MIXED NORM ESTIMATES OF SCHRÖDINGER WAVES AND THEIR APPLICATIONS

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ABSTRACT. In this paper we establish mixed norm estimates of interactive Schrödinger waves and apply them to study smoothing properties and global well-posedness of the nonlinear Schrödinger equations with mass critical nonlinearity.

1. INTRODUCTION

The Strichartz estimate shows the dispersive nature of Schrödinger waves, which can be formulated via mixed norms ([21, 14]). More precisely, for admissible (q, r)

$$\|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2}.$$

Here a pair (q, r) is said to be *admissible* if it satisfies $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r})$, $q, r \geq 2$ with exception $(q, r) = (2, \infty)$ when $n = 2$ and $e^{it\Delta}$ denotes the free propagator of Schrödinger equation.

Due to scaling, the frequency localization via Littewood-Paley decomposition does not give any improvement to the aforementioned Strichartz estimates. However, it was observed by Bourgain [1] that by considering low and high frequency interactions of two Schrödinger waves, namely bilinear control of $e^{it\Delta} f e^{it\Delta} g$, it is possible to obtain a refinement of Strichartz estimate in $L^2_{t,x}(\mathbb{R} \times \mathbb{R}^2)$ (note that $(4, 4)$ is an admissible pair when $n = 2$). In [16] Keraani and Vargas recently extended Bourgain's results to higher dimensions by showing that a sharp $L_{t,x}^{(n+2)/n}(\mathbb{R} \times \mathbb{R}^n), n \geq 1$ estimate holds for the interactive Schrödinger waves.

Our first result is that such refinement of Strichartz estimate is also valid in the mixed norm setting for $n \geq 3$. It is stated as follows:

Theorem 1.1. *Let $n \geq 2$. Let (q, r) satisfy that $2/q = n(1/2 - 1/r)$, $2 < r \leq 4$, and $q > 2$. Then for $|s| < 1 - 2/r$,*

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_t^{q/2} L_x^{r/2}} \lesssim \|f\|_{\dot{H}^s} \|g\|_{\dot{H}^{-s}}. \tag{1.1}$$

Compared with non-mixed norm estimates ([1, 16]), the estimate (1.1) actually gives a stronger interactive estimate in that spatial derivatives of higher order can

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be traded between two waves when $n \geq 3$ and $q < r$. By Plancherel's theorem (1.1) trivially holds for $(r, s) = (2, 0)$ and when $n = 2$ it was obtained in [16] including $(q, r) = (4, 4)$. This estimate obviously has a scaling structure in L^2 space so that the estimate is invariant along the admissible (q, r). Making use of (1.1) it is possible to move a certain amount of derivative on one to the other function. So (1.1) is useful when one studies the smoothing property of nonlinear Schrödinger equations of power type. The range on s is sharp, since (1.1) fails for $|s| > 1 - 2/r$ (see the discussion below Proposition 2.1).

The estimate (1.1) is strongly connected to the bilinear restriction estimates for the paraboloid (see [16, 18, 22]). In fact, for the proof of Theorem 1.1 we establish estimates for bilinear interactions between waves at different frequencies. They rely on the argument used to prove the bilinear restriction estimate for the paraboloid [18, 22], which makes use of wave packet decomposition and induction on scales (see Proposition 2.1 and Corollary 2.4 below). It is also possible to obtain generalization of (1.1) which can be compared with null form estimates for wave equations (see [13, 18]). Some of such estimates can be directly deduced from (2.1) via simple rescaling argument. However, we do not pursue it here since we are mainly concerned with applications to mass critical problems in which the natural scaling structure is important.

Aside from the power type, one of the most typical nonlinearity is that of Hartree in the study of nonlinear Schrödinger equations (see (1.4) below). To handle the Hartree type nonlinearity, we consider the trilinear operator $\mathcal{H}$ which is given by

$$\mathcal{H}(f, g, h) \equiv |\nabla|^{2-n}(e^{it\Delta} f e^{it\Delta} g) e^{it\Delta} h.$$

Here $|\nabla|^{2-n}$ is the pseudo-differential operator with symbol $|\xi|^{2-n}$ which is the convolution with $c_n|x|^{-2}$. To make the operator defined properly, we assume $n \geq 3$ throughout the paper when we use the notation $|\nabla|^{2-n}$. As it is turned out (see Theorem 1.3 and Theorem 1.4), the trilinear estimate enables us to control the interaction of waves arising in Hartree type nonlinearity more effectively. It is stated as follows:

Theorem 1.2. *Let $n \geq 3$ and let $(\tilde{q}, \tilde{r})$ be admissible. Suppose that s_1, s_2, s_3 are positive numbers satisfying $\sum s_i = 1$. Then, if $s_3 > \frac{1}{2}$*

$$\|\mathcal{H}(f, g, \nabla h)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}} \|h\|_{\dot{H}^{s_3}}. \tag{1.2}$$

If $s_1 > \frac{1}{2}$, then

$$\|\mathcal{H}(\nabla f, g, h)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}} \|h\|_{\dot{H}^{s_3}}. \tag{1.3}$$

It should be noticed that the estimates (1.2), (1.3) are invariant under scaling for all admissible $(\tilde{q}, \tilde{r})$ (cf. Lemma 2.5). For the proof we first show frequency localized estimates (Proposition 2.6 below) which also rely on the bilinear interaction estimates and the scaling structure of $\mathcal{H}$. Compared with (1.1), a stronger interaction estimate is possible thanks to the operator $|\nabla|^{2-n}$ which gives additional decay in frequency space.

Now we consider applications of Theorem 1.1 and 1.2 to nonlinear Schrödinger equations. We are concerned with the Cauchy problem of L^2 critical nonlinear Schrödinger equation in $\mathbb{R}^n, n \geq 3$, of which nonlinear part is given by the nonlinear potential $V(u)$ of Hartree or power type:

$$\begin{cases} iu_t + \Delta u = V(u)u, & (t,x) \in [0,T] \times \mathbb{R}^n,\ T > 0, \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}^n). \end{cases} \tag{1.4}$$

That is to say, $V(u) = \kappa |x|^{-2} * |u|^2$ or $V(u) = \kappa |u|^{\frac{4}{n}}$ with $\kappa = \pm 1$. Here $u : [0,T] \times \mathbb{R}^n \to \mathbb{C}$ is a complex valued function. If u is a solution to (1.4), the scaled function $\lambda^{\frac{n}{2}} u(\lambda^2 t, \lambda x)$, $\lambda > 0$ is also a solution. Hence (1.4) is invariant under the scaling in L^2 space (i.e. L^2 critical). By the Duhamel's principle the problem (1.4) is equivalent to solving the integral equation for $t \in [0,T]$;

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-t')\Delta} (V(u)u)(t')\, dt'. \tag{1.5}$$

It is well known that the problem (1.4) is locally wellposed for every $s \geq 0$ (see [2, 23]). The lifespan of solution u depends on the H^s norm of u_0 if $u_0 \in H^s, s > 0$, and the profile of u_0 if $u_0 \in L^2$, respectively. The solution $u \in C([0,T]; H^s)$ to (1.4) satisfies some conservation laws, namely, mass and energy; for any $t \in [0,T]$, if $s \geq 0$

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2,$$

and if $s \geq 1$

$$E(u(t)) \equiv \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \omega \int V(u(t)) |u(t)|^2\, dx = E(u_0),$$

where $\omega = 1/4$ if $V(u) = \kappa |x|^{-2} * |u|^2$ and $\omega = \frac{n}{4+2n}$ if $V(u) = \kappa |u|^{\frac{4}{n}}$. If the data is sufficiently smooth ($s \geq 1$), various results were established by using the classical energy argument. However, it does not work any longer when $0 \leq s < 1$ and there have been a lot of works devoted to extending those results to lower regularity initial data (for instance see [1, 6, 9]).

We firstly apply Theorem 1.1, 1.2 to study the smoothing properties of solutions to the Cauchy problem (1.4). We consider a strong global (in x-space) smoothing effect such that the Duhamel's part

$$D(t) \equiv u(t) - e^{it\Delta}u_0 \in C([0,T]; H^1) \tag{1.6}$$

for all T within the lifespan when the initial data u_0 is in H^s, $0 \le s < 1$. The smoothing actually stems from the interaction of Schrödinger waves arising in the nonlinear term. It was first observed by Bourgain [1] for $V(u) = \kappa|u|^2, n = 2, s > 2/3$ and later extended by Keraani and Vargas [16] for $V(u) = \kappa|u|^{\frac{4}{n}}, n \ge 1, s > s_n$, where $s_1 = 3/4, s_n = n/(n+2)$ for $2 \le n \le 4$, $s_n = (n^2+2n-8)/n(n+2)$. To utilize the interaction, they established refined bilinear Strichartz estimates in $L_{t,x}^{\frac{n+2}{n}}$ as mentioned above. In the following, we get better smoothing effects that (1.6) holds for a rougher u_0, using the Theorems 1.1 and 1.2 together with the duality arguments based on the Bourgain space ([1, 16]).

Theorem 1.3. *Let $n \ge 3$. (1) If $u_0 \in H^s(\mathbb{R}^n)$ and $1/2 < s < 1$, then there is a maximal existence time $T^* > 0$ such that a unique solution u to (1.4) with $V(u) = \kappa|x|^{-2} * |u|^2$ exists in $C([0,T^*); H^s)$ and D satisfies (1.6) for all $T < T^*$.*
(2) Let $s_n = \frac{1}{2}$ for $n = 3, 4$ and $s_n = 1 - \frac{8}{n^2}$ for $n \ge 5$. If $u_0 \in H^s(\mathbb{R}^s)$, $s_n < s < 1$, then there is a maximal existence time T^* such that there exists a unique solution u in $C([0,T^*); H^s)$ to (1.4) with $V(u) = \kappa|u|^{\frac{4}{n}}$ and D satisfies (1.6) for all $T < T^*$.

In part (2) we do not have any improvement on 2-d result which was obtained in [16] ($s_2 = \frac{1}{2}$). The above result shows that the Hartree type interaction is more effective than the power type when $n \ge 5$, which may be interpreted as weaker (of lower power) nonlinearity causes a lower interaction between the waves. The smoothing effect can be used to show an H^1 mechanism for the blowup phenomenon of the Cauchy problem (1.4) (see Remark 1.3 of [16]). In [16], it was shown that if T^* is finite, then

$$\|\nabla D(t)\|_{L^2} \gtrsim (T^* - t)^{-1/2}$$

for power type NLS provided that (1.6) holds for all $T < T^*$. Hence, part (2) of Theorem 1.3 extends the possible range of s. Similarly, using part (1) of Theorem 1.3 and the argument in [16] together with well-known scaling argument, one can also get the same blowup rate of $D(t)$ for the finite time blowup solution of Hartree type NLS as long as $u_0 \in H^s(\mathbb{R}^n)$ and $1/2 < s < 1$.

We now consider the global well-posedness of defocusing L^2 critical Hartree equation, (1.4) with $\kappa = +1$, for rough initial data in $H^s, 0 < s < 1$. Recently Chae and Kwon [3] considered the same problem (1.4) and they got global well-posedness for $u_0 \in H^s$, $2(n-2)/(3n-4) < s < 1$. Their result is based on the so-called I-method. (For details and recent development of I-method, we refer readers to [4, 5, 6, 7, 9, 10, 11, 12].) We here make further improvement. By exploiting the interaction of Schrödinger waves systematically (Proposition 2.6), we obtain better decay estimates for almost energy conservation and interaction Morawetz inequality (see Proposition 4.1, 4.3) which are the major estimates for I-method. As a consequence we get the following global well-posedness theorem.

