An endpoint space-time estimate for the Schrödinger equation

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Abstract

We obtain endpoint estimates for the Schrödinger operator $f \to e^{it\Delta}f$ in $L^q_x(\mathbb{R}^n, L^r_t(\mathbb{R}))$ with initial data f in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$. The exponents and regularity index satisfy $\frac{n+1}{q} + \frac{1}{r} = \frac{n}{2}$ and $s = \frac{n}{2} - \frac{n}{q} - \frac{2}{r}$. For $n = 2$ we prove the estimates in the range $q > 16/5$, and for $n \geq 3$ in the range $q > 2 + 4/(n+1)$.

Key words: Schrödinger equation; Strichartz estimates

1. Introduction

The solution to the Schrödinger equation, $i\partial_t u + \Delta u = 0$, in $\mathbb{R}^{n+1}$, with initial datum f, a Schwartz function, can be written as

$$e^{it\Delta}f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i(x\cdot\xi - 2\pi t|\xi|^2)} d\xi. \qquad (1)$$

The space-time integrability of $e^{it\Delta}f$ has played an important role in the study of nonlinear Schrödinger equations (see for example [4] or [24]). The integrability is usually measured with estimates for $e^{it\Delta}f$ in the mixed norm spaces $L^r_t(\mathbb{R}, L^q_x(\mathbb{R}^n))$. By the work of Stein [20], Tomas [28], Strichartz [22], Ginibre–Velo [4], and Keel–Tao [7] the following theorem is now well known. Scaling dictates the regularity of the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$, and for the endpoint estimates the initial data belong to $L^2(\mathbb{R}^n)$.

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[1] Supported in part by NRF grant 2009-0072531 (Korea), and MEC/MICINN grants MTM2007-60952 and MTM2010-16518 (Spain).

Theorem 1 [7] *Let* $r \geq 2$ *and* $\frac{n}{q} + \frac{2}{r} \leqslant \frac{n}{2}$. *Then*[2]

$$\|e^{it\Delta} f\|_{L_t^r(\mathbb{R}, L_x^q(\mathbb{R}^n))} \leqslant C \|f\|_{\dot{H}^s(\mathbb{R}^n)}, \quad s = \frac{n}{2} - \frac{n}{q} - \frac{2}{r}.$$

Another way of measuring the integrability is to consider $L_x^q(\mathbb{R}^n, L_t^r(\mathbb{R}))$. As before the condition $r \geq 2$ is necessary, however the second condition changes as is easily verified by considering a Knapp example that is Fourier supported in $\{\xi : 1/2 \leqslant |\xi| \leqslant 2\}$. In this case the endpoint estimates have data contained in $\dot{H}^s(\mathbb{R}^n)$.

Conjecture 1 [13] *Let* $r \geq 2$ *and* $\frac{n+1}{q} + \frac{1}{r} \leqslant \frac{n}{2}$. *Then*[3]

$$\|e^{it\Delta} f\|_{L_x^q(\mathbb{R}^n, L_t^r(\mathbb{R}))} \leqslant C \|f\|_{\dot{H}^s(\mathbb{R}^n)}, \quad s = \frac{n}{2} - \frac{n}{q} - \frac{2}{r}. \tag{$S_{q,r}$}$$

In one spatial dimension, the conjecture was proven by Kenig, Ponce and Vega [10]. In higher dimensions, Vega [29] (see also [6], [13], [17]) proved that the conjecture is true when $q \geq \frac{2(n+2)}{n}$. The bilinear restriction estimate of Tao [23] and arguments of Planchon [13] (see also [16], [26]) can be combined to yield $(S_{q,r})$ when $r \in [2, \infty)$ and $\frac{n+1}{q} + \frac{1}{r} < \frac{n}{2}$, in the range $q > \frac{2(n+3)}{n+1}$.

In two spatial dimensions, this was further improved by the second author [14] so that $(S_{q,r})$ holds when $r \in [2, \infty)$ and $\frac{3}{q} + \frac{1}{r} < 1$, in the range $q > 16/5$. In Planchon's article [13], estimates on the sharp line $\frac{3}{q} + \frac{1}{r} = 1$ were also proven, using real interpolation techniques, but for the argument it was necessary to sacrifice part of the range in q.

In this article we prove estimates on the sharp line $\frac{n+1}{q} + \frac{1}{r} = \frac{n}{2}$ without loss in the range of q. Indeed, we show that local bilinear estimates yield endpoint linear estimates, from which we obtain the following theorem.

Theorem 2 *Let* $r \geq 2$ *and* $\frac{n+1}{q} + \frac{1}{r} = \frac{n}{2}$. *Then*

(i) $(S_{q,r})$ *holds when* $n = 2$ *and* $q > 16/5$

(ii) $(S_{q,r})$ *holds when* $n \geq 3$ *and* $q > \frac{2(n+3)}{n+1}$.

These kind of estimates have been applied to nonlinear dispersive equations (see for example [8], [9], [13]). By Sobolev embedding, one can also obtain estimates for the maximal operator with data in the inhomogeneous Sobolev space $H^s(\mathbb{R}^n)$, and such estimates imply almost everywhere convergence to the initial data as time tends to zero;

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n.$$

We recover the best known results (see [11], [19], [30]) in two and three spatial dimensions.

The major difference between the estimates in $L_t^r(\mathbb{R}, L_x^q(\mathbb{R}^n))$ and $L_x^q(\mathbb{R}^n, L_t^r(\mathbb{R}))$ is Galilean invariance, which is enjoyed by the former but not by the latter. That is to say, when the temporal integral is evaluated before the spatial integral, the estimates are not invariant under translation on the frequency side. This means that we cannot use the usual rescaling and translation arguments which simplify matters.

[2] occasionally excluding $q = \infty$

[3] occasionally excluding $r = \infty$

Imposing a separation condition on the Fourier supports, we first obtain bounds for the bilinear operator $(f,g) \to e^{it\Delta}fe^{it\Delta}g$ with f,g Fourier supported in a ball of radius one, and at a large distance from the origin. To get the endpoint linear estimates, we require bilinear bounds with a precise dependency on this distance from the origin. The following section will be dedicated to proving globalization lemmas which preserve this precise dependency. First we globalize estimates restricted to parallelepipeds to estimates which are global in space using decay properties and Schur's test. Then we globalize in time via an 'epsilon removal' argument. In the third section, we obtain the linear estimates from the bilinear ones in the spirit of [27].

In order to prove Theorem 2 (ii), only the third section is required. Combining the two sections reduces Conjecture 1 to local bilinear estimates, which enables the proof of Theorem 2 (i).

Throughout, c and C will denote positive constants that may depend on the dimensions and exponents of the Lebesgue spaces. Their values may change from line to line. The following are notations that will be used frequently:

$L^q_x(\mathbb{R}^n, L^r_t(I))$: the Lebesgue space with norm $\left(\int_{\mathbb{R}^n}\left[\int_I |f(x,t)|^r dt\right]^{q/r} dx\right)^{1/q}$
$\mathbb{A}^n := \{\, \xi \in \mathbb{R}^n \,:\, 1/2 \leqslant |\xi| \leqslant 2 \,\}$
$B_1(Ne_1) := \{\, \xi \in \mathbb{R}^n \,:\, |\xi - (N,0,\dots,0)| \leqslant 1 \,\}$
$s := \frac{n}{2} - \frac{n}{q} - \frac{2}{r}$
$\widehat{f}(\xi) := \int f(x)e^{-2\pi i x\cdot\xi}dx$
$\dot{H}^s(\mathbb{R}^n)$: the homogeneous Sobolev space with norm $\left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2|\xi|^{2s}d\xi\right)^{1/2}$

2. Globalization lemmas

We partition $\mathbb{R}^n$ into cubes $Q_{\mathbf{j}}$ of side R, centred at $R\mathbf{j} \in R\mathbb{Z}^n$, and for $N \gg 1$, we define parallelepipeds $P_{\mathbf{j}}$ by

$$P_{\mathbf{j}} = \{(x,t) \in \mathbb{R}^n \times [0,R] \,:\, x - 4\pi t N e_1 \in Q_{\mathbf{j}}\}. \tag{2}$$

Thus, $\{P_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^n}$ forms a partition of $\mathbb{R}^n \times [0,R]$.

Definition 1 *We say that E_1 and E_2 are λ-separated if they are measurable sets that satisfy*

$$\inf\{\, |\xi_1 - \xi_2| \,:\, \xi_1 \in E_1, \quad \xi_2 \in E_2\} \geq \lambda/2.$$

By adapting the Wolff–Tao induction on scales argument, the following bilinear estimate was proven in [14] (see also [11]).

