration by parts in w (followed by an integration by parts in ξ when z is large) to show that the kernels $\mathcal{K}_{k\tilde{k}}$ of $L_{\tilde{k}}T_k$ satisfy the estimate

$$|\mathcal{K}_{k\tilde{k}}(z,y)| \le C_N 2^{-kN} (1+|z|)^{-N}$$

and vanish for y in the complement of a fixed compact set. Thus (32) follows. Similarly if $\{\mathcal{L}_k\}_{k=0}^{\infty}$ is the corresponding frequency decomposition in \mathbb{R}^d we also see that $T_k \mathcal{L}_{\widetilde{k}}$ has $L^q(\mathbb{R}^d) \to L^q(\mathbb{R}^{d+1})$ operator norm $\leq C_N \min\{2^{-kN}, 2^{-\widetilde{k}N}\}$ if $|k - \widetilde{k}| > 4$.

From these preliminary remarks it follows quickly that for the proof of Theorem 3.1 it suffices to prove the inequalities

$$\left\| \left(\sum_{k} 2^{kbr} \left| 2^{k\mu} L_{k+i_1} T_k \mathcal{L}_{k+i_2} g \right|^r \right)^{1/r} \right\|_q \lesssim \left\| \left(\sum_{k} 2^{kaq} \left\| \mathcal{L}_{k+i_2} g \right\|_q^q \right)^{1/q} \right\|_q, \ |i_1| \le A_0, \ |i_2| \le 4$$

with $a = \mu + b + d(1/2 - 1/q) - 1/2$. Setting $\mathcal{T}_k = 2^{-k\frac{d-1}{2}}T_k$ and $f_k = 2^{ka}\mathcal{L}_{k+i_2}g$, the preceding inequality follows from

(33)
$$\left\| \left(\sum_{k} 2^{k \frac{d}{q} r} \left| L_{k+i_1} \mathcal{T}_k f_k \right|^r \right)^{1/r} \right\|_q \lesssim \left\| \left(\sum_{k} \|f_k\|_q^q \right)^{1/q} \right\|_q, \quad |i_1| \le A_0,$$

for $q_0 < q < \infty$, where $q_0 > \frac{2\ell}{\ell-2}$. This in turn follows from an application of Theorem 3.2 with $d_1 = d$, $d_2 = d + 1$, $\gamma = d$, and $P_k = L_{k+i_1}$. The hypothesis (29) follows from the calculations in [16] (*cf.* also §2.1 above). The hypothesis (28) for $\frac{2\ell}{\ell-2} < q < \infty$ follows from Theorem 2.1 and (29) by interpolation.

If diam $(Q) > \varepsilon$ then \mathcal{W}_Q is simply an expanded cube and (31) follows by the support assumption of K_k . If diam $(Q) < \varepsilon$ the exceptional sets are formed as in §2.1. For $\theta \in S^{d-1}$ we set

$$W_{\theta}(z_Q, C) = \left\{ y : |\langle \varphi'_{\xi}(z_Q, \theta) - y, \theta \rangle| \le C \operatorname{diam}(Q), \ |\Pi_{\theta^{\perp}}(\varphi'_{\xi}(z, \theta) - y)| \le C (\operatorname{diam}(Q))^{1/2} \right\}$$

and if Θ_Q is a maximal set of $(\operatorname{diam}(Q))^{1/2}$ separated unit vectors we set

$$\mathcal{W}_Q = \bigcup_{\theta \in \Theta_Q} W_{\theta}(z_Q, C).$$

Then the measure of \mathcal{W}_Q is $O(\operatorname{diam}(Q)) = O(|Q|^{1-\frac{d}{d+1}})$ so that (30) holds. The hypothesis (31) (even with large δ) holds by the calculations in §2.1 (cf. (14)).