Theorem 1.4. *Let $n \geq 3$ and $V(u) = |x|^{-2} * |u|^2$. Then the initial value problem of* (1.4) *is globally well-posed for data $u_0 \in H^s(\mathbb{R}^n)$ when $\frac{4(n-2)}{7n-8} < s < 1$.*

The global well-posedness for the spherically symmetric data in L^2 was shown by Miao, Xu and Zhao [19]. They adopted the method due to Killip, Tao and Visan [15]. For the 2-d cubic NLS, Colliander and Roy [9] recently combined the improved estimate in [7] with a Mowawetz error estimate by using the double layer bootstrap in time, and established the global well-posedness for the L^2 critical NLS on $\mathbb{R}^2$ with data in H^s, $s > 1/3$. It seems highly possible that such approach also makes further progress for the Hartree equations if it is combined with the results of this paper. We hope to address such issues somewhere else. Compared to the previous works, the use of mixed norm estimates makes our proof of almost energy conservation and interaction Morawetz inequality more systematic and flexible (see Section 4). We believe that it may be useful in studies of various related problems.

This paper is organized as follows: In Section 2 we will obtain the bilinear interaction estimate, trilinear Hartree type interaction estimate, and prove Theorem 1.1, 1.2. In Section 3 we will show the local well-posedness and smoothing effect of Duhamel's part of solutions to (1.4). The Section 4 is devoted to showing the global well-posedness of defocusing Hartree equation. Lastly we append a brief introduction to wave packet decomposition of Schrödinger wave, which will be used in Section 2.

We finally list the notations which are frequently used in the paper:

• $A \lesssim B$ means that $A \leq CB$ for some constant $C > 0$ which may vary from lines to lines. We also write $A \sim B$ when $A \lesssim B$ and $B \lesssim A$.

• The symbol ∇ denotes the gradient $(\partial/\partial_1, \cdots, \partial/\partial_n)$ and Δ the Laplacian $\nabla \cdot \nabla = \sum_j \partial^2/\partial_j^2$. We also denote $(-\Delta)^{\frac{1}{2}}$ by $|\nabla|$.

• Let J_T be the time interval $[0, T]$. For a measurable function F the mixed norm is defined by $\|F\|_{L_t^q L_x^r(J_T \times \mathbb{R}^n)} = \left(\int_{J_T} \left(\int_{\mathbb{R}^n} |F(t,x)|^r \, dx \right)^{\frac{q}{r}} \right)^{\frac{1}{q}}$. We use $\|F\|_{L_t^q L_x^r}$ to denote $\|F\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)}$. L^p is the usual Lebesgue space $L_x^p(\mathbb{R}^n)$.

• The Fourier transform of f is defined by $\mathcal{F}(f)(\xi) = \widehat{f}(\xi) \equiv \int e^{-ix\cdot\xi} f(x)\, dx$ and its inverse by $\mathcal{F}^{-1}(g)(x) \equiv (2\pi)^{-n} \int e^{ix\cdot\xi} g(\xi)\, d\xi$. Hence $e^{it\Delta} f(x) = \mathcal{F}^{-1}(e^{-it|\cdot|^2} \mathcal{F}(f))(x)$ $= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \widehat{f}(\xi) d\xi$.

• Let N denote dyadic number and let P_N be the Littlewood-Paley projection operator with symbol $\chi(\xi/N) \in C_0^\infty$ such that χ is supported in the set $\{1/2 \le |\xi| \le 2\}$ and $\sum_{N:\,dyadic} \chi(\xi/N) = 1$, $\xi \ne 0$. We also define $\widetilde{P}_1 = id - \sum_{N>1} P_N$.

• The inhomogeneous Sobolev space $H^s (= H^s(\mathbb{R}^n), s \in \mathbb{R})$ denotes the space $\{f \in \mathcal{S}' : \|f\|_{H^s} < \infty\}$, where $\|f\|_{H^s} \equiv (\sum_{N>1} N^{2s}\|P_N f\|_{L^2}^2 + \|\widetilde{P}_1 f\|_2^2)^{\frac12} \sim \|\langle\nabla\rangle^s f\|_{L^2} = \left(\int \langle\xi\rangle^{2s}|\widehat{f}(\xi)|^2\, d\xi\right)^{\frac12}$. Here $\langle A\rangle = \sqrt{1+|A|^2}$. We will also use the homogeneous Sobolev space $\dot{H}^s = \{f \in \mathcal{S}'/\mathcal{P} : \|f\|_{\dot{H}^s} < \infty\}$, where $\mathcal{P}$ is the totality of polynomials and the seminorm $\|f\|_{\dot{H}^s} \equiv (\sum_{N>0} N^{2s}\|P_N f\|_{L^2}^2)^{\frac12} \sim \||\nabla|^s f\|_{L^2}$. We note here that if $|s| < n/2$, the definition of $\dot{H}^s$ makes sense in $\mathcal{S}'$ and C_0^∞ is dense in $\dot{H}^s$ (cf. [20]).

2. Mixed norm interaction estimates for the Schrödinger waves

In this section we first prove bilinear interaction estimates for the Schrödinger waves. Considering the mixed norm space, it is possible to get a better interaction estimate than the one obtained in [16]. We denote by $B(\xi,\rho)$ the ball centered at ξ with radius ρ.

Proposition 2.1. *Let $n \ge 2$. Suppose that supp $\widehat{f} \subset B(\xi_0,\rho_1)$ and supp $\widehat{g} \subset B(\eta_0,\rho_2)$ for some $|\xi_0|, |\eta_0| \le 1$. If $|\xi_0 - \eta_0| \sim 1$ and $0 < \rho_1, \rho_2 \ll 1$, then for $\epsilon > 0$ and (q,r) satisfying that $r \le 4$, $2 < q$ and $1 - \frac{2}{r} \le \frac{2}{q} < (n+1)(\frac12 - \frac1r)$*

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^{q/2}L_x^{r/2}} \lesssim \min(\rho_1,\rho_2)^{\alpha(q,r)-\epsilon}\|f\|_{L^2}\|g\|_{L^2},$$

where $\alpha(q,r) = (n+1)(1-2/r) - 4/q$.

It can be shown that the bounds in the above estimates are sharp up to ϵ. Indeed, assuming $\rho_1 \le \rho_2$, let us consider the functions f and g given by $\widehat{f} = \chi_A$ and $\widehat{g} = \chi_B$ with $A = \{\xi : |\xi_1 - 1| \le \rho_1^2,\ |\xi_i| \le \rho_1, i = 2,\dots,n\}$ and $B = \{\xi : |\xi_1 + 1| \le \rho_1^2,\ |\xi_i| \le \rho_1, i = 2,\dots,n\}$. Then it is easy to see that $|e^{it\Delta}f(x)|, |e^{it\Delta}g(x)| \ge c\rho_1^{n+1}$ if $|x_1|, |t| \le c\rho_1^{-2}$ and $|x_i| \le c\rho_1^{-1}$, $i = 2,\dots,n$ for some $c > 0$. Hence

$$\rho_1^{n+1-\frac{2(n+1)}{r}-\frac{4}{q}} \lesssim \frac{\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^{q/2}L_x^{r/2}}}{\|f\|_{L^2}\|g\|_{L^2}}.$$

Letting $\rho_1 \to 0$ we can see that this implies $\alpha(q,r) - \epsilon \le n + 1 - \frac{2(n+1)}{r} - \frac{4}{q}$ for any $\epsilon > 0$. It also shows the failure of the estimates when $\frac{2}{q} > (n+1)(\frac12 - \frac1r)$. The example above is actually the squashed cap function which was used to show the sharpness of the bilinear restriction estimate [24].

Remark 2.2. When $n \ge 2$, using (2.2) below and Plancherel's theorem, we can show that for (q,r) satisfying $1 - 2/r \ge 2/q$ and $r \le 4$,

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^{q/2}L_x^{r/2}} \lesssim \min(\rho_1,\rho_2)^{n(1-\frac{2}{r})-\frac{2}{q}}\|f\|_{L^2}\|g\|_{L^2}.$$

It is also sharp as it can be shown by using the functions f and g given by $\widehat{f} = \chi_A$ and $\widehat{g} = \chi_B$ with $A = \{\xi : |\xi - e_1| \le \rho_1\}$ and $B = \{\xi : |\xi + e_1| \le \rho_1\}$.

For the proof of Proposition 2.1 we will use the wave packet decomposition for the Schrödinger operator (see the appendix). Such decomposition was used to study Fourier restriction estimates [18, 22, 26].

Proof of Proposition 2.1. By symmetry we may assume $\rho_1 \le \rho_2$. We start with recalling the estimates

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_t^{q/2} L_x^{r/2}} \lesssim \|f\|_{L^2} \|g\|_{L^2} \tag{2.1}$$

for $\frac{2}{q} < (n+1)(\frac{1}{2} - \frac{1}{r})$, $2 < q, r \le 4$. See Theorem 2.3 of [18]. Also we make use of the estimate*

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_t^2 L_x^2} \lesssim \rho_1^{\frac{n-1}{2}} \|f\|_{L^2} \|g\|_{L^2}, \tag{2.2}$$

which already appeared in the previous literature (for instance see [1] and [16]). For the convenience of reader we give a simple proof based on Plancherel's theorem.

Using an affine transformation we may assume $\xi_0 = 0$. By decomposing the Fourier support of g into finite number of sets, rotation and dilation, it is enough to show (2.2) whenever f and g are Fourier-supported in $B(0, \rho_1)$ and $B(e_1, \delta)$ for some $0 < \delta \ll 1$, respectively. We write

$$e^{it\Delta} f(x) e^{it\Delta} g(x) = \int e^{i(x\cdot(\xi+\eta) - t(|\xi|^2 + |\eta|^2))} \widehat{f}(\xi) \widehat{g}(\eta) d\xi d\eta.$$

Freezing $\bar{\xi} = (\xi_2, \dots, \xi_n)$, we consider a bilinear operator

$$B_{\bar{\xi}}(f, g) = \int e^{i(x\cdot(\xi+\eta) - t(|\xi|^2 + |\eta|^2))} \widehat{f}(\xi_1, \bar{\xi}) \widehat{g}(\eta) d\xi_1 d\eta.$$

We make the change of variables $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_{n+1}) = (\xi + \eta, |\xi|^2 + |\eta|^2)$. Then by direct computation one can see that $\left|\frac{\partial \zeta}{\partial(\xi_1, \eta)}\right| = 2|\xi_1 - \eta_1| \sim 1$ on the supports of $\widehat{f}$ and $\widehat{g}$. Hence making the change of variables $(\xi_1, \eta) \to \zeta$, applying Plancherel's theorem and reversing the change variables ($\zeta \to (\xi_1, \eta)$), we have

$$\|B_{\bar{\xi}}(f, g)\|_{L_t^2 L_x^2} \lesssim \|\widehat{f}(\xi_1, \bar{\xi}) \widehat{g}(\eta)\|_{L^2_{\xi_1, \eta}}.$$

Since $e^{it\Delta} f(x) e^{it\Delta} g(x) = \int B_{\bar{\xi}}(\hat{f}(\cdot, \bar{\xi}), \hat{g}(\cdot)) d\bar{\xi}$, by Minkowski's inequality we get

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_t^2 L_x^2} \lesssim \int \|\hat{f}(\cdot, \bar{\xi}) \hat{g}(\cdot)\|_{L^2_{\xi_1, \eta}} d\bar{\xi}.$$

This gives the desired estimate (2.2) by Cauchy-Schwarz inequality because $|\bar{\xi}| \le \rho_1$.

*It actually reads as $\|e^{it\Delta} f e^{it\Delta} g\|_{L_t^2 L_x^2} \lesssim \min(\rho_1, \rho_2)^{\frac{n-1}{2}} \|f\|_{L^2} \|g\|_{L^2}$ since we are assuming $\rho_1 \le \rho_2$.

When $n = 2, 3$ we only need to interpolate (2.1), (2.2) and $\|e^{it\Delta}fe^{it\Delta}g\|_{L^\infty_t L^1_x} \lesssim \|f\|_{L^2}\|g\|_{L^2}$ for the proof of Proposition 2.1. It gives all the desired estimates. Hence, similarly when $n \ge 4$, it is sufficient to show that for $\epsilon > 0$

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^{q_\epsilon/2}L_x^{r_\epsilon/2}} \lesssim \rho_1^{\frac{n-3}{2}-\epsilon}\|f\|_{L^2}\|g\|_{L^2}. \tag{2.3}$$

Here (q_ϵ, r_ϵ) converges to $(2, 4)$ as $\epsilon \to 0$. A similar estimate already appeared in [18] for the wave operator and its proof is based on the induction on scale argument. We also follow the same lines of argument.