Proposition 2.1 [14] *Let $n = 2$. Then for all $\varepsilon > 0$,*

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^{8/5}_x L^2_t(P_{\mathbf{0}})} \leqslant C_\varepsilon R^\varepsilon N^{1/8}\|f_1\|_2\|f_2\|_2$$

whenever $R, N \gg 1$, and $\widehat{f}_1, \widehat{f}_2$ are supported on 1-separated subsets of $B_1(Ne_1)$.

The Schrödinger wave does not have finite speed of propagation, however it behaves as if it had finite speed when the Fourier support of the initial datum is confined to a compact set. This can be made rigorous using the wave packet decomposition (see [11]). Since the initial data in the above estimates is Fourier supported in $B_1(Ne_1)$, the waves roughly propagate at speed N in the direction e_1. Hence, decomposing the initial data properly, the Schrödinger wave can be localized in space-time. This observation allows us to globalize the above estimate in space first.

Lemma 1 *Let $r \geq q$, $\varepsilon > 0$, and let $R, N \gg 1$. Suppose that*

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^q_x L^r_t(P_{\mathbf{0}})} \leqslant CR^{\varepsilon}N^{\frac{1}{q}-\frac{1}{r}}\|f_1\|_2\|f_2\|_2$$

whenever $\widehat{f_1}, \widehat{f_2}$ are supported on 1-separated subsets of $B_1(Ne_1)$. Then

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^q_x(\mathbb{R}^n, L^r_t[0,R])} \leqslant CR^{\varepsilon}N^{\frac{1}{q}-\frac{1}{r}}\|f_1\|_2\|f_2\|_2$$

whenever $\widehat{f_1}, \widehat{f_2}$ are supported on 1-separated subsets of $B_1(Ne_1)$.

Proof. Let η_1 and η_2 be smooth functions, that are equal to one on the supports of $\widehat{f_1}$ and $\widehat{f_2}$, respectively, and supported on slightly larger 4/5–separated sets. Define the projection operators $\mathcal{P}_i g = (\eta_i\, \widehat{g})^\vee$, where $i = 1, 2$, and the extension operators S_1 and S_2 by

$$S_1 g(x,t) = e^{it\Delta}\mathcal{P}_1 g(x) \quad \text{and} \quad S_2 h(x,t) = e^{it\Delta}\mathcal{P}_2 h(x).$$

As the projection operators are bounded in L^2, by scaling the hypothesis mildly, we have

$$\|S_1 g\, S_2 h\|_{L^q_x L^r_t(P_{\mathbf{0}})} \leqslant CR^{\varepsilon}N^{\frac{1}{q}-\frac{1}{r}}\|g\|_2\|h\|_2, \tag{3}$$

with no restriction on the Fourier supports of g and h.

As in [3] and [15], we write

$$g_{\mathbf{j}} = g\chi_{Q_{\mathbf{j}}},$$

where $\{Q_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^n}$ is a partition of $\mathbb{R}^n$ in cubes of side R, centred at $R\mathbf{j} \in R\mathbb{Z}^n$. For all $\mathbf{l} \in \mathbb{Z}^n$, we have the decomposition

$$g = \sum_{\mathbf{j}\in\mathbb{Z}^n} g_{\mathbf{j}} = \sum_{\mathbf{j}\,:\,|\mathbf{j}-\mathbf{l}|\leqslant 50n} g_{\mathbf{j}} \;+ \sum_{\mathbf{j}\,:\,|\mathbf{j}-\mathbf{l}|>50n} g_{\mathbf{j}}.$$

Now by Minkowski's inequality,

$$\begin{aligned}\|S_1 g\, S_2 h\|_{L^q(\mathbb{R}^n, L^r_t[0,R])} &\leqslant \sum_{\mathbf{l}} \|S_1 g\, S_2 h\|_{L^q_x L^r_t(P_{\mathbf{l}}))}\\ &\leqslant I + II + III + IV,\end{aligned}$$

where the parallelepipeds $P_{\mathbf{l}}$ are defined as in (2), and

$$\begin{aligned}
I &= \sum_{\substack{\mathbf{j},\,\mathbf{k},\,\mathbf{l}\,:\\ |\mathbf{j}-\mathbf{l}|\leqslant 50n,\ |\mathbf{k}-\mathbf{l}|\leqslant 50n}} \|S_1 g_{\mathbf{j}}\, S_2 h_{\mathbf{k}}\|_{L^q_x L^r_t(P_{\mathbf{l}})},\\
II &= \sum_{\substack{\mathbf{j},\,\mathbf{k},\,\mathbf{l}\,:\\ |\mathbf{j}-\mathbf{l}|> 50n,\ |\mathbf{k}-\mathbf{l}|\leqslant 50n}} \|S_1 g_{\mathbf{j}}\, S_2 h_{\mathbf{k}}\|_{L^q_x L^r_t(P_{\mathbf{l}})},\\
III &= \sum_{\substack{\mathbf{j},\,\mathbf{k},\,\mathbf{l}\,:\\ |\mathbf{j}-\mathbf{l}|\leqslant 50n,\ |\mathbf{k}-\mathbf{l}|> 50n}} \|S_1 g_{\mathbf{j}}\, S_2 h_{\mathbf{k}}\|_{L^q_x L^r_t(P_{\mathbf{l}})},\\
IV &= \sum_{\substack{\mathbf{j},\,\mathbf{k},\,\mathbf{l}\,:\\ |\mathbf{j}-\mathbf{l}|> 50n,\ |\mathbf{k}-\mathbf{l}|> 50n}} \|S_1 g_{\mathbf{j}}\, S_2 h_{\mathbf{k}}\|_{L^q_x L^r_t(P_{\mathbf{l}})}.
\end{aligned} \tag{4}$$

First we consider the main term I. By spatial translation invariance and (3),

$$\|S_1 g_{\mathbf{j}}\, S_2 h_{\mathbf{k}}\|_{L^q_x L^r_t(P_1)} \leqslant C R^{\varepsilon} N^{\frac{1}{q}-\frac{1}{r}} \|g_{\mathbf{j}}\|_2 \|h_{\mathbf{k}}\|_2,$$

so that, by three applications of Cauchy-Schwarz,

$$\begin{aligned} I &\leqslant C R^{\varepsilon} N^{\frac{1}{q}-\frac{1}{r}} \Bigg(\sum_{\mathbf{l}} \sum_{\mathbf{j}\,:\,|\mathbf{j}-\mathbf{l}|\leqslant 50n} \|g_{\mathbf{j}}\|_2^2\Bigg)^{1/2} \Bigg(\sum_{\mathbf{l}} \sum_{\mathbf{k}\,:\,|\mathbf{k}-\mathbf{l}|\leqslant 50n} \|h_{\mathbf{k}}\|_2^2\Bigg)^{1/2} \\ &\leqslant C R^{\varepsilon} N^{\frac{1}{q}-\frac{1}{r}} \|g\|_2\, \|h\|_2. \end{aligned}$$

Next we consider II. Since we are assuming $r \geq q$, by applications of Hölder and Fubini, we see that

$$\|S_1 g_{\mathbf{j}}\, S_2 h_{\mathbf{k}}\|_{L^q_x L^r_t(P_1))} \leqslant (R^n N)^{\frac{1}{q}-\frac{1}{r}} \|S_1 g_{\mathbf{j}}\|_{L^{2r}_t L^{2r}_x(P_1)} \|S_2 h_{\mathbf{k}}\|_{L^{2r}_t L^{2r}_x(P_1)}.$$

By Young's inequality followed by the L^2–boundedness of $e^{it\Delta}$,

$$\|S_2 h_{\mathbf{k}}(\,\cdot\,,t)\|_{L^{2r}_x} = \|\eta_2^{\vee} * e^{it\Delta} h_{\mathbf{k}}\|_{L^{2r}_x} \leqslant C\|h_{\mathbf{k}}\|_2,$$

so that

$$\|S_2 h_{\mathbf{k}}\|_{L^{2r}_t L^{2r}_x(P_1)} \leqslant C R^{\frac{1}{2r}} \|h_{\mathbf{k}}\|_2. \tag{5}$$

For $g_{\mathbf{j}}$ with $|\mathbf{j}-\mathbf{l}| > 50n$ we obtain the improved estimate

$$\|S_1 g_{\mathbf{j}}\|_{L^{2r}_t L^{2r}_x(P_1)} \leqslant C_M R^{-M} |\mathbf{j}-\mathbf{l}|^{-M} \|g_{\mathbf{j}}\|_2, \quad M \in \mathbb{N}. \tag{6}$$