4. Remarks on the constant coefficient case

We now let ρ be a $C^{\infty}(\mathbb{R}^d \setminus \{0\})$ function which is homogeneous of degree 1, so that $\rho(\xi) \neq 0$ for $\xi \neq 0$. We are interested in space time estimates for $U_t \equiv e^{-it\rho(D)}$ defined by

$$\widehat{U_t f}(\xi) = e^{it\rho(\xi)} \widehat{f}(\xi)$$

and obtain a result under a decay assumption for the Fourier transform of surface carried measure on

$$\Sigma_{\rho} = \{\xi : \rho(\xi) = 1\}.$$

Theorem 4.1. Let $\kappa > 1$ and let ρ be as above such that the surface measure $d\sigma$ of Σ_{ρ} satisfies

(34)
$$\sup_{\epsilon} (1+|\xi|)^{\kappa} |\widehat{d\sigma}(\xi)| < \infty.$$

Let $\frac{2\kappa}{\kappa-1} < q < \infty$ and let I be a compact time interval. There is C > 0 such that

$$\left(\int_{I} \left\| e^{it\rho(D)} f \right\|_{L^{q}(\mathbb{R}^{d})}^{q} dt \right)^{1/q} \leq C \|f\|_{B^{q}_{\alpha,q}(\mathbb{R}^{d})}, \quad \alpha = \frac{d-1}{2} - \frac{d}{q}$$

for all $f \in B^q_{\alpha,q}(\mathbb{R}^d)$.

Remark. Under the assumption that Σ_{ρ} has nonvanishing curvature everywhere this follows from Theorem 3.1 above. We note that in this particular case a weaker result with $\alpha > (d-1)/2 - d/q$ is already in [7].

Sketch of proof. We will assume that I = [-1, 1], as one can use rescaling to reduce to this case. Fix k and define $S_{x,t} \equiv S_{x,t}^k$ by

$$S_{x,t}(y) = \int e^{i\langle x-y,\xi\rangle + it\rho(\xi)} \chi(2^{-k}\xi) \, d\xi$$

and as before we may assume that the support of χ has diameter $\leq \varepsilon^2$, for sufficiently small $\varepsilon > 0$, and is contained in $\{1/2 < |\xi| \leq 2\}$. Fix $\xi_{\circ} \in \Sigma_{\rho}$ and $\rho_{\circ} > 0$ so that $\rho_{\circ}\xi_{\circ} \in \text{supp}(\chi)$. Let $u \mapsto \Xi(u)$ be a parametrization of Σ_{ρ} near ξ_{\circ} with parameter $u \in \mathbb{R}^{d-1}$ near u_{\circ} , and $\Xi(u_{\circ}) = \xi_{\circ}$. Let \mathfrak{n}_{\circ} be the outer normal unit vector to Σ_{ρ} at ξ_{\circ} . Let Γ be the cone formed by the $(\rho \Xi(u), \rho)$ with $\rho > 0$ and u near u_{\circ} . Let $\vec{N}_{\circ} = \mathfrak{n}_{\circ} - \langle \xi_{\circ}, \mathfrak{n}_{\circ} \rangle \vec{e}_{d+1}$, which is a normal vector to Γ at $(\rho \Xi(u_{\circ}), \rho)$. By finite decompositions of χ we may further assume that

$$S_{x,t}(y) = \int \beta(2^{-k}\xi) e^{i(\langle x-y,\xi\rangle + it\rho(\xi))} d\xi$$

where $\beta \in C_c^{\infty}$ supported in an ε^2 -neighborhood of $\rho_{\circ} \Xi(u_{\circ})$.

The proof of Theorem 4.1 is a straightforward variant of the proof of Theorem 3.1 once we have established the two appropriate replacements for the scalar product bounds (18) and (17), namely

(35)
$$|\langle S_{x,t}, S_{\tilde{x},\tilde{t}}\rangle| \lesssim 2^{kd} (1+2^k |x-\tilde{x}|+2^k |t-\tilde{t}|)^{-\kappa}$$

and a better estimate when $(x - \tilde{x}, t - \tilde{t})$ is orthogonal (or near orthogonal) to $\vec{N_o}$:

$$(36) \quad |\langle S_{x,t}, S_{\tilde{x},\tilde{t}} \rangle| \leq C_M 2^{kd} (1+2^k |x-\tilde{x}|+2^k |t-\tilde{t}|)^{-M}$$

if $|\langle x-\tilde{x}, \mathfrak{n}_{\circ} \rangle - (t-\tilde{t}) \langle \xi_{\circ}, \mathfrak{n}_{\circ} \rangle| \leq \epsilon_{\circ} |(x-\tilde{x}, t-\tilde{t})|$

for a small $\epsilon_{\circ} > 0$.

