# REMARKS ON THE MULTIPLIER OPERATORS ASSOCIATED WITH A CYLINDRICAL DISTANCE FUNCTION 

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#### Abstract

In this note, we consider $L^{p}$ and maximal $L^{p}$ estimates for the generalized Riesz means which are associated with the cylindrical distance function $\rho(\xi)=$ $\max \left\{\left|\xi^{\prime}\right|,\left|\xi_{d+1}\right|\right\},\left(\xi^{\prime}, \xi_{d+1}\right) \in \mathbb{R}^{d} \times \mathbb{R}$. We prove these estimates up to the currently known range of the spherical Bochner-Riesz and its maximal operators. This is done by establishing implications between the corresponding estimates for the spherical Bochner-Riesz and the cylindrical multiplier operators.


## 1. Introduction and statement of results

In this paper we consider multiplier operators associated with a rough distance function, which are known as the cylinder multiplier operators. More precisely, we define a distance function $\rho$ by

$$
\rho(\xi)=\max \left\{\left|\xi^{\prime}\right|,\left|\xi_{d+1}\right|\right\}, \xi=\left(\xi^{\prime}, \xi_{d+1}\right) \in \mathbb{R}^{d} \times \mathbb{R}
$$

The generalized Riesz means of order $\alpha \geq 0$ which are associated with $\rho$ is defined by

$$
\widehat{S_{t}^{\alpha} f}(\xi)=\left(1-\frac{\rho(\xi)}{t}\right)_{+}^{\alpha} \widehat{f}(\xi)
$$

Here $a_{+}^{\alpha}=a^{\alpha}$ for $a>0$ and $a_{+}^{\alpha}=0$ otherwise. In connection with the convergence of $S_{t} f \rightarrow f$ in $L^{p}$ as $t \rightarrow \infty$, the inequality

$$
\begin{equation*}
\left\|S_{1}^{\alpha} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \tag{1.1}
\end{equation*}
$$

has been studied by some authors [16, 22].
As it was shown in [22], $L^{p}$ boundedness of $S_{1}^{\alpha}$ is closely related to those of the spherical Bochner-Riesz and the cone multiplier operators. For $1 \leq p \leq \infty$, let

$$
\alpha(p)=\max \left\{d\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}, 0\right\}
$$

be the critical exponent for $L^{p}$ boundedness of Bochner-Riesz operator in $\mathbb{R}^{d}$ and the cone multiplier operator in $\mathbb{R}^{d+1}$. We now set

$$
\widehat{T_{t}^{\alpha}} g\left(\xi^{\prime}\right)=\left(1-\frac{\left|\xi^{\prime}\right|^{2}}{t^{2}}\right)_{+}^{\alpha} \widehat{g}\left(\xi^{\prime}\right), \quad \xi^{\prime} \in \mathbb{R}^{d}
$$

The conjecture which is known as Bochner-Riesz conjecture is that for $1 \leq p \leq \infty$

$$
\begin{equation*}
\left\|T_{1}^{\alpha} g\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq C\|g\|_{L^{p}\left(\mathbb{R}^{d}\right)} \tag{1.2}
\end{equation*}
$$

if and only if $\alpha>\alpha(p)$. When $d=2$, it was verified by Carleson and Sjölin [5]. In higher dimensions it is still open and some partial results are known. Indeed, $L^{p}$ boundedness on the range $(2 d+2) /(d-1) \leq p \leq \infty$ and $1 \leq p \leq(2 d+2) /(d+1)$ is due to the sharp

[^0]$L^{2}$ restriction estimate [23] and the argument of Stein (for example, see p.422-p.423, [18]). Beyond these results, progresses have been made (see [1, 2, 13, 21] and references therein). Most recent results are due to the third author [13] (also see [14]) when $d=3,4$, and Bourgain and Guth [3] when $d \geq 5$.

By de Leeuw's restriction theorem the boundedness of $S_{1}^{\alpha}$ on $L^{p}\left(\mathbb{R}^{d+1}\right)$ implies that of Bochner-Riesz operator of the same order on $L^{p}\left(\mathbb{R}^{d}\right)$. From the known necessary condition for (1.2), it follows that $S_{1}^{\alpha}$ is bounded on $L^{p}$ only if $\alpha>\alpha(p)$. When $d \geq 3$ there is an additional necessary condition that $\frac{2 d}{d+3}<p<\frac{2 d}{d-3}$. It is due to the fact that near the surface $\left|\xi^{\prime}\right|=\xi_{d+1}$, the multiplier $\left(1-\frac{\rho(\xi)}{t}\right)_{+}^{\alpha}$ behaves similarly as the cone multiplier of order 1. So it was conjectured ( $[16,22]$ ) that (1.1) holds if and only if $\alpha>\alpha(p)$ and $\frac{2 d}{d+3}<p<\frac{2 d}{d-3}$ when $d \geq 3$ and $1<p<\infty$ when $d=2$. In [22] the problem was settled when $d=2$, and some partial results were obtained when $d \geq 3$. For further progress in higher dimensions, one should improve boundedness of Bochner-Riesz operators. However, thanks to recent progress on the problem of the cone multiplier [7] (also see [8] and [15]), it is possible to show that $L^{p}$ boundedness of $S_{1}^{\alpha}$ is equivalent to that of $T_{1}^{\alpha}$.

Theorem 1.1. Let $1<p<\infty$ when $d=2$, and let $\frac{2 d}{d+3}<p<\frac{2 d}{d-3}$ when $d \geq 3$. (1.2) holds for $\alpha>\alpha(p)$ if and only if (1.1) holds for $\alpha>\alpha(p)$.

So this establishes $L^{p}$ boundedness of the cylinder multiplier operators up to the currently known range of Bochner-Riesz operators. That is to say, (1.1) holds if $p_{\circ} \leq p \leq \infty$, $1 \leq p \leq p_{\circ}^{\prime}$, and $\alpha>\alpha(p)$ where $p_{\circ}$ is given by

$$
p_{\circ}=2+\frac{12}{4 d-3-k} \quad \text { if } \quad d \equiv k(\bmod 3), k=-1,0,1
$$

Nextly we consider the maximal operator

$$
S_{*}^{\alpha} f(x)=\sup _{t>0}\left|S_{t}^{\alpha} f(x)\right| .
$$

In general, $L^{p}$ estimate for $S_{*}^{\alpha} f$ has been of interest in connection with almost everywhere convergence of $S_{t}^{\alpha} f$ as $t \rightarrow \infty$ and it is also an obvious extension of (1.1). The same problems for the maximal Bochner-Riesz operator $T_{*}^{\alpha} g\left(x^{\prime}\right)=\sup _{t>0}\left|T_{t}^{\alpha} g\left(x^{\prime}\right)\right|$ have been studied in $[4,6,13]$ and it is conjectured that for $2 \leq p \leq \infty$

$$
\begin{equation*}
\left\|T_{*}^{\alpha} g\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq C\|g\|_{L^{p}\left(\mathbb{R}^{d}\right)} \tag{1.3}
\end{equation*}
$$

holds if and only if $\alpha>\alpha(p)$. This was settled by Carbery [4] when $d=2$. Partial results are known $[6,13]$ when $d \geq 3$ so that the conjecture is verified for $p \geq 2+\frac{4}{d}$. It seems possible that recent progress [3] leads to further improvement. On the contrary, for $p<2$ Tao [19] showed that the $L^{p}$ boundedness of $T_{*}^{\alpha}$ is different from that of $T_{1}^{\alpha}$, and when $d=2$ he [20] also obtained some improvement upon the classical result [17].