Let λ be a large number so that $\lambda \gg \rho_1^{-2}$ and let us set $\mathcal{Q}(\lambda) = Q(\lambda) \times (-\lambda, \lambda)$, where $Q(\lambda)$ is the cube centered at the origin with side length 2λ. We make an assumption that

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^1 L_x^2(\mathcal{Q}(\lambda))} \lesssim \rho_1^{\frac{n-3}{2}}\lambda^\alpha\|f\|_{L^2}\|g\|_{L^2}. \tag{2.4}$$

Due to (2.2) and Hölder's inequality the above is valid with $\alpha = 1/2$. Now we attempt to suppress α as small as possible.

Let $0 < \delta \ll 1$ and let $\{b\}$ be the collection of the $\lambda^{1-\delta}$-cubes b partitioning $\mathcal{Q}(\lambda)$. We make use of the wave packet decomposition and Lemma 4.8 which had crucial role in the proof of the sharp bilinear restriction estimates for the paraboloids [22]. We provide some basic properties of wave packets in the appendix. To proceed we make use of the notations in the appendix.

Using the wave packet decomposition at scale λ and triangle inequality, we have

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^1 L_x^2(\mathcal{Q}(\lambda))} \le \sum_b \|\sum_{T,T'} e^{it\Delta}f_T e^{it\Delta}g_{T'}\|_{L_t^1 L_x^2(b)}.$$

Using the relation $\sim$, we break the mixed integration over b so that

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^1 L_x^2(\mathcal{Q}(\lambda))} \le I + II,$$

where

$$I = \sum_b \|\sum_{T\sim b \text{ and } T'\sim b} e^{it\Delta}f_T e^{it\Delta}g_{T'}\|_{L_t^1 L_x^2(b)},$$
$$II = \sum_b \|\sum_{T\not\sim b \text{ or } T'\not\sim b} e^{it\Delta}f_T e^{it\Delta}g_{T'}\|_{L_t^1 L_x^2(b)}.$$

For the first we use the induction assumption (2.4) to get

$$I \lesssim \rho_1^{\frac{n-3}{2}}\lambda^{\alpha(1-\delta)}\sum_b \|\sum_{T\sim b} f_T\|_{L^2}\|\sum_{T\sim b} g_T\|_{L^2}$$

because b is a cube of size $\sim \lambda^{1-\delta}$. Hence by (4.17) and Schwarz's inequality

$$I \lesssim \rho_1^{\frac{n-3}{2}}\lambda^\epsilon\lambda^{\alpha(1-\delta)}\|f\|_{L^2}\|g\|_{L^2}.$$

Hölder's inequality and (4.18) give

$$\| \sum_{T\not\sim b \text{ or } T'\not\sim b} e^{it\Delta} f_T e^{it\Delta} g_{T'} \|_{L_t^1 L_x^2(b)} \lesssim \lambda^\epsilon \lambda^{c\delta-(n-3)/4} \|f\|_{L^2} \|g\|_{L^2}.$$

Since there are only $\lambda^{c\delta}$-cubes b and $\rho_1^2 \gg \lambda^{-1}$, it follows that

$$II \lesssim \rho_1^{\frac{n-3}{2}} \lambda^\epsilon \lambda^{c\delta} \|f\|_{L^2} \|g\|_{L^2}.$$

Combining two estimates for I and II, we get

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_t^1 L_x^2(\mathcal{Q}(\lambda))} \lesssim \rho_1^{\frac{n-3}{2}} \lambda^\epsilon (\lambda^{(1-\delta)\alpha} + \lambda^{c\delta}) \|f\|_{L^2} \|g\|_{L^2}. \tag{2.5}$$

Therefore we see that the assumption (2.4) implies the above estimate (2.5). Since $\epsilon, \delta > 0$ can be chosen to be arbitrarily small, we get for any $\alpha > 0$

$$\|e^{it\Delta} f e^{it\Delta} g\|_{L_t^1 L_x^2(\mathcal{Q}(\lambda))} \lesssim \rho_1^{\frac{n-3}{2}} \lambda^\alpha \|f\|_{L^2} \|g\|_{L^2}$$

by iterating the implication (2.4) $\to$ (2.5) finitely many times[†]. To upgrade this to the global one, we need the following globalization lemma in [18].

Lemma 2.3. *Let $S_i = \{(\xi, \phi_i(\xi)) : \xi \in U_i\}$, $i = 1, 2$ be compact surfaces with boundary and the induced Lebesgue measures $d\sigma_i(\xi) = d\xi$, which satisfy $\|d\sigma_i\| \le C_i$, $\sigma_i(B(z,\rho)) \le C\rho^{n-1}$ for any $z, \rho > 0$ and $|\widehat{d\sigma_i}(x,t)| \le C_i(1+|x|+|t|)^{-\sigma}$ for some $C_i \ge 1$ and $\sigma > 0$. Suppose that for some $\frac{2+2\sigma}{\sigma} \ge q_0, r_0 \ge 1$ and $0 < \epsilon \ll \sigma$,*

$$\| \prod_{i=1}^{2} \widehat{f_i d\sigma_i} \|_{L_t^{q_0} L_x^{r_0}(\mathcal{Q}(\lambda))} \le C_0 \lambda^\epsilon \prod_{i=1}^{2} \|f_i\|_{L^2(d\sigma_i)}.$$

Let $\frac{1}{q_1} = \frac{1}{q_0} - \frac{2\epsilon}{2\epsilon+\sigma}\big(\frac{1}{q_0} - \frac{\sigma}{2(\sigma+1)}\big)$, $\frac{1}{r_1} = \frac{1}{r_0} - \frac{2\epsilon}{2\epsilon+\sigma}\big(\frac{1}{r_0} - \frac{\sigma}{2(\sigma+1)}\big)$. Then, for $q > q_1$

$$\| \prod_{i=1}^{2} \widehat{f_i d\sigma_i} \|_{L_t^{q} L_x^{r_1}} \lesssim C_0^{1-\frac{\epsilon}{\sigma}} (\max(C_1, C_2))^{a\epsilon + (1-\frac{q_1}{q})(1-\frac{1}{r_1})} \prod_{i=1}^{2} \|f_i\|_{L^2(d\sigma_i)}$$

with some $a > 0$ depending on σ.

Let us define two extension operators by

$$\widehat{hd\sigma_1} = \int e^{i(x\cdot\xi - t|\xi|^2)} \beta\Big(\frac{\xi-\xi_0}{\rho_1}\Big) h(\xi) d\xi, \quad \widehat{hd\sigma_2} = \int e^{i(x\cdot\xi - t|\xi|^2)} \beta\Big(\frac{\xi-\eta_0}{\rho_2}\Big) h(\xi) d\xi$$

for smooth β supported in $B(0,2)$ and $\beta = 1$ on $B(0,1)$. Since supp $\widehat{f} \subset B(\xi_0, \rho_1)$ and supp $\widehat{g} \subset B(\eta_0, \rho_2)$, by Plancherel's theorem it is sufficient to show that the estimate

$$\| \prod_{i=1}^{2} \widehat{h_i d\sigma_i} \|_{L_t^1 L_x^2(\mathcal{Q}(\lambda))} \lesssim \rho_1^{\frac{n-3}{2}} \lambda^\alpha \prod_{i=1}^{2} \|h_i\|_2$$

[†]For this one should note that the constant c in (2.5) is independent of ϵ, λ.

implies the global estimate

$$\|\prod_{i=1}^{2}\widehat{h_i d\sigma_i}\|_{L_t^{q(\alpha)/2}L_x^{r(\alpha)/2}} \lesssim \rho_1^{\frac{n-3}{2}-\epsilon(\alpha)}\prod_{i=1}^{2}\|h_i\|_2$$

with $q(\alpha)\to 1$, $r(\alpha)\to 2$ and $\epsilon(\alpha)\to 0$ as $\alpha\to 0$. Hence in view of Lemma 2.3, we only need to check that

$$|\widehat{d\sigma_i}(x,t)| \lesssim (1+|x|+|t|)^{-\frac{n}{2}}.$$

This is easy to see by using stationary phase method because $\rho_1,\rho_2 \ll 1$. It completes the proof of Proposition 2.1.

2.1. **Proof of Theorem 1.1: Bilinear interaction estimates.** We note that the bilinear estimate in Proposition 2.1 is invariant under rescaling when $2/q = n(1/2-1/r)$. Hence, by Proposition 2.1 and rescaling one can easily see the following Corollary 2.4, which shows that there is an interactive compensation when one considers the Schrödinger waves of different frequency levels. Throughout the paper we denote by $A(\rho)$ the set $\{\xi : |\xi|\sim\rho\}$.

Corollary 2.4. *Let $n\ge 2$. Let (q,r) satisfy that $2/q = n(1/2-1/r)$, $2\le r\le 4$, $q>2$. If supp $\widehat{f}\subset A(N_1)$ and supp $\widehat{g}\subset A(N_2)$ for $0<N_1\le N_2$, then for any $\epsilon>0$,*

$$\|e^{it\Delta}fe^{it\Delta}g\|_{L_t^{q/2}L_x^{r/2}} \lesssim \left(\frac{N_1}{N_2}\right)^{1-2/r-\epsilon}\|f\|_{L^2}\|g\|_{L^2}.$$

We now give the proof of the Theorem 1.1. The assertion for $s=0$ follows from the Hölder's inequality and Strichartz estimate. By symmetry we may assume that $s>0$. Let P_N be the Littlewood-Paley projection as stated in the introduction. For simplicity we set $f_N = P_N f$ and break $e^{it\Delta}fe^{it\Delta}g$ so that

$$e^{it\Delta}fe^{it\Delta}g = \sum_{N_1,\,N_2:\,dyadic} e^{it\Delta}f_{N_1}e^{it\Delta}g_{N_2}.$$

for any $f\in\dot{H}^s$ and $g\in\dot{H}^{-s}$. Since $|\nabla|\sim N_2$ on the Fourier support of g_{N_2}, it is enough to show that

$$\|\sum_{N_1,\,N_2} e^{it\Delta}f_{N_1}e^{it\Delta}g_{N_2}\|_{L_t^{q/2}L_x^{r/2}} \lesssim \Big(\sum_{N_1} N_1^{2s}\|f_{N_1}\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{N_2} N_2^{-2s}\|g_{N_2}\|_{L^2}^2\Big)^{\frac12}.$$

Let us set $N_{12}=N_1N_2$. By the triangle inequality

$$\|\sum_{N_1,\,N_2} e^{it\Delta}f_{N_1}e^{it\Delta}g_{N_2}\|_{L_t^{q/2}L_x^{r/2}} \le I + II,$$

where

$$I = \| \sum_{N_1 \ge 1} \sum_{N_2} e^{it\Delta} f_{N_{12}} e^{it\Delta} g_{N_2} \|_{L_t^{q/2} L_x^{r/2}}, \quad II = \| \sum_{N_1 < 1} \sum_{N_2} e^{it\Delta} f_{N_{12}} e^{it\Delta} g_{N_2} \|_{L_t^{q/2} L_x^{r/2}}.$$