To see this, by an affine change of variables,

$$\|S_1 g_{\mathbf{j}}\|_{L^{2r}_t L^{2r}_x(P_1)} = \|\widetilde{S}_1 \widetilde{g}_{\mathbf{j}}\|_{L^{2r}_t L^{2r}_x(Q_1 \times [0,R])};$$

here $\widetilde{g}_{\mathbf{j}}(x) = e^{-2\pi i N x_1} g_{\mathbf{j}}(x)$ and $\widetilde{S}_1 = e^{it\Delta}\widetilde{\mathcal{P}}_1$, where $\widetilde{\mathcal{P}}_1$ is the projection operator associated to $\widetilde{\eta}_1 = \eta_1(\,\cdot\, + N_1 e_1)$. Writing $\widetilde{S}_1 g(\cdot,t) = K_t * g$, the decay properties of the kernel K_t are well-known. Indeed, on the support of $\widetilde{\eta}_1$, we have

$$|\nabla(y\cdot\xi - 2\pi t|\xi|^2)| \geq c|y|, \quad |y| \geq 15R, \quad t \in [0,R],$$

so that by integrating by parts,

$$|K_t(y)| = \left|\int_{\mathbb{R}^n} \widetilde{\eta}_1(\xi) e^{2\pi i(y\cdot\xi - 2\pi t|\xi|^2)} d\xi\right| \leqslant C_M |y|^{-M}, \quad |y| \geq 15R, \quad t \in [0,R].$$

From this we see that

$$\begin{aligned} \|\widetilde{S}_1 \widetilde{g}_{\mathbf{j}}\|_{L^{2r}_t L^{2r}_x(Q_1\times[0,R])} &\leqslant C_M \Big(\int_0^R \int_{\mathbb{R}^n} \Big|\int_{|y|\geq \frac{1}{2}|\mathbf{j}-\mathbf{l}|R} |y|^{-M} |g_{\mathbf{j}}|(x-y)\, dy\Big|^{2r} dxdt\Big)^{\frac{1}{2r}} \\ &\leqslant C_M |\mathbf{j}-\mathbf{l}|^{n-M} R^{n+1-M} \|g_{\mathbf{j}}\|_2, \end{aligned}$$

where the second inequality is by Young's inequality. This yields (6).

Substituting (5) and (6) into (4), we see that

$$\begin{aligned} II &\leqslant C_M R^{-M} N^{\frac{1}{q}-\frac{1}{r}} \sum_{\substack{\mathbf{j},\,\mathbf{k},\,\mathbf{l}\,: \\ \mathbf{j}\neq\mathbf{l},\ |\mathbf{k}-\mathbf{l}|\leqslant 50n}} |\mathbf{j}-\mathbf{l}|^{-M} \|g_{\mathbf{j}}\|_2 \|h_{\mathbf{k}}\|_2 \\ &\leqslant C_M R^{-M} N^{\frac{1}{q}-\frac{1}{r}} \sum_{\substack{\mathbf{j},\,\mathbf{l}\,: \\ \mathbf{j}\neq\mathbf{l}}} |\mathbf{j}-\mathbf{l}|^{-M} \|g_{\mathbf{j}}\|_2 \Big(\sum_{\mathbf{k}\,:\,|\mathbf{k}-\mathbf{l}|\leqslant 50n} \|h_{\mathbf{k}}\|_2^2\Big)^{1/2}. \end{aligned}$$

Finally, by Schur's test (see for example [5]), we see that

$$II \leqslant CR^{-M}N^{\frac{1}{q}-\frac{1}{r}}\|g\|_2\,\|h\|_2,$$

and by symmetry this is also true of III.

Now we consider IV. Substituting (6) into (4), we have

$$IV \leqslant C_M R^{-M}N^{\frac{1}{q}-\frac{1}{r}} \sum_{\substack{\mathbf{j},\mathbf{k},\mathbf{l}\\ \mathbf{j}\neq\mathbf{l},\,\mathbf{k}\neq\mathbf{l}}} |\mathbf{j}-\mathbf{l}|^{-M}|\mathbf{k}-\mathbf{l}|^{-M}\|g_{\mathbf{j}}\|_2\|h_{\mathbf{k}}\|_2.$$

For sufficiently large M, by three applications of Cauchy–Schwarz and orthogonality,

$$\begin{aligned} IV &\leqslant C_M R^{-M}N^{\frac{1}{q}-\frac{1}{r}} \sum_{\mathbf{l}} \Big(\sum_{\mathbf{j}\,:\,\mathbf{j}\neq\mathbf{l}} |\mathbf{j}-\mathbf{l}|^{-M}\|g_{\mathbf{j}}\|_2^2\Big)^{\frac{1}{2}} \Big(\sum_{\mathbf{k}\,:\,\mathbf{k}\neq\mathbf{l}} |\mathbf{k}-\mathbf{l}|^{-M}\|h_{\mathbf{k}}\|_2^2\Big)^{\frac{1}{2}} \\ &\leqslant C_M R^{-M}N^{\frac{1}{q}-\frac{1}{r}}\|g\|_2\|h\|_2. \end{aligned}$$

Putting the estimates for $I-IV$ together, we get

$$\|S_1 g\, S_2 h\|_{L^q(\mathbb{R}^n,\,L^r_t[0,R])} \leqslant C\,R^{\varepsilon}N^{\frac{1}{q}-\frac{1}{r}}\|g\|_2\,\|h\|_2.$$

Finally, taking $g = f_1$, $h = f_2$, we have $S_1 f_1 = e^{it\Delta}f_1$, $S_2 f_2 = e^{it\Delta}f_2$, and we are done. $\square$

For interpolation purposes, we will use the following elementary lemma which can be shown by applying Plancherel's theorem in t and interpolation. For a proof see [14].

Lemma 2 *Let $r \geq 1$. For $N \gg 1$ and f_1, f_2 Fourier supported in $B_1(Ne_1)$,*

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^\infty_x(\mathbb{R}^n,L^r_t(\mathbb{R}))} \leqslant CN^{-1/r}\|f_1\|_2\,\|f_2\|_2.$$

The following lemma is similar to one contained in [12] where the spatial integral is evaluated before the temporal integral. In [14], a version with the order reversed is presented, but with a loss in the power of N. In both articles, the hypothesis supposes an estimate which is local in both space and time. By the previous Lemma 1, we can suppose an estimate which is global in space, and this enables us to conserve the power of N.

Lemma 3 *Let $r_0 > q_0$ and $R, N \gg 1$. Suppose that for all $\varepsilon > 0$*

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^{q_0}_x(\mathbb{R}^n,L^{r_0}_t[0,R])} \leqslant C_\varepsilon R^{\varepsilon}N^{\frac{1}{q_0}-\frac{1}{r_0}}\|f_1\|_2\|f_2\|_2$$

whenever $\widehat{f_1}, \widehat{f_2}$ are supported on 1-separated subsets of $B_1(Ne_1)$. Then provided that $q > q_0$ and $r > r_0$,

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^q_x L^r_t(\mathbb{R}^{n+1})} \leqslant C_{q,r}N^{\frac{1}{q}-\frac{1}{r}}\|f_1\|_2\|f_2\|_2,$$

whenever $\widehat{f_1}, \widehat{f_2}$ are supported on 1-separated subsets of $B_1(Ne_1)$.

Proof. One can calculate that the temporal Fourier transform of $e^{it\Delta}f_1$ is contained in an interval of length $\leqslant CN$. Similarly this is true of $e^{it\Delta}f_1\, e^{it\Delta}f_2$. Thus, by Bernstein's inequality,

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^{r_1}_t(\mathbb{R})} \leqslant CN^{\frac{1}{r_2}-\frac{1}{r_1}}\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^{r_2}_t(\mathbb{R})}, \quad r_1 > r_2,$$

and so, by interpolation with Lemma 2, it will suffice to prove

$$\|e^{it\Delta}f_1\, e^{it\Delta}f_2\|_{L^q_xL^r_t(\mathbb{R}^{n+1})} \leqslant C\, N^{\frac{1}{q}-\frac{1}{r}}\|f_1\|_2\|f_2\|_2.$$

for some q and r arbitrarily close to q_0 and r_0.

Define the measure $d\sigma$ to be the canonical pull-back measure on

$$S = \{(\xi, -2\pi|\xi|^2) \in \mathbb{R}^{n+1} \,:\, \xi \in B_1(Ne_1)\}.$$

It is well-known (see [20]) that the Fourier transform decays,

$$|(d\sigma)^\vee(x,t)| \leqslant C_\sigma(1+|x-4\pi tNe_1|+|t|)^{-n/2} \tag{7}$$

Writing

$$g_1(\xi,-2\pi|\xi|^2) = \widehat{f}_1(\xi), \quad \text{and} \quad g_2(\xi,-2\pi|\xi|^2) = \widehat{f}_2(\xi),$$

by Plancherel, it will suffice to prove that

$$\|\prod_{i=1}^2 (g_i d\sigma)^\vee\|_{L^q_xL^r_t(\mathbb{R}^{n+1})} \leqslant CN^{\frac{1}{q}-\frac{1}{r}}\prod_{i=1}^2\|g_i\|_{L^2(d\sigma)},$$

for q and r arbitrarily close to q_0 and r_0. For notational convenience we normalize the measure so that (7) is satisfied with $C_\sigma = 1/2$.