It is natural to expect that the maximal estimate

$$
\begin{equation*}
\left\|S_{*}^{\alpha} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \tag{1.4}
\end{equation*}
$$

holds provided that $\alpha>\alpha(p)$, and $2 \leq p \leq \infty$ when $d=2$ and $2 \leq p<\frac{2 d}{d-3}$ when $d \geq 3$. For $0<p<1$ the boundedness of $S_{*}^{\alpha}$ from $H^{p}$ to $L^{p, \infty}$ was shown in [11]. But, as far as the authors know, nothing is known about (1.4) for $p \geq 1$. In what follows we shall show that the similar implication also holds for the maximal estimates.

Theorem 1.2. Let $2 \leq p \leq \infty$ when $d=2$ and let $2 \leq p<\frac{2 d}{d-3}$ when $d \geq 3$. If (1.3) holds for $\alpha>\alpha(p)$, then (1.4) holds for $\alpha>\alpha(p)$.

Hence, this establishes the boundedness of $S_{*}^{\alpha}$ up to that of currently known range of maximal Bochner-Riesz operator. So, (1.4) holds for $p>2+\frac{4}{d}$ (see [13]).

## 2. Preliminaries

In this section we present various preliminary estimates which will be used for the proof of theorems. We need to obtain the sharp estimates for the multiplier operators of which multipliers are essentially supported in a $\delta$-neighborhood of the sphere and the cone. They are crucial for the proof of theorems.

Specifically, let $\phi \in C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$ and for $0<\delta \ll 1$ we define

$$
\widehat{\mathcal{T}_{\delta}^{t}} g\left(\xi^{\prime}\right)=\phi\left(\delta^{-1}\left(1-\frac{\left|\xi^{\prime}\right|}{t}\right)\right) \widehat{g}\left(\xi^{\prime}\right), \quad \xi^{\prime} \in \mathbb{R}^{d}
$$

The sharp bounds for $\mathcal{T}_{\delta}^{1}$ can be deduced from (1.2).
Lemma 2.1. Let $2 \leq p \leq \infty$. Suppose $\left\|T_{1}^{\alpha} g\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq C\|g\|_{L^{p}\left(\mathbb{R}^{d}\right)}$ for $\alpha>\alpha(p)$. Then for $\epsilon>0$

$$
\begin{equation*}
\left\|\mathcal{T}_{\delta}^{1} g\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq C \delta^{-\alpha(p)-\epsilon}\|g\|_{L^{p}\left(\mathbb{R}^{d}\right)} . \tag{2.1}
\end{equation*}
$$

Here the constant $C$ remains uniform as long as $\|\phi\|_{C^{N}} \leq C$ for some large $N$.
This can be proven by making use of the standard Carleson-Sjölin-Hörmander reduction which involves asymptotic expansion of the kernel (see [19] for details). For the convenience of the reader we include a proof.

Proof. By the standard Carleson-Sjölin reduction and rescaling the assumption implies that

$$
\begin{equation*}
\left\|\int_{\mathbb{R}^{d}} e^{ \pm i \lambda\left|x^{\prime}-y\right|} a\left(\left|x^{\prime}-y\right|\right) f(y) d y\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq C \lambda^{-\frac{d}{p}+\epsilon}\|f\|_{L^{p}\left(\mathbb{R}^{d}\right)} \tag{2.2}
\end{equation*}
$$

whenever $a \in C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$ and the bound $C$ remains uniform as long as $a$ is uniformly contained in $C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$. By a simple argument with dyadic decomposition it can be further extended to any $a \in \mathcal{S}$ and it is also possible to replace $a$ with a singular function. In fact, let $a \in \mathcal{S}$ and $0 \leq \kappa<\frac{d}{2}$. Then (2.2) implies

$$
\begin{equation*}
\left\|\int e^{ \pm i \lambda\left|x^{\prime}-y\right|} a\left(\left|x^{\prime}-y\right|\right)\left|x^{\prime}-y\right|^{-\kappa} f(y) d y\right\|_{p} \leq C \lambda^{-\frac{d}{p}+\epsilon}\|f\|_{p} \tag{2.3}
\end{equation*}
$$

Let $\beta \in C_{c}^{\infty}([-7 / 8,-3 / 8] \cup[3 / 8,7 / 8])$ satisfying $\sum_{j=-\infty}^{\infty} \beta\left(2^{j} t\right)=1$ for $t \neq 0$. To show (2.3) we decompose

$$
a\left(\left|x^{\prime}-y\right|\right)\left|x^{\prime}-y\right|^{-\kappa}=a_{0}\left(\left|x^{\prime}-y\right|\right)+a_{1}\left(\left|x^{\prime}-y\right|\right)+a_{2}\left(\left|x^{\prime}-y\right|\right)
$$

where

$$
\begin{gathered}
a_{0}(t)=\sum_{2^{j} \leq \lambda^{-1}} \beta\left(2^{-j} t\right) a(t) t^{-\kappa}, \quad a_{1}(t)=\sum_{\lambda^{-1}<2^{j} \leq \lambda^{\epsilon}} \beta\left(2^{-j} t\right) a(t) t^{-\kappa} \\
a_{2}(t)=\sum_{\lambda^{\epsilon}<2^{j}} \beta\left(2^{-j} t\right) a(t) t^{-\kappa}
\end{gathered}
$$

It is easy to see $\left\|a_{0}(|\cdot|)\right\|_{1} \lesssim \lambda^{-\frac{d}{2}}$ and $\left\|a_{2}(|\cdot|)\right\|_{1} \lesssim \lambda^{-N}$ for any large $N$. By scaling and (2.2) it follows that if $\lambda^{-1}<2^{j} \leq 1$,

$$
\left\|\int e^{ \pm i \lambda\left|x^{\prime}-y\right|} a\left(\left|x^{\prime}-y\right|\right) \beta\left(2^{-j}\left|x^{\prime}-y\right|\right)\left|x^{\prime}-y\right|^{-\kappa} f(y) d y\right\|_{p} \leq C 2^{j\left(d-\kappa-\frac{d}{p}+\epsilon\right)} \lambda^{-\frac{d}{p}+\epsilon}\|f\|_{p}
$$

and if $1<2^{j} \leq \lambda^{\epsilon}$

$$
\left\|\int e^{ \pm i \lambda\left|x^{\prime}-y\right|} a\left(\left|x^{\prime}-y\right|\right) \beta\left(2^{-j}\left|x^{\prime}-y\right|\right)\left|x^{\prime}-y\right|^{-\kappa} f(y) d y\right\|_{p} \leq C 2^{-N j} \lambda^{-\frac{d}{p}+\epsilon}\|f\|_{p}
$$

For the second we use the rapid decay of $a$. Hence we get (2.3).
Now by rescaling we have

$$
\begin{equation*}
\left\|\int e^{ \pm i\left|x^{\prime}-y\right|} a\left(\frac{\left|x^{\prime}-y\right|}{\lambda}\right)\left(\frac{\left|x^{\prime}-y\right|}{\lambda}\right)^{-\kappa} f(y) d y\right\|_{p} \leq C \lambda^{d-\frac{d}{p}+\epsilon}\|f\|_{p} \tag{2.4}
\end{equation*}
$$

Set $\lambda=\delta^{-1}$. Now let us consider the kernel $K=\mathcal{F}^{-1}\left(\phi\left(\lambda\left(1-\left|\xi^{\prime}\right|\right)\right)\right)$ with $\phi \in C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$. Here $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$ denote the Fourier and the inverse Fourier transforms of $f$, respectively. Then

$$
\begin{aligned}
K\left(x^{\prime}\right) & =(2 \pi)^{-d} \iint_{\mathbb{S}^{d-1}} e^{i x^{\prime} \cdot r \theta} d \theta \phi(\lambda(1-r)) r^{d-1} d r \\
& =-(2 \pi)^{-d} \lambda^{-1} \iint_{\mathbb{S}^{d}-1} e^{i\left(1-r \lambda^{-1}\right) x^{\prime} \cdot \theta}\left(1-r \lambda^{-1}\right)^{d-1} d \theta \phi(r) d r
\end{aligned}
$$