As before $(2\pi)^d \langle S_{x,t}, S_{\tilde{x},\tilde{t}} \rangle = \langle \widehat{S}_{x,t}, \widehat{S}_{\tilde{x},\tilde{t}} \rangle$. We scale and then use generalized polar coordinates $\xi = \rho \Xi(u)$ to write

(37)
$$\langle \widehat{S}_{x,t}, \widehat{S}_{\tilde{x},\tilde{t}} \rangle = 2^{kd} \iint b(\rho, u) e^{i2^k \rho(\langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t})} du \, d\rho$$

where b is smooth and supported near $(\rho_{\circ}, u_{\circ})$. Now for any $\chi \in C_c^{\infty}$ we have $|\chi d\sigma(\xi)| \leq (1 + |\xi|)^{\kappa}$, by assumption (34). We apply this to the inner integral of (37) and obtain

(38)
$$|\langle S_{x,t}, S_{\tilde{x},\tilde{t}}\rangle| \lesssim 2^{kd} (1+2^k |x-\tilde{x}|)^{-\kappa}.$$

Let $B = 2 \max\{|\xi| : \xi \in \Sigma_{\rho}\}$. By an integration by parts in ρ (after interchanging the order of integration in (37)) we obtain

(39)
$$|\langle S_{x,t}, S_{\tilde{x},\tilde{t}}\rangle| \le C_N 2^{kd} (1+2^k |t-\tilde{t}|)^{-N} \text{ if } |t-\tilde{t}| \ge B|x-\tilde{x}|,$$

and (35) follows from (38) and (39).

We now prove (36) and in view of (39) we may assume $|t - \tilde{t}| < B|x - \tilde{x}|$. We distinguish two cases. In the first case we assume $|\langle \frac{x-\tilde{x}}{|x-\tilde{x}|}, \mathfrak{n}_{\circ} \rangle| \leq 1 - \varepsilon$. We note that $|\Xi(u) - \Xi(u_{\circ})| = O(\varepsilon^2)$ and $|(\nabla_u \langle x - \tilde{x}, \Xi(u) \rangle)_{u=u_{\circ}}| \sim |\Pi_{\mathfrak{n}_{\circ}^{\perp}}(x - \tilde{x})|$. Hence we have $|\nabla_u \langle x - \tilde{x}, \Xi(u) \rangle| \geq c\varepsilon |x - \tilde{x}|$ on the support of *b* provided that ε is sufficiently small, and higher derivatives of $\langle x - \tilde{x}, \Xi(u) \rangle$ are $O(|x - \tilde{x}|)$. Thus, integrating by parts in the inner *u*-integral in (37), we get (36). We now consider the second case $|\langle \frac{x-\tilde{x}}{|x-\tilde{x}|}, \mathfrak{n}_{\circ} \rangle| \geq 1 - \varepsilon$. This means that $\frac{x-\tilde{x}}{|x-\tilde{x}|} = s\mathfrak{n}_{\circ} + O(\varepsilon)$ where s = 1 or s = -1. From the condition on $(x - \tilde{x}, t - \tilde{t})$ in (36) we see that the ρ -derivative of the phase is

$$\begin{aligned} \langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t} &= \langle x - \tilde{x}, \xi_{\circ} \rangle + \frac{\langle x - \tilde{x}, \mathfrak{n}_{\circ} \rangle}{\langle \xi_{\circ}, \mathfrak{n}_{\circ} \rangle} + O((\varepsilon^{2} + \epsilon_{\circ})|x - \tilde{x}|) \\ &= s|x - \tilde{x}| \Big(\langle \xi_{\circ}, \mathfrak{n}_{\circ} \rangle + \frac{1}{\langle \xi_{\circ}, \mathfrak{n}_{\circ} \rangle} \Big) + O((\varepsilon + \epsilon_{\circ})||x - \tilde{x}|) \,. \end{aligned}$$

Now $|\langle \xi_{\circ}, \mathfrak{n}_{\circ} \rangle| \geq c > 0$ which is a consequence of the homogeneity relation $\rho(\xi) = \langle \xi, \nabla \rho(\xi) \rangle$. Hence, $|\langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t}| \gtrsim |x - \tilde{x}|$ if ε and ϵ_{\circ} are small enough, and another integration by parts in ρ gives (36).

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