We consider E defined by

$$E = \left\{ (x,t) \in \mathbb{R}^{n+1} \;:\; \prod_{i=1}^2 |(g_i d\sigma)^\vee(x,t)| > \lambda \right\},$$

and for each x, we set $E_x = \{\, t \,:\, (x,t) \in E\,\}$. For a fixed $\nu > 0$, we define

$$E(\nu) = \bigcup_{x\,:\,\nu\leqslant|E_x|<2\nu} \{\,(x,t) \,:\, t \in E_x\,\}, \tag{8}$$

and we also set

$$X(\nu) = \{\, x \in \mathbb{R}^n \,:\, \nu \leqslant |E_x| < 2\nu\,\}, \quad \mu = |X(\nu)|,$$

so that $\mu\nu \leqslant |E(\nu)| \leqslant 2\mu\nu$.

First we use the hypothesis to prove that

$$\|\chi_{E(\nu)}h_1^\vee h_2^\vee\|_1 \leqslant C_\varepsilon R^{\varepsilon-1}N^{\frac{1}{q_0}-\frac{1}{r_0}}\mu^{\frac{1}{q_0'}}\nu^{\frac{1}{r_0'}}\|h_1\|_{L^2(S_R)}\|h_2\|_{L^2(S_R)}, \quad R \gg 1 \tag{9}$$

whenever h_1, h_2 are supported in 1-separated subsets of

$$S_R = \{\,(\xi,\tau) \in \mathbb{R}^{n+1} \,:\, \xi \in B_1(Ne_1), \quad |\tau + 2\pi|\xi|^2| \leqslant R^{-1}\,\}.$$

Proof of (9). Let $\widehat{\phi}$ be a smooth function supported in $(-1,1)$ and equal to one on $[-4/5, 4/5]$ such that

$$\sum_{k\in\mathbb{Z}} \phi^4(\,\cdot\, - k) = 1.$$

As $\phi^2(R^{-1}\,\cdot\,) \leqslant C\sum_j 2^{-100j}\chi_{[-2^jR,\,2^jR]}$, by the hypothesis and temporal translation invariance,

$$\|\phi_k^2 \prod_{i=1}^2 (g_i\, d\sigma)^\vee\|_{L^{q_0}_x L^{r_0}_t} \leqslant C_\varepsilon R^\varepsilon N^{\frac{1}{q_0}-\frac{1}{r_0}}\prod_{i=1}^2\|g_i\|_{L^2(d\sigma)}, \tag{10}$$

where $\phi_k(t) = \phi(R^{-1}t - k)$ for $k \in \mathbb{Z}$. For h_i supported in S_R we can write

$$\phi_k h_i^\vee(x,t) = \int_{-2R^{-1}}^{2R^{-1}} (H_i^{\tau_i} d\sigma)^\vee(x,t) e^{2\pi i t\tau_i}\, d\tau_i,$$

where $H_i^{\tau_i} = \widehat{\phi}_k * h_i(\xi, \tau_i - 2\pi|\xi|^2)$.[4] Thus, by Minkowski's integral inequality and (10),

$$\begin{aligned}\|\, \phi_k^4 h_1^\vee h_2^\vee \|_{L^{q_0} L^{r_0}} &\leqslant \int_{-2R^{-1}}^{2R^{-1}} \int_{-2R^{-1}}^{2R^{-1}} \Big\| \phi_k^2 \prod_{i=1}^{2} (H_i^{\tau_i} d\sigma)^\vee \Big\|_{L^{q_0} L^{r_0}} d\tau_1 d\tau_2 \\ &\leqslant C_\varepsilon R^\varepsilon N^{\frac{1}{q_0} - \frac{1}{r_0}} \int_{-2R^{-1}}^{2R^{-1}} \int_{-2R^{-1}}^{2R^{-1}} \prod_{i=1}^{2} \|H_i^{\tau_i}\|_{L^2(d\sigma)}\, d\tau_1 d\tau_2\end{aligned}$$

whenever h_1, h_2 are supported in 1-separated subsets of S_R. By Cauchy–Schwarz and Plancherel's theorem, we get

$$\|\, \phi_k^4 h_1^\vee h_2^\vee \|_{L^{q_0} L^{r_0}} \leqslant C_\varepsilon R^{\varepsilon - 1} N^{\frac{1}{q_0} - \frac{1}{r_0}} \|\phi_k h_1^\vee\|_2 \|\phi_k h_2^\vee\|_2,$$

which, by Cauchy–Schwarz, almost orthogonality and Plancherel, yields

$$\|h_1^\vee h_2^\vee\|_{L^{q_0} L^{r_0}} \leqslant \sum_k \|\phi_k^4 h_1^\vee h_2^\vee\|_{L^{q_0} L^{r_0}} \leqslant C_\varepsilon R^{\varepsilon-1} N^{\frac{1}{q_0} - \frac{1}{r_0}} \|h_1\|_2 \|h_2\|_2.$$

Finally, by Hölder's inequality $\|\chi_{E(\nu)} h_1^\vee h_2^\vee\|_{L^1} \leqslant \mu^{\frac{1}{q_0'}} \nu^{\frac{1}{r_0'}} \|h_1^\vee h_2^\vee\|_{L^{q_0} L^{r_0}}$, which completes the proof of (9).

We now use the decay of the Fourier transform of the measure to remove the epsilon. We will prove that whenever $R \gg 1$,

$$\|\chi_{E(\nu)} \prod_{i=1}^{2} (g_i d\sigma)^\vee\|_{L^1} \leqslant \left[C_0 R^{-\frac{n}{4}} (\mu\nu)^{\frac{n+4}{2(n+2)}} + C_\varepsilon R^\varepsilon N^{\frac{1}{q_0} - \frac{1}{r_0}} \mu^{\frac{1}{q_0'}} \nu^{\frac{1}{r_0'}} \right] \prod_{i=1}^{2} \|g_i\|_{L^2(d\sigma)}. \quad (11)$$

Proof of (11). (see also [25]) In order to apply some duality arguments we will prove

$$\|\chi_{E(\nu)} (h_1 d\sigma)^\vee (h_2 d\sigma)^\vee\|_1 \leqslant A \|h_1\|_{L^2(d\sigma)} \|h_2\|_{L^2(d\sigma)}, \quad (12)$$

where

$$A = C_0 R^{-\frac{n}{4}} (\mu\nu)^{\frac{n+4}{2(n+2)}} + C_\varepsilon R^\varepsilon N^{\frac{1}{q_0} - \frac{1}{r_0}} \mu^{\frac{1}{q_0'}} \nu^{\frac{1}{r_0'}}$$

and h_1, h_2 are supported[5] on 1-separated subsets of S, and are completely independent of g_1, g_2. We will also assume that $\|h_i\|_{L^2(d\sigma)} = 1$ for $i = 1, 2$.

Defining the linear operator T by

$$Th_1 = \chi_{E(\nu)} (h_1 d\sigma)^\vee (h_2 d\sigma)^\vee,$$

it will suffice to prove that

$$\|Th_1\|_1 \leqslant A \|h_1\|_{L^2(d\sigma)}.$$

By duality, this follows from showing that for $\|F\|_\infty \leqslant 1$,

$$\|T^* F\|_{L^2(d\sigma)} \leqslant A$$

[4] here the convolution is in the t variable

[5] note that sometimes h_1 and h_2 will be supported on S and at other times on S_R.

where T^* is the adjoint operator defined by

$$T^*F = \left(\chi_{E(\nu)}(h_2 d\sigma)^\vee F\right)^\wedge.$$

By squaring and applying Plancherel, it will suffice to prove that

$$|\langle G * (d\sigma)^\vee, G\rangle| \leqslant A^2,$$

where $G = \chi_{E(\nu)}(h_2 d\sigma)^\vee F$. Note that by Theorem 1

$$\|G\|_1 \leqslant \|\chi_{E(\nu)}\|_{\frac{2(n+2)}{n+4}} \|(h_2 d\sigma)^\vee\|_{\frac{2(n+2)}{n}} \|F\|_\infty \leqslant C_0(\mu\nu)^{\frac{n+4}{2(n+2)}}.$$