Since $\int_{\mathbb{S}^{d-1}} e^{i x^{\prime} \cdot \theta} d \theta=c_{d}\left|x^{\prime}\right|^{-\frac{d-2}{2}} J_{(d-2) / 2}\left(\left|x^{\prime}\right|\right)$, using the asymptotic expansion of Bessel functions (see [18, p.347, p.356]) we see that

$$
\begin{aligned}
K\left(x^{\prime}\right) & =c_{ \pm} \lambda^{-1} \frac{1}{\left|x^{\prime}\right|^{\frac{d-1}{2}}} e^{ \pm i\left|x^{\prime}\right|} \int e^{\mp i \lambda^{-1}\left|x^{\prime}\right| r} \phi(r) d r+\text { less singular terms } \\
& =c_{ \pm} \lambda^{-\frac{d+1}{2}} e^{ \pm i\left|x^{\prime}\right|}\left(\frac{\left|x^{\prime}\right|}{\lambda}\right)^{-\frac{d-1}{2}} \widehat{\phi}\left(\frac{\mp\left|x^{\prime}\right|}{\lambda}\right)+\text { less singular terms. }
\end{aligned}
$$

We now apply (2.4) to get the desired bound.
As before, by the standard Carleson-Sjölin reduction, rescaling, and the assumption (1.3) it follows that

$$
\left\|\sup _{1 \leq t \leq 2}\left|\int e^{ \pm i t \lambda\left|x^{\prime}-y\right|} a\left(t\left|x^{\prime}-y\right|\right) f(y) d y\right|\right\|_{p} \leq C \lambda^{-\frac{d}{p}+\epsilon}\|f\|_{p}
$$

whenever $a \in C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$ and the bound $C$ remains uniform as long as $a$ is uniformly contained in $C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$. By a similar argument as above, it is easy to see the following.
Lemma 2.2. Suppose $\left\|T_{*}^{\alpha} f\right\|_{p} \leq C\|f\|_{p}$ for $\alpha>\alpha(p)$. Then for $\epsilon>0$

$$
\begin{equation*}
\left\|\sup _{1 \leq t \leq 2}\left|\mathcal{T}_{\delta}^{t} f\right|\right\|_{L^{p}\left(\mathbb{R}^{d}\right)} \leq C \delta^{-\alpha(p)-\epsilon}\|f\|_{L^{p}\left(\mathbb{R}^{d}\right)} \tag{2.5}
\end{equation*}
$$

Here the constant $C$ remains uniform as long as $\|\phi\|_{C^{N}} \leq C$ for some large $N$.
Let $m$ be a bounded measurable function on $\mathbb{R}^{d+1}$. Let us denote by $m(D)$ the multiplier operator given by

$$
\mathcal{F}(m(D) f)(\xi)=m(\xi) \widehat{f}(\xi)
$$

Let $b \in\left[\frac{1}{2}, 2\right]$ and $I(b, j)=\left[b-2^{1-j}, b+2^{1-j}\right]$. Let $\mathfrak{B}_{j}$ be a function in $C_{c}^{\infty}\left(\left[\frac{1}{2}, 2\right] \times \mathbb{R}\right)$ with the following properties:

$$
\begin{align*}
& \operatorname{supp} \mathfrak{B}_{j}(\rho, \cdot) \subset I(b, j) \\
& \left|\partial_{\tau}^{\gamma} \mathfrak{B}_{j}(\rho, \tau)\right| \leq C_{\gamma} 2^{\gamma j}  \tag{2.6}\\
& \left\|\partial_{\tau}^{\gamma} \mathfrak{B}_{j}(\cdot, \tau)\right\|_{C^{N}} \leq C_{\gamma} \text { uniformly for } \tau \in I(b, j)
\end{align*}
$$

for sufficiently large $N$ where $C_{\gamma}$ is independent of $j$ and $\rho$. The following is a slight modification of [22, Lemma 2.12].
Lemma 2.3. Let $\mathfrak{B}_{j}$ be a smooth function in $C_{c}^{\infty}\left(\left[\frac{1}{2}, 2\right] \times \mathbb{R}\right)$ which satisfies (2.6). Then (2.1) implies

$$
\left\|\mathfrak{B}_{j}\left(2^{j}\left(1-\left|D^{\prime}\right|\right), D_{d+1}\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{j(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Proof. Let $\omega \in C_{c}^{\infty}$ be supported in $(-1,1)$ such that $\sum_{\nu \in \mathbb{Z}} \omega(\cdot-\nu) \equiv 1$ and denote $\omega_{\nu}(\tau)=\omega\left(2^{j(1+\epsilon)}\left(\tau-\frac{\nu}{2^{j(1+\epsilon)}}\right)\right)$. We decompose $\mathfrak{B}_{j}\left(2^{j}\left(1-\left|D^{\prime}\right|\right), D_{d+1}\right)=\sum_{\nu} \mathfrak{B}_{j}\left(2^{j}(1-\right.$ $\left.\left.\left|D^{\prime}\right|\right), D_{d+1}\right) \omega_{\nu}\left(D_{d+1}\right)$. By the first condition of (2.6) there are $O\left(2^{\epsilon j}\right)$ nonzero terms. Hence, it is enough to show that

$$
\begin{equation*}
\left\|\mathfrak{B}_{j}\left(2^{j}\left(1-\left|D^{\prime}\right|\right), D_{d+1}\right) \omega_{\nu}\left(D_{d+1}\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{j(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \tag{2.7}
\end{equation*}
$$

To obtain the above estimate from (2.1) we use Taylor expansion in the $\tau$ variable around $\frac{\nu}{2^{j(1+\epsilon)}}$ (see Lemma 2.12 in [22]). More precisely, we have

$$
\begin{aligned}
\mathfrak{B}_{j}(\rho, \tau)= & \sum_{m=0}^{N-1} \frac{1}{m!} \partial_{\tau}^{m} \mathfrak{B}_{j}\left(\rho, \frac{\nu}{2^{j(1+\epsilon)}}\right)\left(\tau-\frac{\nu}{2^{j(1+\epsilon)}}\right)^{m}+\mathfrak{R}_{N}(\rho, \tau)\left(\tau-\frac{\nu}{2^{j(1+\epsilon)}}\right)^{N} \\
= & \sum_{m=0}^{N-1} 2^{-\epsilon m} \frac{2^{-j m}}{m!} \partial_{\tau}^{m} \mathfrak{B}_{j}\left(\rho, \frac{\nu}{2^{j(1+\epsilon)}}\right)\left(2^{j(1+\epsilon)}\left(\tau-\frac{\nu}{2^{j(1+\epsilon)}}\right)\right)^{m} \\
& +2^{-\epsilon N j}\left[2^{-N j} \mathfrak{R}_{N}(\rho, \tau)\right]\left(2^{j(1+\epsilon)}\left(\tau-\frac{\nu}{2^{j(1+\epsilon)}}\right)\right)^{N} .
\end{aligned}
$$

Now let us set

$$
\mathfrak{B}_{j}^{m}(\rho)=\frac{2^{-j m}}{m!} \partial_{\tau}^{m} \mathfrak{B}_{j}\left(\rho, \frac{\nu}{2^{j(1+\epsilon)}}\right), \omega_{\nu}^{m}(\tau)=\left(2^{j(1+\epsilon)}\left(\tau-\frac{\nu}{2^{j(1+\epsilon)}}\right)\right)^{m} \omega_{\nu}(\tau) .
$$

Then, it follows that

$$
\begin{aligned}
\mathfrak{B}_{j}\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right), \xi_{d+1}\right) \omega_{\nu}\left(\xi_{d+1}\right)= & \sum_{m=0}^{N-1} 2^{-\epsilon \epsilon} \mathfrak{B}_{j}^{m}\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right) \omega_{\nu}^{m}\left(\xi_{d+1}\right) \\
& +2^{-\epsilon N j}\left[2^{-N j} \mathfrak{\Re}_{N}\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right), \xi_{d+1}\right)\right] \omega_{\nu}^{N}\left(\xi_{d+1}\right) .
\end{aligned}
$$

It is easy to see that $\mathfrak{B}_{j}^{m}$ satisfies (2.6). Since $\omega_{\nu}^{m}$ is smooth, by Lemma 2.1 we get

$$
\| \mathfrak{B}_{j}^{m}\left(2^{j}\left(1-\left|D^{\prime}\right|\right) \omega_{\nu}^{m}\left(D_{d+1}\right) f\left\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{j(\alpha(p)+\epsilon)}\right\| f \|_{L^{p}\left(\mathbb{R}^{d+1}\right)} .\right.
$$

If $N$ is sufficiently large, using the condition (2.6), one can see that the contribution from $2^{-\epsilon N j}\left[2^{-N j} \mathfrak{R}_{N}\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right), \xi_{d+1}\right)\right] \omega_{\nu}^{N}\left(\xi_{d+1}\right)$ is negligible. Hence we get (2.7) by summation.