Since $I \lesssim \sum_{N_1 \ge 1} N_1^{2/r-1+\epsilon} \sum_{N_2} \|f_{N_{12}}\|_{L^2} \|g_{N_2}\|_{L^2}$ by the triangle inequality and Corollary 2.4, we see that

$$\begin{aligned} I &\lesssim \sum_{N_1 \ge 1} N_1^{2/r-1-s+\epsilon} \sum_{N_2} (N_{12})^s \|f_{N_{12}}\|_{L^2} N_2^{-s} \|g_{N_2}\|_{L^2} \\ &\lesssim \sum_{N_1 \ge 1} N_1^{2/r-1-s+\epsilon} \big(\sum_{N_1} N_1^{2s} \|f_{N_1}\|_{L^2}^2 \big)^{\frac12} \big(\sum_{N_2} N_2^{-2s} \|g_{N_2}\|_{L^2}^2 \big)^{\frac12} \\ &\lesssim \big(\sum_{N_1} N_1^{2s} \|f_{N_1}\|_{L^2}^2 \big)^{\frac12} \big(\sum_{N_2} N_2^{-2s} \|g_{N_2}\|_{L^2}^2 \big)^{\frac12}, \end{aligned}$$

provided $2/r - 1 - s + \epsilon < 0$. We now turn to II. By the triangle inequality and Corollary 2.4, $II \lesssim \sum_{N_1 < 1} N_1^{1-2/r-\epsilon} \sum_{N_2} \|f_{N_{12}}\|_{L^2} \|g_{N_2}\|_{L^2}$. Hence, by Schwarz's inequality

$$\begin{aligned} II &\lesssim \sum_{N_1 < 1} N_1^{1-2/r-\epsilon-s} \big(\sum_{N_1} N_1^{2s} \|f_{N_1}\|_{L^2}^2 \big)^{\frac12} \big(\sum_{N_2} N_2^{-2s} \|g_{N_2}\|_{L^2}^2 \big)^{\frac12} \\ &\lesssim \big(\sum_{N_1} N_1^{2s} \|f_{N_1}\|_{L^2}^2 \big)^{\frac12} \big(\sum_{N_2} N_2^{-2s} \|g_{N_2}\|_{L^2}^2 \big)^{\frac12} \end{aligned}$$

as long as $1 - 2/r - \epsilon - s > 0$. This completes the proof of Theorem 1.1.

2.2. **Proof of Theorem 1.2: Trilinear interaction of Hartree type nonlinearity.** First we recall the following which is a consequence of Strichartz estimate and Hardy-Littlewood-Sobolev inequality.

Lemma 2.5. *For any admissible* $(\tilde q, \tilde r)$*,*

$$\|\mathcal{H}(f, g, h)\|_{L_t^{\tilde q'} L_x^{\tilde r'}} \lesssim \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

and the estimates are invariant under the rescaling $(f, g, h) \to (f_\lambda, g_\lambda, h_\lambda) = (\lambda^{\frac n2} f(\lambda \cdot),$ $\lambda^{\frac n2} g(\lambda \cdot), \lambda^{\frac n2} h(\lambda \cdot))$ *for any* $\lambda > 0$.

To show this, observe that for any admissible $(\tilde q, \tilde r)$ there is an admissible (q, r) such that $(1/\tilde q', 1/\tilde r') + (0, (n-2)/n) = 3\,(1/q, 1/r)$. Then, using Hölder's and Hardy-Littlewood-Sobolev inequalities one can get the desired estimate.

Via frequency localization on annulus we first obtain the following trilinear interaction estimate.

Proposition 2.6. *Let $n \geq 3$ and N_1, N_2, N_3 be positive numbers. Suppose that supp $\widehat{f}$, supp $\widehat{g}$, supp $\widehat{h}$ are contained in $A(N_1)$, $A(N_2)$, $A(N_3)$, respectively. Then for any admissible pair $(\widetilde{q}, \widetilde{r})$,*

$$\|\mathcal{H}(f,g,h)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim C(N_1, N_2, N_3) \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}, \tag{2.6}$$

where

$$C(N_1, N_2, N_3) = \left(\frac{\min(N_1, N_2, N_3)}{\max(N_1, N_2, N_3)} \right)^{1/2}.$$

For the proof of Proposition 2.6 it is enough to consider two endpoints $(\widetilde{q}', \widetilde{r}') = (1, 2)$, $(2, \frac{2n}{n+2})$ because interpolation gives the remaining estimates. By symmetry we may assume that $N_1 \geq N_2$. On account of scaling structure (Lemma 2.5) we may also assume that

$$1 = \max(N_1, N_3).$$

Hence we can further assume that $1 \gg \min(N_1, N_2, N_3)$ since the desired estimates are already contained in Lemma 2.5 when $N_1 \sim N_2 \sim N_3$.

We now prove Proposition 2.6 by considering the cases $N_1 \gg N_2$ and $N_1 \sim N_2$, separately. To begin with, we recall the following simple lemma which can be easily shown by using the Strichartz estimates and rescaling.

Lemma 2.7. *If supp $\widehat{f} \subset A(N)$, for $q, r \geq 2$ satisfying $n/r + 2/q \leq n/2$*

$$\|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim N^{\frac{n}{2} - \frac{n}{r} - \frac{2}{q}} \|f\|_2.$$

Case $N_1 \gg N_2$. In this case the spatial Fourier support of $e^{it\Delta} f e^{it\Delta} g$ is contained in $A(2N_1)$ because $N_1 \gg N_2$. Hence, $|\nabla|^{2-n} \sim N_1^{2-n}$ on the Fourier support of $e^{it\Delta} f e^{it\Delta} g$. Using Hörmander-Mikhlin multiplier theorem we see that

$$\|\mathcal{H}(f,g,h)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim N_1^{2-n} \|e^{it\Delta} f e^{it\Delta} g e^{it\Delta} h\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{2.7}$$

We now have three subcases (i) $1 = N_3 \geq N_1 \gg N_2$, (ii) $1 = N_1 \geq N_3 \geq N_2$, (iii) $1 = N_1 \gg N_2 \geq N_3$.

We consider the case (i) first. Taking $(\widetilde{q}', \widetilde{r}') = (2, \frac{2n}{n+2})$ in (2.7) and using Hölder's inequality, it follows that

$$\|\mathcal{H}(f,g,h)\|_{L_t^2 L_x^{\frac{2n}{n+2}}} \lesssim N_1^{2-n} \|e^{it\Delta} g e^{it\Delta} h\|_{L_t^2 L_x^2} \|e^{it\Delta} f\|_{L_t^\infty L_x^{\frac{n}{2}}}.$$

Since $1 \gg N_2$, using (2.2) and Bernstein's inequality (or Lemma 2.7), we get

$$\|\mathcal{H}(f,g,h)\|_{L_t^2 L_x^{\frac{2n}{n+2}}} \lesssim N_2^{\frac{1}{2}} \left(\frac{N_2}{N_1} \right)^{\frac{n-2}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.$$

This gives the desired estimate for $(\widetilde{q}', \widetilde{r}') = (2, \frac{2n}{n+2})$. Similarly, taking $(\widetilde{q}', \widetilde{r}') = (1, 2)$ in (2.7) and using Hölder's inequality, we have

$$\|\mathcal{H}(f,g,h)\|_{L_t^1 L_x^2} \lesssim N_1^{2-n} \|e^{it\Delta} f\|_{L_t^2 L_x^\infty} \|e^{it\Delta} g e^{it\Delta} h\|_{L_t^2 L_x^2}.$$

By (2.2) and Lemma 2.7, we get

$$\|\mathcal{H}(f,g,h)\|_{L^1_tL^2_x} \lesssim N_2^{\frac12}\left(\frac{N_2}{N_1}\right)^{\frac{n-2}{2}}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.$$

Hence we get the desired bound for $(\widetilde{q}',\widetilde{r}')=(2,1)$ because $N_2\le N_1$.

The remaining two cases $(ii),(iii)$ can be handled similarly. In fact, for the case (ii), repeating the same argument with (2.7), (2.2) and Lemma 2.7 we see that

$$\|\mathcal{H}(f,g,h)\|_{L^2_tL^{\frac{2n}{n+2}}_x} \lesssim \|e^{it\Delta}fe^{it\Delta}g\|_{L^2_tL^2_x}\|e^{it\Delta}h\|_{L^\infty_tL^n_x} \lesssim N_2^{\frac{n-1}{2}}N_3^{\frac{n-2}{2}}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$$

and

$$\|\mathcal{H}(f,g,h)\|_{L^1_tL^2_x} \lesssim \|e^{it\Delta}fe^{it\Delta}g\|_{L^2_tL^2_x}\|e^{it\Delta}h\|_{L^2_tL^\infty_x} \lesssim N_2^{\frac{n-1}{2}}N_3^{\frac{n-2}{2}}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}$$

because $1\gg N_2$. So we get the desired estimates for $(\widetilde{q}',\widetilde{r}')=(1,2),(2,\frac{2n}{n+2})$.

Finally, for the case (iii), by repeating the same argument one can show that for $(\widetilde{q}',\widetilde{r}')=(1,2),\ (2,\frac{2n}{n+2})$,

$$\|\mathcal{H}(f,g,h)\|_{L^{\widetilde{q}'}_tL^{\widetilde{r}'}_x} \lesssim N_3^{\frac{n-1}{2}}N_2^{\frac{n-2}{2}}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.$$

This completes the proof for the case $N_1\gg N_2$. Now we turn to the remaining case $N_1\sim N_2$.

Case $N_1\sim N_2$. In this case $|\nabla|^{2-n}$ can not be handled simply as before. So we need an additional argument to handle this. We begin with decomposing $|\nabla|^{2-n}$ so that

$$|\nabla|^{2-n}=\sum_{N:\text{dyadic}}N^{2-n}\psi(|\nabla|/N)$$

with a cut-off ψ supported in $A(1)$ [†]. Here $m(|\nabla|)$ is the multiplier operator defined by $m(|\nabla|)f=\mathcal{F}^{-1}(m(|\cdot|)\widehat{f})$ for a measurable function m. Then we have

$$\mathcal{H}(f,g,h)=\sum_{N:\text{dyadic}}N^{2-n}\psi(|\nabla|/N)(e^{it\Delta}fe^{it\Delta}g)e^{it\Delta}h. \tag{2.8}$$

We first try to obtain estimates for $\psi(|\nabla|/N)(e^{it\Delta}fe^{it\Delta}g)e^{it\Delta}h$. We claim that for $(\widetilde{q}',\widetilde{r}')=(1,2),\ (2,\frac{2n}{n+2})$,

$$\begin{aligned}&\|\psi(|\nabla|/N)(e^{it\Delta}fe^{it\Delta}g)e^{it\Delta}h\|_{L^{\widetilde{q}'}_tL^{\widetilde{r}'}_x}\\ &\quad\lesssim N^{\frac{n-2}{2}}\big(\min(N,N_3)\big)^{\frac{n-1}{2}}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.\end{aligned} \tag{2.9}$$

[†]Actually the sum is taken over $N\lesssim\max(N_1,N_2)$ because of the supports of $\widehat{f}$, $\widehat{g}$.

To show this we break f and g into functions having Fourier supports in cubes of side length $2^{-2}N$. Let $\{Q\}$ be a collection of (essentially) disjoint cubes of side length $2^{-2}N$ covering $A(N_1)$ and we set

$$\widehat{f}_Q = \chi_Q(\xi)\widehat{f}, \quad \widehat{g}_Q = \chi_Q(\xi)\widehat{g}.$$

Then we have $f = \sum_Q f_Q$ and $g = \sum_Q g_Q$, and we may assume that $Q \subset A(N_1)$ because $N_1 \sim N_2$. Then it follows that

$$\begin{aligned}\text{LHS of (2.9)} &\lesssim \sum_{Q,Q'} \|\psi(|\nabla|/N)(e^{it\Delta} f_Q e^{it\Delta} g_{Q'}) e^{it\Delta} h\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \\ &\lesssim \sum_{\text{dist }(Q,-Q')\le 4N} \|\psi(|\nabla|/N)(e^{it\Delta} f_Q e^{it\Delta} g_{Q'}) e^{it\Delta} h\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}}\end{aligned} \tag{2.10}$$

because $\psi(|\nabla|/N)(e^{it\Delta} f_Q e^{it\Delta} g_{Q'}) = 0$ if dist $(Q, -Q') > 4N$. Hence it is enough to show that for $(\widetilde{q}', \widetilde{r}') = (1, 2),\ (2, \frac{2n}{n+2})$,

$$\begin{aligned}&\|\psi(|\nabla|/N)(e^{it\Delta} f_Q e^{it\Delta} g_{Q'}) e^{it\Delta} h\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \\ &\qquad \lesssim N^{\frac{n-2}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \|f_Q\|_{L^2} \|g_{Q'}\|_{L^2} \|h\|_{L^2}.\end{aligned} \tag{2.11}$$

Indeed, from (2.10) and (2.11) we get

$$\begin{aligned}\text{LHS of (2.9)} &\lesssim N^{\frac{n-2}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \sum_{\text{dist }(Q,-Q')\le 4N} \|f_Q\|_{L^2} \|g_{Q'}\|_{L^2} \|h\|_{L^2} \\ &\lesssim N^{\frac{n-2}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \big(\sum_Q \|f_Q\|_{L^2}^2\big)^{\frac12} \big(\sum_{Q'} \|g_{Q'}\|_{L^2}^2\big)^{\frac12} \|h\|_{L^2} \\ &\lesssim N^{\frac{n-2}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.\end{aligned}$$

For the second and third inequalities we used Cauchy-Schwarz inequality and orthogonality, respectively. We are reduced to showing (2.11).