Now for ϕ as before we write $d\sigma = d\sigma^R + d\sigma_R$, where

$$(d\sigma_R)^\vee(x,t) = \phi^\vee(R^{-1}t)(d\sigma)^\vee(x,t).$$

From (7), we have $\|(d\sigma^R)^\vee\|_\infty \leqslant R^{-\frac{n}{2}}$, so that

$$|\langle G * (d\sigma^R)^\vee, G\rangle| \leqslant C_0^2 R^{-\frac{n}{2}}(\mu\nu)^{\frac{n+4}{(n+2)}}.$$

It remains to prove that

$$|\langle G * (d\sigma_R)^\vee, G\rangle| \leqslant A^2.$$

Now

$$d\sigma_R(\xi,\tau) = R\phi\left(R(\tau + 2\pi|\xi|^2)\right) d\xi d\tau,$$

so that by Plancherel and decomposing dyadically,

$$|\langle G * (d\sigma_R)^\vee, G\rangle| \leqslant C \sum_{j\geq 0} R 2^{-100j} \|\widehat{G}\|^2_{L^2(S_{2^{-j}R})},$$

where for $j \geq 1$,

$$S_{2^{-j}R} = \{(\xi,\tau) \in \mathbb{R}^{n+1} \,:\, \xi \in B_1(Ne_1), \quad 2^{j-1}R^{-1} \leqslant |\tau + 2\pi|\xi|^2| \leqslant 2^j R^{-1}\}.$$

We treat the $j = 0$ case; the others are aided by the 2^{-100j} factor. It suffices to show that

$$\|\widehat{G}\|_{L^2(S_R)} \leqslant R^{-\frac{1}{2}} A.$$

By the definition of G, this would follow from

$$\|(\chi_{E(\nu)}(h_2 d\sigma)^\vee F)^\wedge\|_{L^2(S_R)} \leqslant R^{-\frac{1}{2}} A \|F\|_\infty,$$

which by duality would follow from

$$\|\chi_{E(\nu)} h_1^\vee (h_2 d\sigma)^\vee\|_1 \leqslant R^{-\frac{1}{2}} A \|h_1\|_{L^2(S_R)} \|h_2\|_{L^2(d\sigma)}.$$

Note that we reduced (12) to the above by fixing $(h_2 d\sigma)^\vee$. Hence, fixing $h_1^\vee$, we can repeat the argument with $(h_2 d\sigma)^\vee$ in the place of $(h_1 d\sigma)^\vee$ so that it will suffice to prove

$$\|\chi_{E(\nu)} h_1^\vee h_2^\vee\|_1 \leqslant C_\varepsilon R^{\varepsilon-1} N^{\frac{1}{q_0} - \frac{1}{r_0}} \mu^{\frac{1}{q_0'}} \nu^{\frac{1}{r_0'}} \|h_1\|_{L^2(S_R)} \|h_2\|_{L^2(S_R)},$$

where h_1, h_2 are supported in S_R, which is (9), which completes the proof of (11).

Letting

$$\frac{1}{q_1} = \frac{1}{q_0} - \frac{4\varepsilon}{n+4\varepsilon}\left(\frac{1}{q_0} - \frac{n}{2(n+2)}\right), \quad \frac{1}{r_1} = \frac{1}{r_0} - \frac{4\varepsilon}{n+4\varepsilon}\left(\frac{1}{r_0} - \frac{n}{2(n+2)}\right),$$

we use (9) to obtain

$$\|\chi_{E(\nu)}\|_{L_x^{q_1}L_t^{r_1}} \leqslant C\lambda^{-1}N^{\frac{1}{q_1}-\frac{1}{r_1}}\prod_{i=1}^{2}\|g_i\|_{L^2(d\sigma)}. \tag{13}$$

Proof of (13). We may assume that $\|g_i\|_{L^2(d\sigma)} = 1$ for $i = 1, 2$. Now, if

$$C_0(\mu\nu)^{\frac{n+4}{2(n+2)}} > C_\varepsilon N^{\frac{1}{q_0}-\frac{1}{r_0}}\mu^{\frac{1}{q_0'}}\nu^{\frac{1}{r_0'}},$$

by choosing $R \gg 1$ so that

$$R^{-\frac{n}{4}}(\mu\nu)^{\frac{n+4}{2(n+2)}} = CR^\varepsilon N^{\frac{1}{q_0}-\frac{1}{r_0}}\mu^{\frac{1}{q_0'}}\nu^{\frac{1}{r_0'}}$$

is satisfied, from (11) we get

$$\Big\|\chi_{E(\nu)}\prod_{i=1}^{2}(g_i d\sigma)^\vee\Big\|_{L^1} \leqslant CN^{(\frac{1}{q_0}-\frac{1}{r_0})(1-\frac{4\varepsilon}{n+4\varepsilon})}\mu^{\frac{1}{q_1'}}\nu^{\frac{1}{r_1'}}.$$

Now (13) follows as

$$\lambda\mu\nu \leqslant \lambda|E(\nu)| \leqslant \Big\|\chi_{E(\nu)}\prod_{i=1}^{2}(g_i d\sigma)^\vee\Big\|_{L^1},$$

and $\|\chi_{E(\nu)}\|_{L^{q_1}L^{r_1}} \leqslant 2\mu^{\frac{1}{q_1}}\nu^{\frac{1}{r_1}}$.

If $C_0(\mu\nu)^{\frac{n+4}{2(n+2)}} < C_\varepsilon N^{\frac{1}{q_0}-\frac{1}{r_0}}\mu^{\frac{1}{q_0'}}\nu^{\frac{1}{r_0'}}$, taking a small $R > 1$ in (11) we get

$$\mu^{\frac{1}{q_0}}\nu^{\frac{1}{r_0}} \leqslant CN^{\frac{1}{q_0}-\frac{1}{r_0}}\lambda^{-1}.$$

On the other hand, by the Theorem 1 combined with Chebychev and Cauchy–Schwarz,

$$(\mu\nu)^{\frac{n}{n+2}} \leqslant C\lambda^{-1}.$$

Now as $\prod_{i=1}^{2}|(g_i d\sigma)^\vee| \leqslant 1$, we may assume that $\lambda \leqslant 1$, so that

$$\begin{aligned}\mu^{\frac{1}{q_1}}\nu^{\frac{1}{r_1}} &= (\mu^{\frac{1}{q_0}}\nu^{\frac{1}{r_0}})^{1-\frac{4\varepsilon}{n+4\varepsilon}}(\mu\nu)^{\frac{n}{2(n+2)}\frac{4\varepsilon}{n+4\varepsilon}}\\ &\leqslant C\lambda^{-1}N^{(\frac{1}{q_0}-\frac{1}{r_0})(1-\frac{4\varepsilon}{n+4\varepsilon})},\end{aligned}$$

which completes the proof of (13).

For $p > q_1$ we now prove the weak type inequality

$$\lambda\|\chi_E\|_{L_x^p L_t^{r_1}} \leqslant CN^{\frac{1}{p}-\frac{1}{r_1}}\prod_{i=1}^{2}\|g_i\|_{L^2(d\sigma)}. \tag{14}$$

Proof of (14). Again, we may assume that $\|g_i\|_{L^2(d\sigma)} = 1$ for $i = 1, 2$ and $\lambda \leqslant 1$. For $\nu = 2^k$, define $E(2^k)$ as in (8) and decompose $E = \bigcup_k E(2^k)$. For each fixed x, by Lemma 2,

$$\lambda\Big|\Big\{\,t : \prod_{i=1}^{2}\big|(g_i d\sigma)^\vee(x,t)\big| > \lambda\,\Big\}\Big| \leqslant \Big\|\prod_{i=1}^{2}(g_i d\sigma)^\vee(x,\cdot)\Big\|_{L_t^1} \leqslant cN^{-1}. \tag{15}$$

Therefore, we only need to consider the case $2^k \leqslant c(N\lambda)^{-1}$. We have,

$$\begin{aligned}\|\chi_E\|^p_{L^p_x L^{r_1}_t} = \int_{\mathbb{R}^n} |E_x|^{p/r_1}\,dx &\leqslant C \sum_{k:\ 2^k \leqslant c(N\lambda)^{-1}} 2^{kp/r_1} |X(2^k)| \\ &\leqslant C \sum_{k:\ 2^k \leqslant c(N\lambda)^{-1}} 2^{k(p-q_1)/r_1} \sup_k |X(2^k)| 2^{kq_1/r_1} \\ &\leqslant C \sum_{k:\ 2^k \leqslant c(N\lambda)^{-1}} 2^{k(p-q_1)/r_1} \sup_k \|\chi_{E(2^k)}\|^{q_1}_{L^{q_1}_x L^{r_1}_t}.\end{aligned}$$

We use (13) and sum the geometric series to obtain

$$\begin{aligned}\|\chi_E\|^p_{L^p_t L^{r_1}_x} &\leqslant C\lambda^{-q_1}(N\lambda)^{-(p-q_1)/r_1} N^{1-\frac{q_1}{r_1}} \\ &\leqslant C\lambda^{-p}\lambda^{(p-q_1)(1-1/r_1)} N^{1-\frac{p}{r_1}}.\end{aligned}$$

Since $p > q_1$ and $\lambda \leqslant 1$ we get (14).