Using the maximal estimate (2.5) instead of (2.1) and repeating the same argument as before, we have a statement which is similar to the one in Lemma 2.3 but for the maximal operator.
Lemma 2.4. Let $\mathfrak{B}_{j}$ be a smooth function in $C_{c}^{\infty}\left(\left[\frac{1}{2}, 2\right] \times \mathbb{R}\right)$ which satisfies (2.6). Then (2.5) implies

$$
\left\|\sup _{1 \leq t \leq 2}\left|\mathfrak{B}_{j}\left(2^{j}\left(1-\frac{\left|D^{\prime}\right|}{t}\right), \frac{D_{d+1}}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{j(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

For $0<\delta \ll 1$ and $\psi \in C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$, we define the operator $\mathcal{C}_{\delta}$ by

$$
\mathcal{C}_{\delta} f=\psi\left(D_{d+1}\right) \beta\left(\delta^{-1}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) f .
$$

Theorem 2.5. Let $d \geq 4$ and $p>\frac{2 d-2}{d-3}$. Then

$$
\begin{equation*}
\left\|\mathcal{C}_{\delta} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C \delta^{-\alpha(p)-\epsilon}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \tag{2.8}
\end{equation*}
$$

This is basically due to Heo [7]. In fact (2.8) can be deduced from Heo's results by the standard Carleson-Sjölin reduction and asymptotic expansion for kernels as before (see Lemma 2.1 and [12]). Alternatively, the estimate without even $\epsilon$-loss can also be deduced from sharp local smoothing estimate for the wave equation which is obtained in $[8]^{1}$ (also see [9]). Now note that $\alpha\left(\frac{2 d-2}{d-3}\right)<1$ when $d \geq 4$. Hence, in particular, this gives sharp $L^{p}$ bound for the cone multiplier of order 1 when $d \geq 3$ (one can use the trivial $L^{\infty}$ bound when $d=3$ ) so that $\left\|\left(\left|D_{d+1}\right|-\left|D^{\prime}\right|\right)+\psi\left(D_{d+1}\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ for $2 \leq p<\frac{2 d}{d-3}$.
Proposition 2.6. (Proposition 2.3 in [22]) Let $2 \leq p_{B} \leq p_{C} \leq \infty$ and $l \geq j-5$. Suppose that (2.1) and (2.8) hold for $p=p_{B}$ and $p=p_{C}$, respectively. Then for $p_{B} \leq p \leq p_{C}<\infty$,
$\left\|\psi\left(D_{d+1}\right) \beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \beta\left(2^{j}\left(1-\left|D^{\prime}\right|\right)\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{(l-j) \lambda\left(p, p_{B}, p_{C}\right)} 2^{l(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ where

$$
\lambda\left(p, p_{B}, p_{C}\right)=\left(\frac{1}{p}-\frac{1}{p_{C}}\right)\left(\frac{1}{p_{B}}-\frac{1}{p_{C}}\right)^{-1}\left(1-\frac{2}{p_{B}}\right) .
$$

The case $p_{C}=\infty$ was excluded in [22] but it is clear that Proposition 2.6 also holds with $p_{C} \leq \infty$. By Lemma 2.3 the estimate (2.1) implies

$$
\begin{aligned}
& \left\|\psi\left(D_{d+1}\right) \beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \omega\left(2^{l(1+\epsilon)}\left(D_{d+1}-\frac{\nu}{2^{l(1+\epsilon)}}\right)\right) \beta\left(2^{j}\left(1-\left|D^{\prime}\right|\right)\right) f\right\|_{p_{B}} \\
& \leq C 2^{l\left(\alpha\left(p_{B}\right)+\epsilon\right)}\|f\|_{p_{B}} .
\end{aligned}
$$

Then interpolation with this (also making use of orthogonality) and (2.8) gives the desired estimate. See [22] for the details of the proof.

## 3. Proof of Theorems 1.1 and 1.2

We decompose the multiplier $(1-\rho(\xi))_{+}^{\alpha}=\left(1-\max \left\{\left|\xi^{\prime}\right|,\left|\xi_{d+1}\right|\right\}\right)_{+}^{\alpha}$ similarly as in [22]. Let us set $\beta_{1}(t)=1-\sum_{j=2}^{\infty} \beta\left(2^{j}(1-t)\right)$. Here we use $\beta$ which is given in Section 2 (see below (2.3)). For $j, k \geq 1$ let us define

$$
\begin{array}{ll}
\beta_{j, k}(\xi)=\beta\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right) \beta\left(2^{k}\left(1-\left|\xi_{d+1}\right|\right)\right) & \text { for } j, k \geq 2, \\
\beta_{1, k}(\xi)=\beta_{1}\left(\left|\xi^{\prime}\right|\right) \beta\left(2^{k}\left(1-\left|\xi_{d+1}\right|\right)\right) & \text { for } \quad k \geq 2, \\
\beta_{j, 1}(\xi)=\beta\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right) \beta_{1}\left(\left|\xi_{d+1}\right|\right) & \text { for } \quad j \geq 2, \\
\beta_{1,1}(\xi)=\beta_{1}\left(\left|\xi^{\prime}\right|\right) \beta_{1}\left(\left|\xi_{d+1}\right|\right) . &
\end{array}
$$

So we have $\sum_{j, k \geq 1} \beta_{j, k}(\xi)=\left(1-\left|\xi^{\prime}\right|\right)_{+}\left(1-\left|\xi_{d+1}\right|\right)_{+}$. We also set

$$
m_{j, k}(\xi)=(1-\rho(\xi))_{+}^{\alpha} \beta_{j, k}(\xi) .
$$

[^1]We decompose the multiplier

$$
\begin{equation*}
(1-\rho(\xi))_{+}^{\alpha}=\sum_{j, k \geq 1} m_{j, k}:=\mathfrak{M}+\mathfrak{N} \tag{3.1}
\end{equation*}
$$

where

$$
\mathfrak{M}=\sum_{k \geq j+2} m_{j, k}, \quad \mathfrak{N}=\sum_{k<j+2} m_{j, k}
$$

3.1. Proof of Theorem 1.1. The implication from (1.1) to (1.2) follows from de Leeuw's theorem as already explained. Hence it is sufficient to show (1.1) by assuming (1.2). By duality we may assume that $2 \leq p<\infty$ when $d=2$ and $2 \leq p<\frac{2 d}{d-3}$ when $d \geq 3$.