Now observe that

$$\begin{aligned}\psi(|\nabla|/N)(e^{it\Delta} f_Q e^{it\Delta} g_{Q'}) = &\iint e^{ix\cdot(\xi+\eta) - it(|\xi|^2+|\eta|^2)} \\ &\times \phi(\xi/N - \xi_0)\psi((\xi+\eta)/N)\phi(\eta/N - \eta_0)\widehat{f}_Q(\xi)\widehat{g}_{Q'}(\eta)\, d\xi d\eta\end{aligned}$$

for some $\xi_0, \eta_0 \in \mathbb{R}^n$ and ϕ supported in $B(0,1)$. Expanding $\Psi(\xi, \eta) = \phi(\xi - \xi_0)\psi(\xi+\eta)\phi(\eta - \eta_0)$ into Fourier series on the cube of side length 2π which contains the support of Ψ, we have

$$\phi(\xi - \xi_0)\psi(\xi+\eta)\phi(\eta - \eta_0) = \sum_{k,\, l \,\in \mathbb{Z}^n} C_{k,\, l}\, e^{i(k\cdot\xi + l\cdot\eta)}$$

with $\sum_{k,l} |C_{k,l}| \le C$, independent of ξ_0, η_0. Plugging this in the above we get

$$\psi(|\nabla|/N)(e^{it\Delta} f_Q e^{it\Delta} g_{Q'}) = \sum_{k,\, l\, \in \mathbb{Z}^n} C_{k,l}\, e^{it\Delta} f_Q^k e^{it\Delta} g_{Q'}^l$$

with $\|f_Q^k\|_{L^2} = \|f_Q\|_{L^2}$ and $\|g_{Q'}^l\|_{L^2} = \|g_{Q'}\|_{L^2}$ for all k, l. To show (2.11) it suffices to show that for $(\widetilde{q}', \widetilde{r}') = (1,2),\ (2, \frac{2n}{n+2})$,

$$\|e^{it\Delta} f e^{it\Delta} g e^{it\Delta} h\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \lesssim N^{\frac{n-2}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}$$

whenever $\widehat{f}, \widehat{g}$ are supported in cubes $Q, Q' \subset A(N_1)$ of side length N and $\widehat{h}$ is supported in $A(N_3)$. Note that dist (supp $\widehat{f}$, supp $\widehat{h}$) ~ 1 and dist (supp $\widehat{g}$, supp $\widehat{h}$) ~ 1 because there are only two possible cases $1 = N_1 \sim N_2 \gg N_3$, $1 = N_3 \gg N_1 \sim N_2$. By Hölder's inequality, (2.2) and Lemma 2.7, we get

$$\begin{aligned}\|e^{it\Delta} f e^{it\Delta} g e^{it\Delta} h\|_{L_t^2 L_x^{\frac{2n}{n+2}}} &\lesssim \|e^{it\Delta} f\|_{L_t^\infty L_x^n} \|e^{it\Delta} g e^{it\Delta} h\|_{L_t^2 L_x^2} \\ &\lesssim N^{\frac{n-2}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}\|e^{it\Delta} f e^{it\Delta} g e^{it\Delta} h\|_{L_t^1 L_x^2} &\lesssim \|e^{it\Delta} f e^{it\Delta} g\|_{L_t^2 L_x^2} \|e^{it\Delta} h\|_{L_t^2 L_x^\infty} \\ &\lesssim N^{\frac{n-2}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.\end{aligned}$$

Therefore we get (2.11).

We now consider two cases $1 = N_1 \sim N_2 \gg N_3$, $1 = N_3 \gg N_1 \sim N_2$, separately. When $1 = N_1 \sim N_2 \gg N_3$, from (2.8), triangle inequality and (2.9) we get

$$\begin{aligned}\|\mathcal{H}(f,g,h)\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} &\lesssim \sum_{N \le 4} N^{2-n} \|\psi(|\nabla|/N)(e^{it\Delta} f e^{it\Delta} g) e^{it\Delta} h\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \\ &\lesssim \sum_{N \le 4} N^{\frac{2-n}{2}} \big(\min(N, N_3)\big)^{\frac{n-1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.\end{aligned}$$

Summation in N gives

$$\|\mathcal{H}(f,g,h)\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \lesssim N_3^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.$$

This proves the case (i) $1 = N_1 \sim N_2 \gg N_3$. When $1 = N_3 \gg N_1 \sim N_2$ note that the summation is taken over $N \lesssim N_1$. By (2.8), triangle inequality and (2.9) we get

$$\|\mathcal{H}(f,g,h)\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \lesssim \sum_{N \lesssim N_1} N^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}.$$

The desired estimate follows from summation in N. This completes the proof of Proposition 2.6.

Before closing this subsection we state a slightly strengthened version of Proposition 2.6 which is to be used in Section 4.

Corollary 2.8. *Let $i = 1, 2, 3$. If $N_i \sim \min(N_1, N_2, N_3)$, then Proposition 2.6 remains valid with $A(N_i)$ replaced by $B(0, N_i)$ in the assumption.*

Proof. When all of $N_1, N_2, N_3 \sim \min(N_1, N_2, N_3)$, (2.6) trivially holds by Lemma 2.5. If only one N_i of $N_1, N_2, N_3 \sim \min(N_1, N_2, N_3)$, then by decomposition of $B(0, N_i)$ into dyadic shells, applying Proposition 2.6 to each dyadic shell and direct summation of geometric series one can easily see that (2.6) holds. The other possibility is that two of $N_1, N_2, N_3 \sim \min(N_1, N_2, N_3)$. In this case we only need to consider two cases $N_2 \sim N_3 \ll N_1$ and $N_1 \sim N_2 \ll N_3$ by symmetry between N_1 and N_2. For both cases one can see without difficulty that the argument for the proof of Proposition 2.6 works for either supp $\widehat{g} \subset B(0, N_2)$ and supp $\widehat{h} \subset B(0, N_3)$ or supp $\widehat{f} \subset B(0, N_1)$ and supp $\widehat{g} \subset B(0, N_2)$. □

We now prove Theorem 1.2 by showing (1.2), (1.3), separately.

2.3. **Proof of** (1.2)**.** For simplicity we denote by $f_{N_j}(j = 1, 2, 3)$ the Littlewood-Paley projection $P_{N_j} f$ of f. Then we decompose

$$\begin{aligned}\mathcal{H}(f, g, \nabla h) &= \sum_{N_1, N_2, N_3} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3}) \\ &= \sum_{N_1 \le N_2} \sum_{N_3} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3}) + \sum_{N_1 > N_2} \sum_{N_3} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3}).\end{aligned}$$

By symmetry it is enough to handle the first one because the second can be handled similarly. Then we have three possible cases; $N_3 \le N_1 \le N_2$, $N_1 \le N_3 \le N_2$ and $N_1 \le N_2 \le N_3$. We separately treat the summation of each case.

Case $N_3 \le N_1 \le N_2$. This case is the easiest. It can be handled by using the Strichartz estimates only. We claim that for any positive s_1, s_2, s_3 with $\sum s_i = 1$,

$$\| \sum_{N_3 \le N_1 \le N_2} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3}) \|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}} \|h\|_{\dot{H}^{s_3}}.$$

By setting $N_2 = N_3 N_4 \equiv N_{34}$ we write

$$\sum_{N_3 \le N_1 \le N_2} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3}) = \sum_{N_4 \ge 1} \sum_{N_3 = -\infty}^{\infty} \sum_{N_3 \le N_1 \le N_{34}} \mathcal{H}(f_{N_1}, g_{N_{34}}, \nabla h_{N_3}).$$

Using Lemma 2.5 we get

$$\|\mathcal{H}(f_{N_1}, g_{N_{34}}, \nabla h_{N_3})\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim N_3 \|f_{N_1}\|_{L^2} \|g_{34}\|_{L^2} \|h_3\|_{L^2}.$$

So, the norm $\| \sum_{N_3 \le N_1 \le N_2} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3})\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$ is bounded by

$$C \sum_{N_4 \ge 1} \sum_{N_3} \sum_{N_3 \le N_1 \le N_{34}} N_3 N_1^{-s_1} N_{34}^{-s_2} N_3^{-s_3} (N_1^{s_1} \|f_{N_1}\|_{L^2})(N_{34}^{s_2} \|g_{N_{34}}\|_{L^2})(N_3^{s_3} \|h_{N_3}\|_{L^2}).$$

It is also bounded again by

$$C\|f\|_{\dot H^{s_1}} \sum_{N_4\geq 1} N_4^{-s_2} \sum_{N_3} (N_{34}^{s_2}\|g_{N_{34}}\|_{L^2})(N_3^{s_3}\|h_{N_3}\|_{L^2}).$$

Then Cauchy-Schwarz inequality yields the desired bound.

Case $N_1 \leq N_3 \leq N_2$**.** In this case we set $N_2 = N_1N_4 \equiv N_{14}$ and write

$$\sum_{N_1\leq N_3\leq N_2} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3}) = \sum_{N_4\geq 1}\sum_{N_1}\sum_{N_1\leq N_3\leq N_{14}} \mathcal{H}(f_{N_1}, g_{N_{14}}, \nabla h_{N_3}).$$

Using triangle inequality and Proposition 2.6 we see

$$\|\mathcal{H}(f_{N_1}, g_{N_{14}}, \nabla h_{N_3})\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \lesssim N_3 N_4^{-\frac12}\|f_{N_1}\|_{L^2}\|g_{N_{14}}\|_{L^2}\|h_{N_3}\|_{L^2}.$$

Hence the norm $\|\sum_{N_1\leq N_3\leq N_2}\mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3})\|_{L_t^{\tilde q'}L_x^{\tilde r'}}$ is bounded by

$$\|h\|_{\dot H^{s_3}} \sum_{N_4\geq 1} N_4^{-\frac12}\sum_{N_1}\sum_{N_1\leq N_3\leq N_{14}} N_3^{1-s_3}N_1^{-s_1}N_{14}^{-s_2}(N_1^{s_1}\|f_{N_1}\|_{L^2})(N_{14}^{s_2}\|g_{N_{14}}\|_{L^2}).$$

Taking summation in N_3 and using Schwarz's inequality in N_1, we bound this by

$$C\|f\|_{\dot H^{s_1}}\|g\|_{\dot H^{s_2}}\|h\|_{\dot H^{s_3}}\sum_{N_4\geq 1} N_4^{s_1-\frac12}.$$

Note that $s_1 < \frac12$ because $s_3 > \frac12$. Hence we get the desired estimate.