We now complete the proof by obtaining the strong type estimate. Again, we may assume that $\|g_i\|_{L^2(d\sigma)} = 1$ for $i = 1, 2$, so that $\prod_{i=1}^2 |(g_i d\sigma)^\vee| \leqslant 1$, and write

$$\prod_{i=1}^2 |(g_i d\sigma)^\vee| \leqslant \sum_{k\geq 0} 2^{-k}\chi_{E_k}$$

where

$$E_k = \Big\{(x,t) : 2^{-k} < \prod_{i=1}^2 |(g_i d\sigma)^\vee(x,t)| \leqslant 2^{-k+1}\Big\}.$$

Since $r_0 > q_0$, we can choose ε sufficiently small so that $q_1 < r_1$, and fix q such that $q_1 < p < q < r_1$. Then,

$$\Big\|\prod_{i=1}^2 (g_i d\sigma)^\vee\Big\|_{L^q L^{r_1}} \leqslant C\Big(\int \Big(\sum_{k\geq 0} 2^{-kr_1}|(E_k)_x|\Big)^{q/r_1} dx\Big)^{1/q}.$$

By concavity we bound this by

$$\Big\|\prod_{i=1}^2 (g_i d\sigma)^\vee\Big\|_{L^q L^{r_1}} \leqslant C\Big(\sum_{k\geq 0} \int 2^{-kq}|(E_k)_x|^{q/r_1}\,dx\Big)^{1/q}.$$

By (15), we have $|(E_k)_x| \leqslant cN^{-1}2^k$, so that

$$\Big\|\prod_{i=1}^2 (g_i d\sigma)^\vee\Big\|^q_{L^q L^{r_1}} \leqslant C\sum_{k\geq 0} 2^{-kq}(N^{-1}2^k)^{\frac{q}{r_1}-\frac{p}{r_1}} \|\chi_{E_k}\|^p_{L^p L^{r_1}},$$

and by (14), the right hand side of the above is bounded by

$$C\sum_{k\geq 0} 2^{-k(q-p)(1-1/r_1)} N^{-\frac{q}{r_1}+\frac{p}{r_1}} N^{(\frac{1}{p}-\frac{1}{r_1})p} \prod_{i=1}^2 \|g_i\|^q_{L^2(d\sigma)}.$$

Thus, by summing the geometric series, we obtain

$$\Big\|\prod_{i=1}^2 (g_i d\sigma)^\vee\Big\|_{L^q L^{r_1}} \leqslant CN^{\frac{1}{q}-\frac{1}{r_1}} \prod_{i=1}^2 \|g_i\|_{L^2(d\sigma)},$$

and we are done. $\square$

Combining Lemmas 1 and 3, we see that global bilinear estimates follow from estimates restricted to parallelepipeds, with no loss in the power of N. Combining with Proposition 2.1, we obtain the following corollary.

Corollary 1 *Let $q > 8/5$ and $r > 2$. Then*

$$\|e^{it\Delta} f_1 \, e^{it\Delta} f_2\|_{L^q_x(\mathbb{R}^2,\, L^r_t(\mathbb{R}))} \leqslant C N^{\frac{1}{q}-\frac{1}{r}} \|f_1\|_2 \|f_2\|_2$$

whenever $N \gg 1$, and $\widehat{f}_1, \widehat{f}_2$ are supported on 1-separated subsets of $B_1(Ne_1)$.

3. Bilinear estimates imply linear estimates

By scaling and rotational invariance, the estimates of the previous section yield estimates of the form:

Definition 2 *We denote by $R^*(2 \times 2 \to (q, r))$ the estimate*

$$\|e^{it\Delta} f_1 \, e^{it\Delta} f_2\|_{L^q_x L^r_t} \leqslant C 2^{j(\frac{n+1}{q}+\frac{1}{r}-n)} \|f_1\|_2 \|f_2\|_2$$

whenever $\widehat{f}_1, \widehat{f}_2$ are supported in 2^{-j}-separated subsets of $B_{2^{-j}}(\xi_0) \subset \mathbb{A}^n$.

Indeed, by an application of Cauchy–Schwarz and interpolation with Lemma 2, the estimates in [6, 13, 17, 29] yield $R^*(2 \times 2 \to (q, 1))$ when $q \geq \frac{n+1}{n-1}$. Similarly, using the localization of the temporal Fourier support as in the proof of Lemma 3, by Bernstein's inequality, Tao's estimate [23] yields $R^*(2 \times 2 \to (q, r))$ when $r \geq q > \frac{n+3}{n+1}$. Interpolating the two we see that $R^*(2 \times 2 \to (q, r))$ holds when

$$q > \frac{n+3}{n+1}, \quad \text{and} \quad \frac{n+1}{q} + \frac{2}{r} < n+1.$$

In two spatial dimensions, this is improved by interpolation with Corollary 1, so that $R^*(2 \times 2 \to (q, r))$ holds when

$$q > 8/5, \quad \frac{3}{q} + \frac{2}{r} < 3, \quad \text{and} \quad \frac{4}{q} + \frac{1}{r} < 3.$$

In particular,

$$R^*\big(2 \times 2 \to \big(\frac{q}{2}, \frac{r}{2}\big)\big) \qquad \text{holds when} \quad q > 16/5, \quad r \geq 4, \quad n = 2, \tag{16}$$

and

$$R^*\big(2 \times 2 \to \big(\frac{q}{2}, \frac{r}{2}\big)\big) \qquad \text{holds when} \quad q > \frac{2(n+3)}{n+1}, \quad r > \frac{2(n+3)}{n+1}. \tag{17}$$

Definition 3 *We denote by $R^*(2 \to (q, r))$ the estimate*

$$\|e^{it\Delta} f\|_{L^q_x L^r_t} \leqslant C \|f\|_2$$

whenever $\widehat{f}$ is supported in $\mathbb{A}^n$.

Let $q \in [2,\infty]$ and $r \in [2,\infty)$. Let ψ be a smooth and positive function, supported in $(1/2,2)$, that satisfies

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}|\cdot|) = 1,$$

and set $\widehat{f}_k = \psi(2^{-k}|\cdot|)\widehat{f}$. Since the temporal Fourier transform of $e^{it\Delta} f_k$ is contained in the set $\{\tau : |\tau| \sim 2^{2k}\}$, by the Littlewood–Paley inequality

$$\Big\| \sum_k e^{it\Delta} f_k(x) \Big\|_{L^r_t} \leqslant C \Big\| \Big(\sum_k |e^{it\Delta} f_k(x)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^r_t}$$

Using Minkowski's inequality in $L^{r/2}_t$ then $L^{q/2}_x$, we have

$$\|e^{it\Delta} f\|^2_{L^q_x L^r_t} \leqslant C \sum_{k=-\infty}^{\infty} \|e^{it\Delta} f_k\|^2_{L^q_x L^r_t}.$$

Assuming $R^*(2 \to (q,r))$ and scaling, we get $\|e^{it\Delta} f_k\|_{L^q_x L^r_t} \leqslant C 2^{k(\frac{n}{2}-\frac{n}{q}-\frac{2}{r})} \|f_k\|_2$, so that

$$\|e^{it\Delta} f\|^2_{L^q_x L^r_t} \leqslant C \sum_{k=-\infty}^{\infty} 2^{2k(\frac{n}{2}-\frac{n}{q}-\frac{2}{r})} \|f_k\|^2_2.$$

By Plancherel's theorem, we get the desired estimate $(S_{q,r})$ of Conjecture 1. Hence in order to prove $(S_{q,r})$ it is enough to prove $R^*(2 \to (q,r))$.

Recalling that the estimates $(S_{q,r})$ for $q \geq \frac{2(n+2)}{n}$ were already proven in [6, 13, 17, 29], Theorem 2 is a consequence of (16) and (17) combined with the following proposition.