It is easy to treat the operator defined by the first sum $\mathfrak{M}$. In fact, when $k \geq 2+j$, the multiplier $m_{j, k}$ is equal to $\left.\left(1-\left|\xi_{d+1}\right|\right)_{+}^{\alpha}\right) \beta\left(2^{k}\left(1-\left|\xi_{d+1}\right|\right)\right) \beta\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right.$. So,

$$
\begin{equation*}
m_{j, k}(\xi)=2^{-k \alpha} \widetilde{\beta}\left(2^{k}\left(1-\left|\xi_{d+1}\right|\right)\right) \beta\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right) \tag{3.2}
\end{equation*}
$$

for some $\widetilde{\beta} \in C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$. Since $\left\|\widetilde{\beta}\left(2^{k}\left(1-\left|\xi_{d+1}\right|\right)\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ for $1 \leq p \leq \infty$, applying Lemma 2.1 , we see that $\left\|m_{j, k}(D) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \lesssim 2^{-k \alpha} 2^{j(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$. Hence, for $\alpha>\alpha(p)$ we get

$$
\begin{aligned}
\|\mathfrak{M}(D) f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} & \leq \sum_{k \geq 2+j}\left\|m_{j, k}(D) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \\
& <C \sum_{k \geq 2+j} 2^{-k \alpha} 2^{j(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
\end{aligned}
$$

We now handle the operator $\mathfrak{N}(D)$. Let us set

$$
\widehat{\widehat{T}_{t} f}(\xi)=\left(1-\frac{\left|\xi^{\prime}\right|}{t}\right)_{+}^{\alpha} \widehat{f}(\xi)
$$

Then from the assumption (1.2) it is obvious that $\left\|\widetilde{T}_{1} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$. Hence it is enough to show that

$$
\left\|\mathfrak{N}(D) f-\widetilde{T}_{1} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Note that

$$
\begin{align*}
\mathfrak{N}(\xi)-\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha}=\sum_{k<j+2} \beta_{j, k}(\xi) & \left((1-\rho(\xi))_{+}^{\alpha}-\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha}\right) \\
& -\sum_{k \geq j+2} \beta_{j, k}(\xi)\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha} \tag{3.3}
\end{align*}
$$

Since $\sum_{k \geq j+2}^{\infty} \beta\left(2^{k} t\right)=\widetilde{\beta}\left(2^{j} t\right)$ for some smooth function $\widetilde{\beta}$ which is supported in $[-2,2]$, $\sum_{k \geq j+2}^{\infty} \beta_{j, k}(\xi)=\beta\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right) \widetilde{\beta}\left(2^{j}\left(1-\left|\xi_{d+1}\right|\right)\right)$. Thus

$$
\begin{equation*}
\sum_{k \geq j+2} \beta_{j, k}(\xi)\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha}=\sum_{j \geq 1} 2^{-j \alpha} \beta\left(2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right) \widetilde{\beta}\left(2^{j}\left(1-\left|\xi_{d+1}\right|\right)\right) \tag{3.4}
\end{equation*}
$$

By Lemma 2.3

$$
\begin{aligned}
\left\|\sum_{k \geq j+2} \beta_{j, k}(D)\left(1-\left|D^{\prime}\right|\right)_{+}^{\alpha} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} & \leq \sum_{j \geq 1} 2^{-\alpha j}\left\|\beta\left(2^{j}\left(1-\left|D^{\prime}\right|\right)\right) \widetilde{\beta}\left(2^{j}\left(1-\left|D_{d+1}\right|\right)\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \\
& \leq C \sum_{j \geq 1} 2^{-\alpha j+\alpha(p) j+\epsilon j}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
\end{aligned}
$$

if $\alpha>\alpha(p)$. Hence we are further reduced to showing that

$$
\left\|\sum_{k<j+2} \beta_{j, k}(D)\left((1-\rho(D))_{+}^{\alpha}-\left(1-\left|D^{\prime}\right|\right)_{+}^{\alpha}\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} .
$$

Note that if $j \geq k+2, \beta_{j, k}(D)\left((1-\rho(D))_{+}^{\alpha}-\left(1-\left|D^{\prime}\right|\right)_{+}^{\alpha}\right)=0$. So, it is sufficient to show that

$$
\left\|\sum_{k-1 \leq j \leq k+1} \beta_{j, k}(D)\left((1-\rho(D))_{+}^{\alpha}-\left(1-\left|D^{\prime}\right|\right)_{+}^{\alpha}\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} .
$$

By the support property of $\beta_{j, k}$ and the mean value theorem

$$
\begin{equation*}
\beta_{j, k}(\xi)\left((1-\rho(\xi))_{+}^{\alpha}-\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha}\right)=\left(\left|\xi_{d+1}\right|-\left|\xi^{\prime}\right|\right)_{+} \beta_{j, k}(\xi) \widetilde{\psi}_{\alpha}(\xi) \tag{3.5}
\end{equation*}
$$

where

$$
\tilde{\psi}_{\alpha}(\xi)=\alpha \int_{0}^{1}\left((1-s)\left(1-\left|\xi_{d+1}\right|\right)+s\left(1-\left|\xi^{\prime}\right|\right)\right)^{\alpha-1} d s
$$

We firstly consider $\beta_{1,1}(D)\left((1-\rho(D))_{+}^{\alpha}-\left(1-\left|D^{\prime}\right|\right)_{+}^{\alpha}\right)$. On the support of $\beta_{1,1}$, $((1-$ $\left.s)\left(1-\left|\xi_{d+1}\right|\right)+s\left(1-\left|\xi^{\prime}\right|\right)\right)^{\alpha-1} \sim 1$. So $\beta_{1,1} \widetilde{\psi}_{\alpha}$ is a smooth function. Hence it is sufficient to show that

$$
\left\|\left(\left|D_{d+1}\right|-\left|D^{\prime}\right|\right)_{+} \phi(D) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

for $2 \leq p<\infty$ when $d=2$ and $2 \leq p<\frac{2 d}{d-3}$ when $d \geq 3$. Here $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{d+1}\right)$. In fact, by dyadic decomposition it follows from $\left\|\beta\left(2^{k}\left|D_{d+1}\right|\right)\left(\left|D_{d+1}\right|-\left|D^{\prime}\right|\right)_{+} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq$ $C 2^{-k}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}, k \geq 0$ which is obvious from rescaling since the (truncated) cone multiplier of order 1 is bounded on $L^{p}$ provided that $2 \leq p<\infty$ when $d=2$ and $2 \leq p<\frac{2 d}{d-3}$ when $d \geq 3$.

It remains to show that

$$
\left\|\sum_{\substack{k-1 \leq j \leq k+1 \\(j, k) \neq(1,1)}} \beta_{j, k}(D)\left((1-\rho(D))_{+}^{\alpha}-\left(1-\left|D^{\prime}\right|\right)_{+}^{\alpha}\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Let us set

$$
\eta_{j, k}(\xi)=2^{j(\alpha-1)} \beta_{j, k}(\xi) \tilde{\psi}_{\alpha}(\xi) .
$$

Then we have

$$
\beta_{j, k}(\xi)\left((1-\rho(\xi))_{+}^{\alpha}-\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha}\right)=2^{-j(\alpha-1)}\left(\left|\xi_{d+1}\right|-\left|\xi^{\prime}\right|\right)_{+} \eta_{j, k}(\xi) .
$$