Case $N_1 \leq N_2 \leq N_3$**.** We set $N_3 = N_1N_4 \equiv N_{14}$ and write

$$\sum_{N_1\leq N_2\leq N_3} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3}) = \sum_{N_4\geq 1}\sum_{N_1}\sum_{N_1\leq N_2\leq N_{14}} \mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_{14}}).$$

Using triangle inequality and Proposition 2.6 we have

$$\begin{aligned}\|\sum_{N_1\leq N_2\leq N_3} &\mathcal{H}(f_{N_1}, g_{N_2}, \nabla h_{N_3})\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \\ &\lesssim \sum_{N_4\geq 1}\sum_{N_1}\sum_{N_1\leq N_2\leq N_{14}} N_{14}N_4^{-\frac12}\|f_{N_1}\|_{L^2}\|g_{N_2}\|_{L^2}\|h_{N_{14}}\|_{L^2}.\end{aligned}$$

The left hand side of the above is

$$\lesssim \|g\|_{\dot H^{s_2}}\sum_{N_4\geq 1}N_4^{-\frac12}\sum_{N_1}\sum_{N_1\leq N_2\leq N_{14}} N_1^{-s_1}N_2^{-s_2}N_{14}^{1-s_3}(N_1^{s_1}\|f_{N_1}\|_{L^2})(N_{14}^{s_3}\|h_{N_{14}}\|_{L^2}).$$

Taking summation in N_2 and using Schwarz's inequality in N_1, the above is bounded by $C\|f\|_{\dot H^{s_1}}\|g\|_{\dot H^{s_2}}\|h\|_{\dot H^{s_3}}\sum_{N_4\geq 1}N_4^{-s_3+\frac12}$. Since $s_3 > \frac12$, we get the desired estimate.

2.4. **Proof of** (1.3)**.** We decompose

$$\mathcal{H}(\nabla f, g, h) = \sum_{N_1,N_2,N_3} \mathcal{H}(\nabla f_{N_1}, g_{N_2}, h_{N_3}).$$

There is no obvious symmetry. We should consider the following six cases:

$$\begin{aligned}&(i)\, N_1 \ge N_2 \ge N_3, \quad (ii)\, N_1 \ge N_3 \ge N_2, \quad (iii)\, N_3 \ge N_1 \ge N_2,\\ &(iv)\, N_2 \ge N_1 \ge N_3, \quad (v)\, N_2 \ge N_3 \ge N_1, \quad (vi)\, N_3 \ge N_2 \ge N_1.\end{aligned}$$

As expected, the cases (i) and (ii) are the major parts. The others are sort of minor terms which is easier to handle. Each of the cases can be handled by the same argument as before.

Case (i) $N_1 \ge N_2 \ge N_3$**:** To begin with, we set $N_1 = N_{34} (= N_3 N_4)$. By the triangle inequality and rearrangement of the summation we get

$$\begin{aligned}&\Big\| \sum_{N_3 \le N_2 \le N_1} \mathcal{H}(\nabla f_{N_1}, g_{N_2}, h_{N_3}) \Big\|_{L_t^{\tilde q'} L_x^{\tilde r'}}\\ &\qquad\qquad \lesssim \sum_{N_4 \ge 1} \sum_{N_1} \sum_{N_{34} \ge N_2 \ge N_3} \| \mathcal{H}(\nabla f_{N_{34}}, g_{N_2}, h_{N_3}) \|_{L_t^{\tilde q'} L_x^{\tilde r'}}.\end{aligned}$$

Applying Proposition 2.6 we bound the left hand side of the above by

$$\|g\|_{\dot H^{s_2}} \sum_{N_4 \ge 1} N_4^{-\frac12} \sum_{N_3} \sum_{N_{34} \ge N_2 \ge N_3} N_{34}^{1-s_1} N_2^{-s_2} N_3^{-s_3} (N_{34}^{s_1} \|f_{N_{34}}\|_{L^2}) (N_3^{s_3} \|h_{N_3}\|_{L^2}).$$

Taking summ in N_2 and using Schwarz's inequality in N_1, we see that

$$\Big\| \sum_{N_3 \le N_2 \le N_1} \mathcal{H}(\nabla f_{N_1}, g_{N_2}, h_{N_3}) \Big\|_{L_t^{\tilde q'} L_x^{\tilde r'}} \lesssim \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}} \|h\|_{\dot H^{s_3}} \sum_{N_4 \ge 1} N_4^{-s_1 + \frac12}.$$

Since $s_1 > \frac12$, we get the desired estimate.

Case (ii) $N_1 \ge N_3 \ge N_2$**:** We set $N_1 = N_{24}$ and rearrange the summation such that

$$\sum_{N_1 \ge N_3 \ge N_2} = \sum_{N_1} \sum_{N_2} \sum_{N_{24} \ge N_3 \ge N_2}.$$

While applying triangle inequality and Proposition 2.6, the only difference to the previous case (i) is that N_3 is replaced by N_2. Hence by the same argument we get

$$\Big\| \sum_{N_1 \ge N_3 \ge N_2} (\cdot) \Big\|_{L_t^{\tilde q'} L_x^{\tilde r'}} \lesssim \|f\|_{\dot H^{s_1}} \|g\|_{\dot H^{s_2}} \|h\|_{\dot H^{s_3}} \sum_{N_4 \ge 1} N_4^{-s_1 + \frac12}.$$

Since $s_1 > \frac12$, we get the desired estimate.

The remaining cases $(iii)-(vi)$ can be handled by the same way. Repeating the argument one can show

$$\|\sum_{N_3\ge N_1\ge N_2}(\cdot)\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \lesssim \|f\|_{\dot H^{s_1}}\|g\|_{\dot H^{s_2}}\|h\|_{\dot H^{s_3}}\sum_{N_4\ge 1}N_4^{-\frac12+s_2},$$
$$\|\sum_{N_2\ge N_1\ge N_3}(\cdot)\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \lesssim \|f\|_{\dot H^{s_1}}\|g\|_{\dot H^{s_2}}\|h\|_{\dot H^{s_3}}\sum_{N_4\ge 1}N_4^{-\frac12+s_3}.$$

Since $s_1>\frac12$, $s_2,s_3<\frac12$, we get the desired bound. One can also show

$$\|\sum_{N_3\ge N_2\ge N_1}(\cdot)\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \lesssim \|f\|_{\dot H^{s_1}}\|g\|_{\dot H^{s_2}}\|h\|_{\dot H^{s_3}}\sum_{N_4\ge 1}N_4^{-\frac12-s_3},$$
$$\|\sum_{N_2\ge N_3\ge N_1}(\cdot)\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \lesssim \|f\|_{\dot H^{s_1}}\|g\|_{\dot H^{s_2}}\|h\|_{\dot H^{s_3}}\sum_{N_4\ge 1}N_4^{-\frac12-s_2}.$$

Therefore this completes the proof of Theorem 1.2.

3. Smoothing properties

In this section we prove Theorem 1.3. The proof relies on the arguments using the Bourgain space $\mathbf{X}^{s,b}$ for $s,b\in\mathbb{R}$. It consists of the functions u such that

$$\|u\|_{\mathbf{X}^{s,b}} \equiv \left(\iint \langle\xi\rangle^{2s}\langle\tau-|\xi|^2\rangle^{2b}|\widetilde u(\tau,\xi)|^2\,d\tau d\xi\right)^{\frac12}<\infty,$$

where $\widetilde u(\tau,\xi)$ is the time-space Fourier transform of u. We also use the norm $\mathbf{X}^{s,b}(J_T)$ for time interval $J_T=[0,T]$ defined as

$$\|u\|_{\mathbf{X}^{s,b}(J_T)}\equiv\inf\{\|\varphi\|_{\mathbf{X}^{s,b}}:\varphi|_{J_T}=u\}.$$

We give proof by considering Hartree type $V(u)=\kappa|x|^{-2}*|u|^2$ and power type $V(u)=\kappa|u|^{\frac4n}$ nonlinearities, separately.

3.1. **Hartree type nonlinearity.** In this subsection $V(u)$ denotes $\kappa|x|^{-2}*|u|^2$. Let us invoke $V(u)=c|\nabla|^{2-n}(|u|^2)$ for some constant c. Using the $\mathbf{X}^{s,b}$ spaces and Theorem 1.2, one can derive the following.

Proposition 3.1. *Let $n\ge 3$. Then for any $s,b>\frac12$ there exists $0<\epsilon\ll 1$ such that*

$$\|[V(u)u]\|_{\mathbf{X}^{1,-\frac12+\epsilon}}\lesssim\|u\|^3_{\mathbf{X}^{s,b}}. \tag{3.1}$$

Proof. We first show that for $s_0>2+\frac n2$

$$\|\mathcal{H}(\nabla f,g,h)\|_{L_t^\infty L_x^\infty}+\|\mathcal{H}(f,g,\nabla h)\|_{L_t^\infty L_x^\infty}\lesssim\|f\|_{H^{s_0}}\|g\|_{H^{s_0}}\|h\|_{H^{s_0}}. \tag{3.2}$$

Here we do not intend to obtain sharp s_0 but we here are content with some crude estimate which is enough for our purpose. From the Sobolev embedding we note that $\|e^{it\Delta}\langle\nabla\rangle f\|_{L_t^\infty L_x^\infty} \lesssim \|f\|_{H^{s_0}}$. Hence it is enough to show that

$$\||\nabla|^{2-n}(e^{it\Delta}\langle\nabla\rangle f e^{it\Delta}g)\|_{L_t^\infty L_x^\infty} \lesssim \|f\|_{H^{s_0}}\|g\|_{H^{s_0}}.$$

But this follows easily from the observation that

$$\||\nabla|^{2-n}G\|_{L_x^\infty} \lesssim \Big(\int_{|\xi|\le 1} + \int_{|\xi|\ge 1}\Big)|\xi|^{2-n}|\widehat{G}(\xi)|d\xi \lesssim \|G\|_{L^1} + \|G\|_{H^1}$$

together with Leibniz rule and Schwarz's inequality.

Now interpolating (3.2) with the estimate (1.2) of Theorem 1.2, we see that if $(\tilde q, \tilde r)$ is any admissible pair, $s_3 > \frac12$ and $\sum_{i=1}^3 s_i > 1$, then there exist $0 < \epsilon_1, \epsilon_2 \ll 1$ such that

$$\|\mathcal{H}(f,g,\nabla h)\|_{L_t^{\tilde q'+\epsilon_1}L_x^{\tilde r'+\epsilon_2}} \lesssim \|f\|_{H^{s_1}}\|g\|_{H^{s_2}}\|h\|_{H^{s_3}}. \tag{3.3}$$

Similarly, using (3.2) and (1.3), for any admissible pair $(\tilde q, \tilde r)$ and the exponents s_1, s_2, s_3 with $s_1 > \frac12, \sum_{i=1}^3 s_i > 1$, we can find ϵ_1 and ϵ_2 such that

$$\|\mathcal{H}(\nabla f,g,h)\|_{L_t^{\tilde q'+\epsilon_1}L_x^{\tilde r'+\epsilon_2}} \lesssim \|f\|_{H^{s_1}}\|g\|_{H^{s_2}}\|h\|_{H^{s_3}}. \tag{3.4}$$

One may write $u(t,x) = c_n \int e^{it\tau}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)}\widetilde{u}(\tau - |\xi|^2,\xi)d\xi d\tau$ by inversion and translation in frequency variables. Hence, we get

$$(|\nabla|^{2-n}(uv)\nabla w)(x,t) = \iiint \mathcal{H}(f_\tau, g_{\tau'}, \nabla h_{\tau''})(x)\,d\tau d\tau' d\tau'',$$

where $\widehat{f_\tau}(\xi) = \widetilde{u}(\tau - |\xi|^2,\xi)$, $\widehat{g}_\tau(\xi) = \widetilde{v}(\tau - |\xi|^2,\xi)$, $h_\tau(\xi) = \widetilde{w}(\tau - |\xi|^2,\xi)$. From (3.3) and Minkowski's inequality it follows that

$$\||\nabla|^{2-n}(uv)\nabla w\|_{L_t^{\tilde q'+\epsilon_1}L_x^{\tilde r'+\epsilon_2}} \lesssim \iiint \|f_\tau\|_{H^{s_1}}\|g_{\tau'}\|_{H^{s_2}}\|h_{\tau''}\|_{H^{s_3}}\,d\tau d\tau' d\tau''.$$

Plancherel's theorem and Schwarz's inequality yield

$$\||\nabla|^{2-n}(uv)\nabla w\|_{L_t^{\tilde q'+\epsilon_1}L_x^{\tilde r'+\epsilon_2}} \lesssim \|u\|_{\mathbf{X}^{s_1,b}}\|v\|_{\mathbf{X}^{s_2,b}}\|w\|_{\mathbf{X}^{s_3,b}} \tag{3.5}$$

for any $b > \frac12$ and for any $(\tilde q, \tilde r)$, s_1, s_2, s_3 as in (3.3). By repeating the same argument, we also get

$$\||\nabla|^{2-n}(\nabla uv)w\|_{L_t^{\tilde q'+\epsilon_1}L_x^{\tilde r'+\epsilon_2}} \lesssim \|u\|_{\mathbf{X}^{s_1,b}}\|v\|_{\mathbf{X}^{s_2,b}}\|w\|_{\mathbf{X}^{s_3,b}} \tag{3.6}$$

for any $b > \frac12$ and for any $(\tilde q, \tilde r)$, s_1, s_2, s_3 as in (3.4).