Proposition 3.1 *Let $q_0, r_0 \in (2,\infty)$ and $\frac{n+1}{q_0} + \frac{1}{r_0} = \frac{n}{2}$. If $R^*(2 \times 2 \to (\frac{q}{2}, \frac{r}{2}))$ holds for (q,r) in a neighbourhood of (q_0,r_0), then $R^*(2 \to (q_0,r_0))$ holds.*

Proof. By Plancherel, it is enough to prove

$$\|e^{it\Delta} \widehat{f}\|_{L^{q_0}_x L^{r_0}_t} \leqslant C \|f\|_2$$

where f is supported in $\mathbb{A}^n$. In order to apply the bilinear estimate, we square the integral as in [3], so that

$$\|e^{it\Delta} \widehat{f}\|^2_{L^{q_0}_x L^{r_0}_t} = \|e^{it\Delta} \widehat{f}\, e^{it\Delta} \widehat{f}\|_{L^{q_0/2}_x L^{r_0/2}_t}.$$

For each $k \in \mathbb{Z}$ we partition $\mathbb{R}^n$ into dyadic cubes τ^k_j of side 2^{-k}. We write $\tau^k_j \sim \tau^k_{j'}$ if τ^k_j and $\tau^k_{j'}$ have adjacent parents, but are not adjacent. As in [1], [18] and [27], we use a Whitney type decomposition of $\mathbb{R}^n \times \mathbb{R}^n$ away from its diagonal D, so that $(\mathbb{R}^n \times \mathbb{R}^n \backslash D) = \cup_k \cup_{\tau^k_j \sim \tau^k_{j'}} \tau^k_j \times \tau^k_{j'}$. Writing $f = \sum_j f^k_j$, where $f^k_j(-\xi) = f(-\xi)\chi_{\tau^k_j}(\xi)$ we have

$$\begin{aligned} e^{it\Delta} \widehat{f}(x)\, e^{it\Delta} \widehat{f}(x) &= \int\!\!\int f(-\xi) f(-y) e^{2\pi i (x\cdot(\xi+y) - 2\pi t(|\xi|^2+|y|^2))} d\xi dy \\ &= \sum_{k,j,j' :\, \tau^k_j \sim \tau^k_{j'}} \int\!\!\int f^k_j(-\xi) f^k_{j'}(-y) e^{2\pi i (x\cdot(\xi+y) - 2\pi t(|\xi|^2+|y|^2))} d\xi dy \\ &= \sum_k \sum_{j,j' :\, \tau^k_j \sim \tau^k_{j'}} e^{it\Delta} \widehat{f}^k_j(x)\, e^{it\Delta} \widehat{f}^k_{j'}(x). \end{aligned}$$

For $k \in \mathbb{Z}$, we define the bilinear operators T_k by

$$T_k(f,g) = \sum_{j,j' :\, \tau_j^k \sim\, \tau_{j'}^k} e^{it\Delta} \widehat{f}_j^k \, e^{it\Delta} \widehat{g}_{j'}^k,$$

so that

$$e^{it\Delta} \widehat{f} e^{it\Delta} \widehat{f} = \sum_k T_k(f,f).$$

We will prove that there exists an $\varepsilon > 0$ such that

$$\|T_k(f,g)\|_{L_x^{q/2} L_t^{r_0/2}} \leqslant C 2^{2k\left((n+1)(\frac{1}{q}-\frac{1}{q_0})-n(\frac{1}{2}-\frac{1}{p})\right)} \|f\|_p \|g\|_p \tag{18}$$

for $q \in (q_0 - \varepsilon, q_0 + \varepsilon)$ and $p \in (2-\varepsilon, 2+\varepsilon)$.

To see this we interpolate the hypothesis which is equivalent to

$$\|e^{it\Delta} \widehat{f}_j^k \, e^{it\Delta} \widehat{g}_{j'}^k\|_{L_x^{q/2} L_t^{r/2}} \leqslant C 2^{2k\left(\frac{n+1}{q}+\frac{1}{r}-\frac{n}{2}\right)} \|f_j^k\|_2 \|g_j^k\|_2$$

with the trivial estimate $\|e^{it\Delta} \widehat{f}_j^k e^{it\Delta} \widehat{g}_{j'}^k\|_{L_x^\infty L_t^\infty} \leqslant \|f_j^k\|_1 \|g_j^k\|_1$, to obtain

$$\|e^{it\Delta} \widehat{f}_j^k \, e^{it\Delta} \widehat{g}_{j'}^k\|_{L_x^{q/2} L_t^{r_0/2}} \leqslant C 2^{2k\left(\frac{n+1}{q}+\frac{1}{r_0}-n(1-\frac{1}{p_0})\right)} \|f_j^k\|_{p_0} \|g_j^k\|_{p_0}$$

for some $p_0 < 2$ and q in a neighborhood of q_0. Since f_j^k and $g_{j'}^k$ are supported in sets of measure $\sim 2^{-nk}$, applying Hölder's inequality, we get for (q,p) in a neighbourhood of $(q_0, 2)$,

$$\|e^{it\Delta} \widehat{f}_j^k \, e^{it\Delta} \widehat{g}_{j'}^k\|_{L_x^{q/2} L_t^{r_0/2}} \leqslant C 2^{2k\left(\frac{n+1}{q}+\frac{1}{r_0}-n(1-\frac{1}{p})\right)} \|f_j^k\|_p \|g_j^k\|_p.$$

Using the relation $\frac{n+1}{q_0} + \frac{1}{r_0} = \frac{n}{2}$, this is the same as

$$\|e^{it\Delta} \widehat{f}_j^k \, e^{it\Delta} \widehat{g}_{j'}^k\|_{L_x^{q/2} L_t^{r_0/2}} \leqslant C 2^{2k\left((n+1)(\frac{1}{q}-\frac{1}{q_0})-n(\frac{1}{2}-\frac{1}{p})\right)} \|f_j^k\|_p \|g_j^k\|_p. \tag{19}$$

By concavity when $p < 2$ and Lemma 4 when $p > 2$ (choosing ε sufficiently small), we have

$$\Big\| \sum_{j,j' : \tau_j^k \sim\, \tau_{j'}^k} e^{it\Delta} \widehat{f}_j^k \, e^{it\Delta} \widehat{g}_{j'}^k \Big\|_{L_x^{q/2} L_t^{r_0/2}} \leqslant C \left(\sum_{j,j' : \tau_j^k \sim\, \tau_{j'}^k} \|e^{it\Delta} \widehat{f}_j^k \, e^{it\Delta} \widehat{g}_{j'}^k\|_{L_x^{q/2} L_t^{r_0/2}}^{p/2} \right)^{2/p}.$$

Combining this with (19) and applying Cauchy–Schwarz inequality gives

$$\begin{aligned} \|T_k(f,g)\|_{L_x^{q/2} L_t^{r_0/2}} &\leqslant C 2^{2k\left((n+1)(\frac{1}{q}-\frac{1}{q_0})-n(\frac{1}{2}-\frac{1}{p})\right)} \left(\sum_j \|f_j^k\|_p^p \right)^{1/p} \left(\sum_j \|g_j^k\|_p^p \right)^{1/p} \\ &\leqslant C 2^{2k\left((n+1)(\frac{1}{q}-\frac{1}{q_0})-n(\frac{1}{2}-\frac{1}{p})\right)} \|f\|_p \, \|g\|_p \end{aligned}$$

for all $q \in (q_0 - \varepsilon, q_0 + \varepsilon)$ and $p \in (2-\varepsilon, 2+\varepsilon)$, which is (18).

Note that Proposition 3.1 would follow from

$$\|e^{it\Delta} \widehat{f}\|_{L_x^{q,\infty} L_t^{r_0}} \leqslant C \, \|f\|_{L^{p,1}}, \quad (n+1)\Big(\frac{1}{q} - \frac{1}{q_0}\Big) = n\Big(\frac{1}{2} - \frac{1}{p}\Big)$$

for q in a small neighbourhood of q_0, where $L^{p,q}$ denotes the Lorentz space. In fact, by real interpolation (see for example [21]) this gives $\|e^{it\Delta}f\|_{L_x^{q_0,2}L_t^{r_0}} \leqslant C\|f\|_{L^{2,2}}$, and since $q_0 \geq 2$ and $L^{q_0} = L^{q_0,q_0} \supset L^{q_0,2}$, we get the desired inequality. We can rewrite the above estimate as

$$\|e^{it\Delta}\widehat{f}e^{it\Delta}\widehat{f}\,\|_{L_x^{q/2,\infty}L_t^{r_0/2}} \leqslant C\,\|f\|_{L^{p,1}}^2,$$

so that it will suffice to prove

$$|\{x \in \mathbb{R}^n\,:\,T\chi_E > \lambda\}| \leqslant C\lambda^{-q/2}|E|^{q/p}, \quad (n+1)(\frac{1}{q}-\frac{1}{q_0}) = n(\frac{1}{2}-\frac{1}{p}) \tag{20}$$

for measurable sets E and q in a neighbourhood of q_0, where

$$Tf := \sum_k \|T_k(f,f)\|_{L_t^{r_0/2}} \geq \|e^{it\Delta}\widehat{f}e^{it\Delta}\widehat{f}\,\|_{L_t^{r_0/2}}.$$