Note that $\eta_{j, k}(\xi)=\beta_{j, k}(\xi) \alpha \int_{0}^{1}\left((1-s) 2^{j}\left(1-\left|\xi_{d+1}\right|\right)+s 2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right)^{\alpha-1} d s$. Since $k-1 \leq$ $j \leq k+1, k, j \neq 1$, and $\left|s 2^{j}\left(1-\left|\xi_{d+1}\right|\right)+(1-s) 2^{j}\left(1-\left|\xi^{\prime}\right|\right)\right| \sim 1$, it is easy to see that $\eta_{j, k}$ is smooth and

$$
\left|\partial_{\xi}^{\gamma} \eta_{j, k}\right| \leq C 2^{j|\gamma|} .
$$

We make further decomposition to treat the singularity near the cone. We can write $\left(1-\left|\xi^{\prime}\right| /\left|\xi_{d+1}\right|\right)_{+}=\sum_{l=-\infty}^{\infty} 2^{-l} \widetilde{\beta}\left(2^{l}\left(1-\left|\xi^{\prime}\right| /\left|\xi_{d+1}\right|\right)\right)$ for some $\widetilde{\beta} \in C_{c}^{\infty}\left[\frac{1}{2}, 2\right]$. In fact, $\widetilde{\beta}(t)=$ $t \beta(t)$. Note that $\beta_{j, k}(\xi) \beta\left(2^{l}\left(1-\left|\xi^{\prime}\right| /\left|\xi_{d+1}\right|\right)\right)=0$ if $l<j-5$. Now we have for $k-1 \leq j \leq$ $k+1, k, j \neq 1$,

$$
\begin{equation*}
\beta_{j, k}(\xi)\left((1-\rho(\xi))_{+}^{\alpha}-\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha}\right)=\sum_{l \geq j-5} 2^{-l} 2^{-j(\alpha-1)}\left|\xi_{d+1}\right| \widetilde{\beta}\left(2^{l}\left(1-\left|\xi^{\prime}\right| /\left|\xi_{d+1}\right|\right)\right) \eta_{j, k}(\xi) \tag{3.6}
\end{equation*}
$$

Since $|k-j| \leq 1$, we see that $\eta_{j, k}$ in (3.6) satisfies the condition (2.6). So, by Proposition 2.6

$$
\left\|\left|D_{d+1}\right| \widetilde{\beta}\left(2^{l}\left(1-\left|D^{\prime}\right| /\left|D_{d+1}\right|\right)\right) \eta_{j, k}(D) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{l(\alpha(p)+\epsilon)} 2^{(l-j) \lambda\left(p, p_{B}, p_{C}\right)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

When $d \geq 4$, by Theorem 2.5 we can take any $p_{C}>\frac{2 d-2}{d-3}$. When $d=2$, 3 , we may set $p_{C}=\infty$ by the obvious $L^{\infty}$ estimate. Thus, by the triangle inequality and the summation along $j$ it follows that

$$
\begin{aligned}
\| \beta_{j, k}(D)\left((1-\rho(D))_{+}^{\alpha}-\right. & \left.\left(1-\left|D^{\prime}\right|\right)_{+}^{\alpha}\right) f \|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \\
& \leq C\left(\sum_{l>j-10}^{\infty} 2^{l(\alpha(p)+\epsilon)} 2^{-j \alpha} 2^{(l-j)\left(\lambda\left(p, p_{B}, p_{C}\right)-1\right)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}\right) \\
& \leq C 2^{-j(\alpha-\alpha(p)-\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
\end{aligned}
$$

For the second inequality we used the fact that $\alpha(p)+\lambda\left(p, p_{B}, p_{C}\right)<1$ for $p_{B} \leq p<\frac{2 d}{d-3}$ if we choose $p_{C}=\frac{2 d}{d-3}$. Therefore we get the desired (3.1) for $p_{B} \leq p<\frac{2 d}{d-3}$.
3.2. Proof of Theorem 1.2. We make use of the decomposition (3.1). The proof of the maximal bound is actually parallel with that of $L^{p}$ bound.

First we show that if $\alpha>\alpha(p)$, then

$$
\left\|\sup _{t>0}\left|\mathfrak{M}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Now observe that supp $\mathfrak{M}$ is contained in $\left\{\xi: \frac{3}{8} \leq|\xi| \leq 1\right\}$. Then for the above it is sufficient to show that

$$
\left\|\sup _{1 \leq t \leq 2}\left|\mathfrak{M}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Here we use the following elementary lemma. We include a proof of it for reader's convenience.

Lemma 3.1. Let $\widetilde{\mathfrak{M}}$ be a bounded measurable function supported in $\left\{\xi: 2^{-B+1} \leq|\xi| \leq\right.$ $\left.2^{B+1}\right\}$ for some $B>1$. Suppose that $\left\|\sup _{1 \leq t \leq 2} \left\lvert\, \widetilde{M}\left(\frac{D}{t}\right) f\right.\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ for $1 \leq p \leq \infty$. Then,

$$
\left\|\sup _{t>0}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Proof. In fact, let $P_{l}$ be the projection operator which is defined by $\widehat{P_{l} f}=\beta\left(2^{-l}|\cdot|\right) \widehat{f}$. We note that

$$
\begin{aligned}
\sup _{t>0}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right) f\right| & \leq \sup _{k} \sup _{2^{-k}<t \leq 2^{-k+1}}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right) f\right| \\
& =\sup _{k} \sup _{2^{-k}<t \leq 2^{-k+1}}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right)\left(\sum_{k-B \leq l \leq k+B} P_{l} f\right)\right| \\
& \leq\left(\sum_{k=-\infty^{2^{-k}<t \leq 2^{-k+1}}}^{\infty} \sup ^{\infty}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right)\left(\sum_{k-B \leq l \leq k+B} P_{l} f\right)\right|^{p}\right)^{\frac{1}{p}}
\end{aligned}
$$

By the assumption and scaling we see that $\left\|\sup _{2^{-k}<t \leq 2^{-k+1}}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ holds uniformly for any $k$. Hence, using the above inequality

$$
\begin{aligned}
\left\|\sup _{t>0}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p} & \leq \sum_{k=-\infty}^{\infty}\left\|\sup _{2^{-k}<t \leq 2^{-k+1}}\left|\widetilde{\mathfrak{M}}\left(\frac{D}{t}\right)\left(\sum_{k-B \leq l \leq k+B} P_{l} f\right)\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p} \\
& \leq C \sum_{k=-\infty}^{\infty}\left\|\sum_{k-B \leq l \leq k+B} P_{l} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p} \\
& \leq C B^{p} \sum_{l=-\infty}^{\infty}\left\|P_{l} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p} \leq C B^{p}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p}
\end{aligned}
$$

For the last inequality we use the inequality $\left(\sum_{l=-\infty}^{\infty}\left\|P_{l} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p}\right)^{1 / p} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ for $2 \leq p \leq \infty$, which follows from interpolation between the estimates with $p=2, p=\infty$. The first is a consequence of Plancherel's theorem and the second is obvious because the kernel of $P_{l}$ is uniformly contained in $L^{1}$.

Recalling (3.2), note that

$$
\begin{equation*}
\sup _{1 \leq t \leq 2}\left|m_{j, k}\left(\frac{D}{t}\right) f\right| \leq 2^{-k \alpha} \sup _{1 \leq t \leq 2}\left|\widetilde{\beta}\left(2^{k}\left(1-\frac{\mid D_{d+1}}{t}\right)\right) \beta\left(2^{j}\left(1-\frac{\left|D^{\prime}\right|}{t}\right)\right) f\right| \tag{3.7}
\end{equation*}
$$

By the smoothness of $\widetilde{\beta}$ and the fact that $k \geq 3$ (here we are assuming $k \geq 2+j$ ) a simple computation shows that the kernel of $\widetilde{\beta}\left(2^{k}\left(1-\left|D_{d+1}\right| / t\right)\right)$ is bounded by $\mathcal{K}=$ $C_{N} 2^{-k}\left(1+2^{-k}\left|x_{d+1}\right|\right)^{-N}$ with $C_{N}$ independent of $t \in[1,2]$. From this the right hand side of (3.7) is bounded by

$$
2^{-k \alpha} \mathcal{K} *_{d+1}\left(\sup _{1 \leq t \leq 2}\left|\beta\left(2^{j}\left(1-\frac{\left|D^{\prime}\right|}{t}\right)\right) f\right|\right)
$$