We now fix $s, b > 1/2$. To show (3.1), we need to show that

$$\|V(u)u\|_{\mathbf{X}^{0,-\frac12+\epsilon}},\ \|\nabla[V(u)u]\|_{\mathbf{X}^{0,-\frac12+\epsilon}} \lesssim \|u\|_{\mathbf{X}^{s,b}}^3.$$

By duality it suffices to show that

$$|\langle \psi, U\rangle| \lesssim \|\psi\|_{\mathbf{X}^{0,\frac12-\epsilon}} \|u\|^3_{\mathbf{X}^{s,b}}$$

for $U = |\nabla|^{2-n}(|u|^2)u$, $|\nabla|^{2-n}(\nabla u\overline{u})u$, $|\nabla|^{2-n}(u\nabla\overline{u})u$ and $|\nabla|^{2-n}(|u|^2)\nabla u$. We first handle the case $U = |\nabla|^{2-n}(\nabla u\overline{u})u$. By Hölder's inequality we have

$$|\langle \psi, |\nabla|^{2-n}(\nabla u\overline{u})u\rangle| \lesssim \|\psi\|_{L_t^{\tilde q}L_x^{\tilde r}} \||\nabla|^{2-n}(\nabla u\overline{u})u\|_{L_t^{\tilde q'}L_x^{\tilde r'}}.$$

By (3.6) we can choose $(1/\tilde q, 1/\tilde r)$ close enough to the Strichartz line $2/q+n/r = n/2$ so that $2/\tilde q+n/\tilde r > n/2$ and $\||\nabla|^{2-n}(\nabla u\overline{u})u\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \lesssim \|u\|^3_{\mathbf{X}^{s,b}}$. By the choice of $(\tilde q, \tilde r)$ and Lemma 3.2 below, we see $\|\psi\|_{L_t^{\tilde q}L_x^{\tilde r}} \lesssim \|\psi\|_{\mathbf{X}^{0,\frac12-\epsilon}}$ for some $\epsilon > 0$. Hence we get the desired estimate. The remaining cases $U = |\nabla|^{2-n}(u\nabla\overline{u})u$, $|\nabla|^{2-n}(|u|^2)\nabla u$, $|\nabla|^{2-n}(|u|^2)u$ can be similarly shown using (3.6), (3.5) and Lemma 3.2. This completes the proof of Proposition 3.1. $\square$

Lemma 3.2. *Let $q, r \ge 2$ with exception $q \ne 2$ when $n = 2$. If $\frac{2}{q} + \frac{n}{r} \le \frac{n}{2}$, then*

$$\|u\|_{L_t^qL_x^r} \lesssim \|u\|_{\mathbf{X}^{s,b}}$$

for $s = s(q,r) = \frac{n}{2} - \frac{2}{q} - \frac{n}{r}$ and any $b > \frac12$. If $\frac{2}{q} + \frac{n}{r} > \frac{n}{2}$,

$$\|u\|_{L_t^qL_x^r} \lesssim \|u\|_{\mathbf{X}^{0,b+\epsilon}}$$

for $b = b(q,r) = \frac{n+2}{4} - \frac{1}{q} - \frac{n}{2r}$ and any $\epsilon > 0$.

The first follows from the estimates $\|e^{it\Delta}f\|_{L_t^qL_x^r} \lesssim \|f\|_{\dot H^{s(q,r)}}$ and the standard argument (for instance see [23]). Interpolation between the first estimate and the trivial $\|u\|_{L_t^2L_x^2} \le \|u\|_{\mathbf{X}^{0,0}}$ gives the second.

By using the Proposition 3.1 and standard fixed point argument in $\mathbf{X}^{1,\frac12+\epsilon}$ for $0 < \epsilon \ll 1$, we prove the first part of Theorem 1.3. Here we note that $\mathbf{X}^{1,\frac12+\epsilon}(J_T) \hookrightarrow C([0,T]; H^1(\mathbb{R}^n))$.

Proof of (1) *of Theorem 1.3.* We first show the local well-posedness. For this purpose we define a nonlinear functional $\mathcal{N}$ by

$$\mathcal{N}(u) = \phi(t)e^{i\Delta}u_0 - i\phi(t/T)\int_0^t e^{i(t-t')\Delta}[V(u(t'))u(t')]\,dt',$$

where ϕ is a fixed smooth cut-off function such that $\phi(t) = 1$ if $|t| < 1$ and $\phi(t) = 0$ if $|t| > 2$, and $0 < T \le 1$ is fixed. Then we use the well-known properties of $\mathbf{X}^{s,b}$ (for instance see Proposition 2.2 of [16]);

$$\|\phi(t)e^{it\Delta}u_0\|_{\mathbf{X}^{s,b}} \lesssim \|u_0\|_{H^s} \tag{3.7}$$

for any s, b, and

$$\|\int_0^t e^{i(t-t')\Delta}F(t',x)\,dt'\|_{\mathbf{X}^{s,b}(J_T)} \lesssim T^{1+b'-b}\|F\|_{\mathbf{X}^{s,b'}(J_T)}, \tag{3.8}$$

for $s \in \mathbb{R}$ and $-\frac{1}{2} < b' \le 0, 0 \le b \le b'+1$.

Let us define a complete metric space $B_{T,\rho}$ by

$$B_{T,\rho} = \{u \in \mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T) : \|u\|_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T)} \le \rho\}$$

with metric d such that $d(u,v) = \|u-v\|_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T)}$. From (3.7) and (3.8) with $b = \frac{1}{2}+\epsilon, b' = -\frac{1}{2}+\epsilon', \epsilon < \epsilon'$ it follows that for any $u \in B_{T,\rho}$

$$\|\mathcal{N}(u)\|_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}} \lesssim \|u_0\|_{H^s} + T^{\epsilon'-\epsilon}\|V(u)u\|_{\mathbf{X}^{1,-\frac{1}{2}+\epsilon'}}.$$

If ϵ' is sufficiently small, then we deduce from Proposition 3.1 that

$$\|\mathcal{N}(u)\|_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}} \lesssim \|u_0\|_{H^s} + T^{\epsilon'-\epsilon}\|u\|^3_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T)} \lesssim \|u_0\|_{H^s} + T^{\epsilon'-\epsilon}\rho^3.$$

Choosing ρ and T such that $\rho \ge 2C\|u_0\|_{H^s}$ and $CT^{\epsilon'-\epsilon}\rho^3 \le \rho/2$ for some constant C, we see that the functional $\mathcal{N}$ is a map from $B_{T,\rho}$ to itself. One can now easily observe that $\mathcal{N}$ is a contraction. In fact, using Proposition 3.1 again, one can easily see that for any $u, v \in B_{T,\rho}$ and for sufficiently small T

$$\begin{aligned} d(\mathcal{N}(u),\mathcal{N}(v)) &\lesssim T^{\epsilon'-\epsilon}\|V(u)u - V(v)v\|_{\mathbf{X}^{s,-\frac{1}{2}+\epsilon'}(J_T)} \\ &\lesssim T^{\epsilon'-\epsilon}(\|u\|_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T)} + \|v\|_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T)})^2 d(u,v) \\ &\lesssim T^{\epsilon'-\epsilon}\rho^2 d(u,v). \end{aligned}$$

Hence a choice of small T makes $\mathcal{N}$ be a contraction map. Therefore there is a unique $u \in \mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T)$ such that $u(t) = e^{it\Delta}u_0 + D(t)$, where

$$D(t) = -i\int_0^t e^{i(t-t')\Delta}[V(u)u(t')]\,dt'.$$

In view of Proposition 3.1 and the estimate (3.8), we have for $s > 1/2$

$$\|D\|_{\mathbf{X}^{1,\frac{1}{2}+\epsilon}(J_T)} \lesssim \|V(u)u\|_{\mathbf{X}^{1,-\frac{1}{2}+\epsilon}(J_T)} \lesssim \|u\|_{\mathbf{X}^{s,\frac{1}{2}+\epsilon}(J_T)} < \infty.$$

Hence the smoothing effect is obtained. □

3.2. **Power type nonlinearity.** Adopting the argument in the proof of Proposition 3.1 and using Theorem 1.1, one can easily get the following.

Corollary 3.3. *Let (q,r) be a admissible pair satisfying that $2 < r \le 4$ when $n = 3$ and $q, r > 2$ when $n \ge 4$. Then for $b > \frac{1}{2}$ and $0 < s < 1-2/r$,*

$$\|u\nabla v\|_{L_t^{q/2}L_x^{r/2}} \lesssim \|u\|_{\mathbf{X}^{s,b}}\|v\|_{\mathbf{X}^{1-s,b}}.$$

The local well-posedness of the Cauchy problem (1.4) with $V(u) = \kappa|u|^{\frac{4}{n}}$ is well-known in H^s space and also in $\mathbf{X}^{s,b}$ space [16]. Hence using Corollary 3.3 and following the lines of argument in [16] we get the proposition.

Proposition 3.4. *Let $n \ge 3$, $s_n = \frac{1}{2}$ if $n = 3$, and $s_n = 1 - (\frac{4}{n})(\frac{2}{n})$ if $n \ge 4$. Then for $b > \frac{1}{2}$ and every $s > s_n$ there is an $\epsilon > 0$ such that*

$$\|\nabla(|u|^{\frac{4}{n}} u)\|_{\mathbf{X}^{0,-\frac{1}{2}+\epsilon}} \lesssim \|u\|^{\frac{4}{n}+1}_{\mathbf{X}^{s,b}}.$$

Once this is established, the proof of the second part of Theorem 1.3 is almost same as the case of Hartree type nonlinearity, part (1). Hence we omit the details.