By (18), we have

$$\Big\|\,\|T_k(\chi_E,\chi_E)\|_{L_t^{r_0/2}}\Big\|_{L_x^{q_1/2}} \leqslant C2^{2\delta_1 k}|E|^{2/p_1},$$
$$\Big\|\,\|T_k(\chi_E,\chi_E)\|_{L_t^{r_0/2}}\Big\|_{L_x^{q_2/2}} \leqslant C2^{-2\delta_2 k}|E|^{2/p_2},$$

for $p_1 \in (2-\varepsilon, 2)$, $q_1 \in (q_0-\varepsilon, q_0)$ and $p_2 \in (2, 2+\varepsilon)$, $q_2 \in (q_0, q_0+\varepsilon)$, where

$$\delta_1 := \Big((n+1)(\frac{1}{q_1}-\frac{1}{q_0}) - n(\frac{1}{2}-\frac{1}{p_1})\Big) > 0,$$
$$\delta_2 := -\Big((n+1)(\frac{1}{q_2}-\frac{1}{q_0}) - n(\frac{1}{2}-\frac{1}{p_2})\Big) > 0.$$

Decomposing $T = T_K + T^K$, where

$$T_K f = \sum_{k\leqslant K} \|T_k(f,f)\|_{L_t^{r_0/2}} \quad \text{and} \quad T^K = \sum_{k>K} \|T_k(f,f)\|_{L_t^{r_0/2}},$$

by Minkowski's inequality and summing the geometric series', we see that

$$\|T_K\chi_E\|_{L_x^{q_1/2}} \leqslant C2^{2\delta_1 K}|E|^{2/p_1}, \qquad \|T^K\chi_E\|_{L_x^{q_2/2}} \leqslant C2^{-2\delta_2 K}|E|^{2/p_2}.$$

Now

$$|\{x\,:\,T\chi_E > \lambda\}| \leqslant |\{x\,:\,T_K\chi_E > \lambda/2\}| + |\{x\,:\,T^K\chi_E > \lambda/2\}|,$$

so that by Tchebyshev's inequality,

$$|\{x\,:\,T\chi_E > \lambda\}| \leqslant C\left(\lambda^{-q_1/2}2^{q_1\delta_1 K}|E|^{q_1/p_1} + \lambda^{-q_2/2}2^{-q_2\delta_2 K}|E|^{q_2/p_2}\right).$$

Optimizing in K, yields (20) for p and q defined by

$$\frac{1}{p} = \frac{\delta_2}{(\delta_1+\delta_2)p_1} + \frac{\delta_1}{(\delta_1+\delta_2)p_2}, \quad \frac{1}{q} = \frac{\delta_2}{(\delta_1+\delta_2)q_1} + \frac{\delta_1}{(\delta_1+\delta_2)q_2}.$$

The condition $(n+1)(\frac{1}{q}-\frac{1}{q_0}) = n(\frac{1}{2}-\frac{1}{p})$ is satisfied, and by varying p_1, p_2, q_1, q_2 we obtain (20) for q in a neighbourhood of q_0. This completes the proof. $\square$

4. Appendix

Lemma 4 *Suppose that the spatial Fourier transforms of $F_k : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ are supported in a sequence B_k of finitely overlapping balls of the same radius. Then for all $\varepsilon > 0$, there exists $\alpha > 1$ such that*

$$\Big\| \sum_k F_k \Big\|_{L^q_x L^r_t} \leqslant C \left(\sum_k \|F_k\|^\alpha_{L^q_x L^r_t} \right)^{1/\alpha} \quad \textit{when} \quad 1/q,\, 1/r \in (\varepsilon, 1-\varepsilon).$$

Proof. By finitely many applications of the Minkowski's inequality, we may suppose that the sequence of balls $2B_k$, with double the radius, are disjoint. Let $\widehat{\phi}$ be a Schwartz function equal to one on $\{|\xi| \leqslant 1\}$ and supported in $\{|\xi| \leqslant 2\}$, and let $\widehat{\phi}_k$ be translates and dilations of $\widehat{\phi}$ such that $\widehat{\phi}_k$ is equal to one on B_k and supported on $2B_k$. Defining the operators P_k by $P_k G_k = \phi_k *_x G_k$, it will suffice to show that

$$\Big\| \sum_k P_k G_k \Big\|_{L^q_x L^r_t} \leqslant C \left(\sum_k \|G_k\|^\alpha_{L^q_x L^r_t} \right)^{1/\alpha}, \tag{21}$$

for general functions G_k, as then we can take $G_k = F_k$ and $P_k G_k = F_k$. Note that when $q = r = 2$, we can take $\alpha = 2$ in (21) by Fubini and Plancherel.

Thus, it remains to prove that

$$\Big\| \sum_k P_k G_k \Big\|_{L^q_x L^r_t} \leqslant C \sum_k \|G_k\|_{L^q_x L^r_t},$$

as then we may interpolate with the case $q = r = 2$ to get the result. Now (21) follows from Minkowski's inequality and the fact that

$$\|\phi_k *_x G_k\|_{L^q_x L^r_t} \leqslant \Big\| \, |\phi_k| *_x \|G_k\|_{L^r_t} \Big\|_{L^q_x} \leqslant \|\phi\|_{L^1} \, \Big\| \|G_k\|_{L^r_t} \Big\|_{L^q_x},$$

which is a consequence of Minkowski's integral inequality followed by Young's inequality and scaling, and so we are done. □

References

[1] J. Bourgain, *Estimates for cone multipliers*, Geometric aspects of functional analysis (Israel, 1992), Oper. Theory Adv. Appl., vol. 77, Birkhäuser, Basel, 1995, pp. 41–60.

[2] L. Carleson, *Some analytic problems related to statistical mechanics*, Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), 1980, pp. 5–45.

[3] C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math. **15** (1973), 44–52.

[4] J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), no. 4, 309–327.

[5] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson, 2004.

[6] T. Kato and K. Yajima, *Some examples of smooth operators and the associated smoothing effect*, Rev. Math. Phys. **1** (1989), no. 4, 481–496.

[7] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.

[8] C.E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. **4** (1991), no. 2, 323–347.

[9] ———, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), no. 4, 527–620.

[10] ———, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40** (1991), no. 1, 33–69.

[11] S. Lee, *On pointwise convergence of the solutions to Schrödinger equations in* $\mathbb{R}^2$, Int. Math. Res. Not. (2006), Art. ID 32597, 21.

[12] S. Lee and A. Vargas, *Sharp null form estimates for the wave equation*, Amer. J. Math. **130** (2008), no. 5, 1279-1326.

[13] F. Planchon, *Dispersive estimates and the 2D cubic NLS equation*, J. Anal. Math. **86** (2002), 319–334.

[14] K.M. Rogers, *Strichartz estimates via the Schrödinger maximal operator*, Math. Ann. **343** (2009), no. 3, 603–622.

[15] K.M. Rogers and A. Seeger, *Endpoint maximal and smoothing estimates for Schrödinger equations*, J. Reine Angew. Math. **640** (2010), 47–66.

[16] K.M. Rogers and P. Villarroya, *Global estimates for the Schrödinger maximal operator*, Ann. Acad. Sci. Fenn. Math. **32** (2007), no. 2, 425–435.

[17] A. Ruiz and L. Vega, *On local regularity of Schrödinger equations*, Internat. Math. Res. Notices **1** (1993), 13–27.

[18] A. Seeger, *Endpoint inequalities for Bochner-Riesz multipliers in the plane*, Pacific J. Math. **174** (1996), 543–553.

[19] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987), no. 3, 699–715.

[20] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, NJ, 1993.

[21] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[22] R.S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), no. 3, 705–714.

[23] T. Tao, *A sharp bilinear restrictions estimate for paraboloids*, Geom. Funct. Anal. **13** (2003), no. 6, 1359–1384.

[24] ———, *Nonlinear dispersive equations, local and global analysis*, Vol. 106 of CBMS Regional Conference Series in Mathematics, Washington, DC, 2006.

[25] T. Tao and A. Vargas, *A bilinear approach to cone multipliers. I. Restriction estimates*, Geom. Funct. Anal. **10** (2000), no. 1, 185–215.

[26] ———, *A bilinear approach to cone multipliers. II. Applications*, Geom. Funct. Anal. **10** (2000), no. 1, 216–258.

[27] T. Tao, A. Vargas, and L. Vega, *A bilinear approach to the restriction and Kakeya conjectures*, J. Amer. Math. Soc. **11** (1998), no. 4, 967–1000.

[28] P.A. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81** (1975), 477–478.

[29] L. Vega, *El multiplicador de Schrödinger. La funcion maximal y los operadores de restricción.* Universidad Autónoma de Madrid (1988).

[30] ———, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), no. 4, 874–878.