Here $*_{d+1}$ denotes the convolution with respect to $(d+1)$-th variables. By Young's convolution inequality it follows that

$$
\left\|\sup _{1 \leq t \leq 2}\left|m_{j, k}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{-k \alpha}\left\|_{1 \leq t \leq 2}\left|\beta\left(2^{j}\left(1-\frac{\left|D^{\prime}\right|}{t}\right)\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Using Lemma 2.2 and summation, we get, for $\alpha>\alpha(p)$,

$$
\begin{aligned}
\left\|\sup _{1 \leq t \leq 2}\left|\mathfrak{M}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} & \leq \sum_{k \geq 2+j}\left\|\sup _{1 \leq t \leq 2} \left\lvert\, m_{j, k}\left(\frac{D}{t}\right) f\right.\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \\
& <C \sum_{k \geq 2+j} 2^{-k \alpha} 2^{j(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
\end{aligned}
$$

Now we handle the operator $f \rightarrow \sup _{t>0}\left|\mathfrak{N}\left(\frac{D}{t}\right) f\right|$. From the assumption it is obvious that $\left\|\sup _{t>0} \mid \widetilde{T}_{t} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$. Hence it is enough to show that

$$
\left\|\sup _{t>0} \left\lvert\,\left(\mathfrak{N}\left(\frac{D}{t}\right)-\widetilde{T}_{t}\right) f\right.\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

We use Lemma 2.4 and Lemma 3.1 to get, for $\alpha>\alpha(p)$,

$$
\begin{aligned}
\| \sup _{t>0} \left\lvert\, \sum_{k \geq j+2} \beta_{j, k}\left(\frac{D}{t}\right)\right. & \left.\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha} f \right\rvert\, \|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \\
& \leq C \sum_{j \geq 1} 2^{-\alpha j}\left\|_{t>0}\left|\beta\left(2^{j}\left(1-\frac{\left|D^{\prime}\right|}{t}\right)\right) \widetilde{\beta}\left(2^{j}\left(1-\frac{\left|D_{d+1}\right|}{t}\right)\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \\
& \leq C \sum_{j \geq 1} 2^{-j(\alpha-\alpha(p)-\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
\end{aligned}
$$

Here $\widetilde{\beta}$ is given as in (3.4). Hence by (3.3) and (3.4) it is sufficient to show that

$$
\left\|\sup _{t>0}\left|\sum_{k-1 \leq j \leq k+1} \beta_{j, k}\left(\frac{D}{t}\right)\left(\left(1-\rho\left(\frac{D}{t}\right)\right)_{+}^{\alpha}-\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

As before, we separately handle $f \rightarrow \sup _{t>0}\left|\beta_{1,1}\left(\frac{D}{t}\right)\left(\left(1-\rho\left(\frac{D}{t}\right)\right)_{+}^{\alpha}-\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha}\right) f\right|$, and claim that

$$
\left\|\sup _{t>0}\left|\beta_{1,1}\left(\frac{D}{t}\right)\left(\left(1-\rho\left(\frac{D}{t}\right)\right)_{+}^{\alpha}-\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

By a simple scaling argument it is sufficient to show that

$$
\left\|\sup _{0<t \leq 1}\left|\beta_{1,1}\left(\frac{D}{t}\right)\left(\left(1-\rho\left(\frac{D}{t}\right)\right)_{+}^{\alpha}-\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

So, we may assume that $\widehat{f}$ is supported in $\{\xi:|\xi| \leq 2\}$. Let $M$ denote the Hardy-Littlewood maximal function. Since $\left|\xi_{d+1}\right| \beta_{1,1} \widetilde{\psi}_{\alpha}$ (see (3.5)) is a smooth function,

$$
\sup _{t>0}\left|\beta_{1,1}\left(\frac{D}{t}\right)\left(\left(1-\rho\left(\frac{D}{t}\right)\right)_{+}^{\alpha}-\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha}\right) f\right| \leq C M\left(\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)_{+} f\right)
$$

Hence it is sufficient to show that

$$
\begin{equation*}
\left\|\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)_{+} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \tag{3.8}
\end{equation*}
$$

for $2 \leq p<\infty$ when $d=2$ and $2 \leq p<\frac{2 d}{d-3}$ when $d \geq 3$.
Let us first consider the case $d=2$ which is easy because we are not concerned with the sharp estimate. In fact, by following the argument in [10] which makes use of the kernel estimate and the Calderón-Zygmund decomposition, it is not difficult to see that

$$
\begin{equation*}
\left\|\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)_{+}^{\alpha} f\right\|_{L^{1, \infty}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{1}\left(\mathbb{R}^{d+1}\right)} \tag{3.9}
\end{equation*}
$$

for $\alpha>\frac{d-1}{2}$. This also remains valid for complex $\alpha$, provided that $\Re(\alpha)>\frac{d-1}{2}$, and the bound is bounded above by $C_{N}(1+|\Im(\alpha)|)^{N}$ for some large $N$. When $d=2$, taking $\alpha=1$ we interpolate (real interpolation) this with the obvious $L^{2}$ estimate to get (3.8) for $1<p \leq 2$. By duality, the desired bound for $2 \leq p<\infty$ follows. When $d=3$, by the complex interpolation between the estimates $L^{2} \rightarrow L^{2}$ for $\Re(\alpha) \geq 0$ and (3.9) for $\Re(\alpha)>1$ we have for $1<p \leq 2$

$$
\begin{equation*}
\left\|\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)_{+} f\right\|_{L^{p, \infty}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \tag{3.10}
\end{equation*}
$$

As for $d \geq 4$ the same estimate can be deduced from the estimate in [9, Corollary 1.3] where the weak type endpoint estimates at the critical exponent

$$
\left\|\left(1-\frac{\left|D^{\prime}\right|^{2}}{D_{d+1}^{2}}\right)_{+}^{\alpha(p)} f\right\|_{L^{p, \infty}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

were shown for $1<p<\frac{2 d-2}{d+1}$. Even though the multipliers are slightly different, the same estimate can be shown for $f \rightarrow\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)_{+}^{\alpha(p)} f$ by following the argument in [9] (the proof of Corollary 1.3) ${ }^{2}$. Hence, in particular, (3.10) holds for $1<p<\frac{2 d}{d+3}$ when $d \geq 3$. Then the desired estimate (3.8) follows from interpolation with the trivial $L^{2}$ estimate and duality.

We now turn to the operator

$$
f \rightarrow \sup _{t>0}\left|\sum_{\substack{k-1 \leq j \leq k+1 \\(j, k) \neq(1,1)}} \beta_{j, k}\left(\frac{D}{t}\right)\left(\left(1-\rho\left(\frac{D}{t}\right)\right)_{+}^{\alpha}-\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha}\right) f\right|
$$

Note that the support of $\sum_{k-1 \leq j \leq k+1} \beta_{j, k}(\xi)\left((1-\rho(\xi))_{+}^{\alpha}-\left(1-\left|\xi^{\prime}\right|\right)_{+}^{\alpha}\right)$ is contained in $\left\{2^{-c} \leq|\xi| \leq 2^{c}\right\}$ for some $c>0$. By Lemma 3.1 it is sufficient to show that

$$
\left\|\sup _{1 \leq t \leq 2} \left\lvert\, \sum_{\substack{k-1 \leq j \leq k+1 \\(j, k) \neq(1,1)}} \beta_{j, k}\left(\frac{D}{t}\right)\left(\left(1-\rho\left(\frac{D}{t}\right)\right)_{+}^{\alpha}-\left(1-\frac{\left|D^{\prime}\right|}{t}\right)_{+}^{\alpha}\right) f\right.\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

As before, this follows by using the decomposition (3.6) and direct summation once we get the estimate

$$
\begin{align*}
\| \sup _{1 \leq t \leq 2} \left\lvert\, \widetilde{\beta}\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right)\right. & \left.\eta_{j, k}\left(\frac{D}{t}\right) f \right\rvert\, \|_{L^{p}\left(\mathbb{R}^{d+1}\right)}  \tag{3.11}\\
& \leq C 2^{(l-j) \lambda\left(p, p_{B}, p_{C}\right)} 2^{l(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
\end{align*}
$$

We now proceed to show (3.11).
Lemma 3.2. Let $|k-j| \leq 1, k, j \neq 1$. Assume that the following estimate holds:

$$
\begin{equation*}
\left\|\sup _{t \in\left[a, a+2^{-j}\right]}\left|\beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq B\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \tag{3.12}
\end{equation*}
$$

for some constant $B$ and $a \sim 1$. Then for $2 \leq p \leq \infty$, we have

$$
\left\|\sup _{t \sim 1}\left|\beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}\left(\frac{D}{t}\right) f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C B\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

Proof. Let $\omega \in C_{c}^{\infty}$ be supported in $(1 / 2,4)$ so that $\sum_{\nu \in \mathbb{Z}} \omega(\cdot-\nu) \equiv 1$. Let us define

$$
P_{\nu} f=\omega\left(2^{j}\left(D_{d+1}-\nu\right)\right) f
$$

where $\nu \in 2^{-j} \mathbb{Z}$. For $\nu \in 2^{-j} \mathbb{Z}$, we denote by $I_{\nu}$ the interval of center $\nu$ with the length $2^{-j}$.

[^2]Now we note that $\operatorname{supp} \eta_{j, k}(\cdot / t)$ is contained in $\left\{\xi: \xi_{d+1} \in\left(t-C 2^{-j}, t+C 2^{-j}\right)\right\}$. So, we see that

$$
\begin{aligned}
\sup _{t \sim 1} \mid \beta\left(2^{l}(1-\right. & \left.\left.\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \left.\eta_{j, k}\left(\frac{D}{t}\right) f \right\rvert\, \\
& \leq \sup _{\nu} \sup _{t \in I_{\nu}}\left|\beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}\left(\frac{D}{t}\right) f\right| \\
& \leq \sup _{\nu} \sup _{t \in I_{\nu}} \sum_{\left|\nu^{\prime}\right| \leq C 2^{-j}}\left|\beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}\left(\frac{D}{t}\right) P_{\nu+\nu^{\prime}} f\right| \\
& \leq \sum_{\left|\nu^{\prime}\right| \leq C 2^{-j}}\left(\sum_{\nu}\left(\sup _{t \in I_{\nu}}\left|\beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}\left(\frac{D}{t}\right) P_{\nu+\nu^{\prime}} f\right|\right)^{p}\right)^{1 / p}
\end{aligned}
$$

From this and the assumption (3.12), we have that

$$
\begin{aligned}
\| \sup _{t \sim 1} \mid \beta\left(2^{l}(1-\right. & \left.\left.\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \left.\eta_{j, k}\left(\frac{D}{t}\right) f \right\rvert\, \|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p} \\
& \leq C \sum_{\left|\nu^{\prime}\right| \leq C 2^{-j}} \sum_{\nu}\left\|\sup _{t \in I_{\nu}}\left|\beta\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}\left(\frac{D}{t}\right) P_{\nu+\nu^{\prime}} f\right|\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p} \\
& \leq C \sum_{\left|\nu^{\prime}\right| \leq C 2^{-j}} \sum_{\nu}\left\|P_{\nu+\nu^{\prime}} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}^{p} \leq C B\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)^{2}}^{p}
\end{aligned}
$$

For the last inequality we use $\left(\sum_{\nu=-\infty}^{\infty}\left\|P_{\nu} f\right\|_{p}^{p}\right)^{1 / p} \leq C\|f\|_{p}$ for $2 \leq p \leq \infty$. It can be shown similarly as before (for example, see Lemma 6.1 in [21]).

By Lemma 3.2 we are now reduced to showing (3.12). To control the maximal function, we recall the following well known simple lemma which is an easy consequence of the fundamental theorem of calculus and Hölder's inequality.

Lemma 3.3. Let $F$ be a smooth function defined on $(a, b)$. Then, for $1 \leq p \leq \infty$,

$$
\sup _{t \in(a, b)}|F(t)| \leq C\left(|F(a)|+|F(b)|+\|F\|_{L^{p}(a, b)}^{(p-1) / p}\left\|\partial_{t} F\right\|_{L^{p}(a, b)}^{1 / p}\right)
$$

Let us denote by $I_{a}=\left[a, a+2^{-j}\right]$ and also set

$$
T_{l, j, k}^{t} f=\widetilde{\beta}\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}\left(\frac{D}{t}\right) f
$$

Using Lemma 3.3 followed by a simple inequality (e.g. Young's inequality), we see that

$$
\sup _{t \in I_{a}}|F(t)| \leq C\left(|F(a)|+\left|F\left(a+2^{-j}\right)\right|+2^{j / p}\|F\|_{L^{p}\left(I_{a}\right)}+2^{-j+j / p}\left\|\partial_{t} F\right\|_{L^{p}\left(I_{a}\right)}\right)
$$

We apply this to $\sup _{t \in I_{a}}\left|T_{l, j, k}^{t} f\right|$. By a direct differentiation and (2.6), we see that

$$
\begin{aligned}
\sup _{t \in I_{a}} \mid \widetilde{\beta}\left(2^{l}(1-\right. & \left.\left.\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \left.\eta_{j, k}\left(\frac{D}{t}\right) f|\leq C| T_{l, j, k}^{a} f|+C| T_{l, j, k}^{a+2^{-j}} f \right\rvert\, \\
& +C 2^{j / p} \sum_{i=1}^{d+1}\left(\int_{I_{a}}\left|\widetilde{\beta}\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}^{i}\left(\frac{D}{t}\right) f^{i}\right|^{p} d t\right)^{1 / p}
\end{aligned}
$$

where $f^{1}, \ldots, f^{d+1}$ satisfying $\left\|f^{1}\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}, \ldots,\left\|f^{d+1}\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ and $\eta_{j, k}^{1}, \ldots$, $\eta_{j, k}^{d+1}$ are smooth functions satisfying (2.6). By making use of Proposition 2.6, we see that
$\left\|T_{l, j, k}^{a} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ and $\left\|T_{l, j, k}^{a+2^{-j}} f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$ are bounded by $C 2^{(l-j) \lambda\left(p, p_{B}, p_{C}\right)} 2^{l(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}$. Hence it is sufficient to show that for $p_{B} \leq p \leq p_{C}$,

$$
\begin{aligned}
\|\left(\int_{I_{a}} \left\lvert\, \widetilde{\beta}\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right)\right.\right. & \left.\left.\eta_{j, k}^{i}\left(\frac{D}{t}\right) f\right|^{p} d t\right)^{1 / p} \|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \\
& \leq C 2^{-j / p} 2^{(l-j) \lambda\left(p, p_{B}, p_{C}\right)} 2^{l(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
\end{aligned}
$$

From Proposition 2.6 and mild rescaling we note that the estimate

$$
\left\|\widetilde{\beta}\left(2^{l}\left(1-\frac{\left|D^{\prime}\right|}{\left|D_{d+1}\right|}\right)\right) \eta_{j, k}^{i}\left(\frac{D}{t}\right) f\right\|_{L^{p}\left(\mathbb{R}^{d+1}\right)} \leq C 2^{(l-j) \lambda\left(p, p_{B}, p_{C}\right)} 2^{l(\alpha(p)+\epsilon)}\|f\|_{L^{p}\left(\mathbb{R}^{d+1}\right)}
$$

holds uniformly for $t \in I_{a}$ because $I_{a} \subset\left[\frac{1}{2}, 2\right]$. Therefore, changing the order of integration and using the above uniform bound, we get the desired inequality. This completes the proof of Theorem 1.2.

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[^1]:    ${ }^{1}$ One also can obtain (2.8) without $\epsilon$ by adopting the argument in [8].

[^2]:    ${ }^{2}$ Actually it is simpler.