Proof of Proposition 3.4. Using duality it is enough to show that

$$|\langle \psi, \nabla(|u|^{\frac{4}{n}} u)\rangle| \lesssim \|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\epsilon}} \|u\|^{\frac{4}{n}+1}_{\mathbf{X}^{s,b}}.$$

By direct differentiation the left hand side is bounded by a constant multiple of

$$\langle |\psi|, |u|^{\frac{4}{n}} |\nabla u|\rangle \lesssim \sum_{N \ge 1} \langle |\psi|, |u|^{\frac{4}{n}} |\nabla u_N|\rangle.$$

Here $u_N = P_N u$ for dyadic $N > 1$ and $u_1 = \widetilde{P}_1 u$ for the projection operator $\widetilde{P}_1$ (recall the notation in introduction). Using Hölder's inequality, we see that for $n = 3$

$$\langle |\psi|, |u|^{\frac{4}{n}} |\nabla u_N|\rangle \lesssim \|\psi\|_{L_t^{q_1} L_x^{r_1}} \|u \nabla u_N\|_{L_t^{q/2} L_x^{r/2}} \|u\|^{\frac{1}{3}}_{L_t^{q_2} L_x^{r_2}},$$

where $(1/q_1, 1/r_1) + (2/q, 2/r) + (1/3)(1/q_2, 1/r_2) = (1, 1)$, and for $n \ge 4$

$$\langle |\psi|, |u|^{\frac{4}{n}} |\nabla u_N|\rangle \lesssim \|\psi\|_{L_t^{q_1} L_x^{r_1}} \|u \nabla u_N\|^{\frac{4}{n}}_{L_t^{q/2} L_x^{r/2}} \|\nabla u_N\|^{1-\frac{4}{n}}_{L_t^{q_2} L_x^{r_2}},$$

where $(1/q_1, 1/r_1) + (4/n)(2/q, 2/r) + (1 - 4/n)(1/q_2, 1/r_2) = 1$. We want to choose admissible (q, r) which is $(\frac{8}{3}, 4)$ for $n = 3$ and arbitrarily close to $(2, \frac{2n}{n-2})$ for $n \ge 4$, respectively, and non-admissible pairs (q_1, r_1) and (q_2, r_2) such that $(1/q_1, 1/r_1)$ and $(1/q_2, 1/r_2)$ are slightly above and below the Strichartz line, respectively. More precisely, $\epsilon_1 > 2/q_1 + n/r_1 - n/2 > 0$ and $0 > 2/q_2 + n/r_2 - n/2 > -\epsilon_2$ for small $\epsilon_1, \epsilon_2 > 0$. With the choices of (q_1, r_1) and (q_2, r_2), using Lemma 3.2 and Proposition 3.3, we have for $|s_1| < 1 - \frac{2}{r}$ and $\epsilon > 0$

$$\begin{aligned}\langle |\psi|, |u|^{\frac{4}{n}} |\nabla u_N|\rangle &\lesssim \|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\epsilon}} \|u\|_{\mathbf{X}^{s_1,b}} \|u_N\|_{\mathbf{X}^{1-s_1,b}} \|u\|^{\frac{1}{3}}_{\mathbf{X}^{\epsilon_0,b}} \\ &\lesssim N^{1-s_1-s} \|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\epsilon}} \|u\|_{\mathbf{X}^{s_1,b}} \|u\|_{\mathbf{X}^{s,b}} \|u\|^{\frac{1}{3}}_{\mathbf{X}^{s,b}}\end{aligned}$$

when $n = 3$, and

$$\begin{aligned}\langle |\psi|, |u|^{\frac{4}{n}} |\nabla u_N|\rangle &\lesssim \|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\epsilon}} \|u\|^{\frac{4}{n}}_{\mathbf{X}^{s_1,b}} \|u_N\|^{\frac{4}{n}}_{\mathbf{X}^{1-s_1,b}} \|\nabla u_N\|^{1-\frac{4}{n}}_{\mathbf{X}^{\epsilon_0,b}} \\ &\lesssim N^{(1-s-s_1 \cdot 4/n + \epsilon_0(1-4/n))} \|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\epsilon}} \|u\|^{\frac{4}{n}}_{\mathbf{X}^{s_1,b}} \|u\|^{\frac{4}{n}}_{\mathbf{X}^{s,b}} \|u\|^{1-\frac{4}{n}}_{\mathbf{X}^{s,b}}\end{aligned}$$

when $n \ge 4$. Therefore, for $s > \max(s_1, 1 - s_1)$ we get

$$|\langle \psi, |u|^{\frac{4}{n}} u\rangle| \lesssim \sum_{N \ge 1} N^{1-s-s_1} \|\psi\|_{\mathbf{X}^{0,\frac{1}{2}-\epsilon}} \|u\|^{\frac{4}{3}+1}_{\mathbf{X}^{s,b}}$$

when $n=3$, and

$$|\langle \psi, |u|^{\frac{4}{n}} u\rangle| \lesssim \sum_{N\ge 1} N^{(1-s-s_1\cdot 4/n+\epsilon_0(1-4/n))}\|\psi\|_{\mathbf{X}^{0,\frac12-\epsilon}}\|u\|_{\mathbf{X}^{s,b}}^{\frac{4}{n}+1}$$

when $n\ge 4$. So we get the desired bound provided $s>1-s_1$ when $n=3$ and $s>1-s_1\frac{4}{n}$ when $n\ge 4$. We now choose admissible pair (q,r) to be $(\frac83,4)$ when $n=3$ and arbitrarily close to $(2,\frac{2n}{n-2})$ when $n\ge 4$. Then we get the desired bound for $s>\frac12$ when $n=3$, and for $s>1-\frac{8}{n^2}$ when $n\ge 4$ because we can choose s_1 to be arbitrarily close to $\frac12$ when $n=3$ and to $\frac2n$ when $n\ge 4$, and $1-\frac{8}{n^2}>\max(\frac2n,1-\frac2n)$. This completes the proof of Proposition 3.4. □

4. Global well-posedess of Hartree equations.

In this section we give the proof of Theorem 1.4 which improves the global well-posedness result in [3]. Based on I-method, the two main ingredients are the almost energy conservation and almost interaction Morawetz inequality. Our improvement results from the better decay control of these crucial estimates (Proposition 4.1, 4.3), which are obtained by exploiting the trilinear interaction estimate (Proposition 2.6). Since we basically follow the usual steps of I-method ([3, 4, 11]), we do not intend to give all the details of the proof. Instead, we are devoted to proving Proposition 4.1, 4.3 after giving a brief explanation about the overall argument.

The I-method, introduced by Colliander *et al.* [5] to handle low regularity initial data, makes use of a smoothing operator I which regularizes a rough solution up to the regularity level of a conservation law by damping the high frequency part. For $0<s<1$ the operator $I: H^s\to H^1$ (depending on a parameter $N\gg 1$) is defined by

$$\widehat{If}(\xi)\equiv m(\xi)\widehat{f}(\xi),$$

where the multiplier $m(\xi)$ is smooth, radially symmetric, nonincreasing in $|\xi|$ and satisfies

$$m(\xi)=\begin{cases} 1 & |\xi|\le N,\\ \left(\frac{N}{|\xi|}\right)^{1-s} & |\xi|\ge 2N.\end{cases}$$

When the solution u of (1.4) is in H^s, $0<s<1$, $E(u)$ may not be finite, but $E(Iu)$ is finite. Since Iu is not a solution to (1.4), $E(Iu)$ is not expected to be conserved. However, it is *almost conserved* and the deviation can be controlled by $O(N^{-\sigma})$, $\sigma>0$, since the operator I gets close to the identity as N increases. In Proposition 4.1 we show that for $p=3/2$

$$E(Iu)(T)=E(Iu_0)+N^{-p+}\Gamma(Z_I(T)), \tag{4.1}$$

where $\Gamma(r) = \sum_{1\le i\le k} O(r^{m_i})$ for some $k, m_1, \ldots, m_k \ge 1$ and $Z_I(T)$ is the iteration space norm defined by

$$Z_I(T) = \sup_{(q,r)\,:\,\text{admissible}} \|I\langle\nabla\rangle u\|_{L_t^q L_x^r(J_T\times\mathbb{R}^n)}.$$

After the Morawetz interaction potential for 3-d NLS was introduced by Colliander *et al.* [6], it was extended to other dimensions [4, 11, 25]. To make use of such estimates (e.g. local in time Morawetz inequality in [11]), the restriction $s > 1/2$ is inevitable. However, this restriction can be removed by using an inequality for Iu ([4]). In fact, it is almost valid in the sense that for some $p > 0$

$$\begin{aligned}
&\|Iu\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}(J_T\times\mathbb{R}^n)} \\
&\lesssim T^{\frac{n-2}{4(n-1)}} \left(\|Iu\|^{\frac{1}{n-1}}_{L_t^\infty L_x^2(J_T\times\mathbb{R}^n)} \|Iu\|^{\frac{n-2}{n-1}}_{L_t^\infty \dot H_x^{\frac12}(J_T\times\mathbb{R}^n)} + \|Iu\|^{\frac{2n-6}{2n-3}}_{L_t^\infty \dot H_x^{\frac12}(J_T\times\mathbb{R}^n)} \right) \\
&\quad + T^{\frac{n-2}{4(n-1)}} N^{-p+}\Gamma(Z_I(T)).
\end{aligned} \tag{4.2}$$

The iteration norm $Z_I(T)$ is controlled initially provided that the critical Strichartz norm of Iu is small (Lemma 3.1 in [3]). More precisely, there is a $\delta > 0$ such that if

$$\|Iu\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}(J_T\times\mathbb{R}^n)} \le \delta, \text{ then } Z_I(T) \lesssim \|\langle\nabla\rangle I u_0\|_{L^2}(\mathbb{R}^n).$$

Therefore, once (4.1), (4.2) are obtained, the global well-posedness follows from the usual accounting argument (see Section 5 in [3] or [4] for details). The threshold regularity s is determined by the decay rate N^{-p+}. Going over the argument in [3], one gets the global well-posedness for $u_0 \in H^s$ as long as $N \gg 1$ can be chosen such that

$$K N^{\frac{1-s}{s}\frac{2(n-2)}{n}} \sim N^{p-}$$

for any arbitrarily large K. This is possible if $s > \frac{2(n-2)}{(2+p)n-4}$. By Proposition 4.1 and Proposition 4.3 with (4.4) we get (4.1) and (4.2) with $p = 3/2$. Therefore we conclude Theorem 1.4.

The rest of this section is devoted to the proofs of the almost energy conservation and almost interaction Morawetz inequality; Proposition 4.1, 4.3.

Proposition 4.1. *Let $0 < s < 1$ and $N \gg 1$. Suppose that $u_0 \in C_0^\infty(\mathbb{R}^n)$ and u is the solution to* (1.4). *Then for any $\epsilon > 0$ and $T > 0$*

$$|E(Iu)(T) - E(Iu_0)| \lesssim N^{-\frac32+\epsilon}(Z_I(T)^4 + Z_I(T)^6 + Z_I(T)^{10} + Z_I(T)^{12}).$$

Now let us consider the I-Hartree equation by

$$i(Iu)_t + \Delta(Iu) = V(Iu)Iu + \big[I(V(u)u) - V(Iu)Iu\big] \equiv \mathcal{N}_{good} + \mathcal{N}_{bad},$$

where $V(u) = |x|^{-2} * |u|^2$. We first recall higher dimensional interaction Morawetz inequality for a general nonlinearity.

Lemma 4.2. *Let u solve*

$$i\partial_t u + \Delta u = \mathcal{N}$$

on $J_T \times \mathbb{R}^n$. Then, we have

$$\begin{aligned}
-(n-1)\int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} &\Delta\big(\frac{1}{|y-x|}\big)|u(x,t)|^2|u(y,t)|^2 dxdydt \\
&+ 2\int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |u(x,t)|^2 \frac{y-x}{|y-x|}\cdot\{\mathcal{N}, u\}(y,t)dxdydt \\
&\lesssim \|u\|^2_{L_t^\infty L_x^2(J_T\times\mathbb{R}^n)}\|u\|^2_{L_t^\infty \dot H^{\frac12}(J_T\times\mathbb{R}^n)} \\
&\qquad + \int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |Im(\mathcal{N}\bar u(t,y))u(t,x)\nabla u(t,x)|dxdydt,
\end{aligned}$$

where $\{f, g\} = Re(f\nabla\overline{g} - g\nabla\overline{f})$.

The above is a slight modification of Proposition 5.5 in [25] which can be similarly shown by making use of a Morawetz inequality

$$|\mathrm{Im}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(t.x)|^2\frac{y-x}{|y-x|}\cdot\nabla u(t,y)\overline{u(t,y)}\,dydx| \lesssim \|u(t)\|^2_{L^2}\|u(t)\|^2_{\dot H^{\frac12}}.$$

By applying the inequality in Lemma 4.2 to I-Hartree equation, we have that

$$\begin{aligned}
-(n-1)\int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} &\Delta\big(\frac{1}{|y-x|}\big)|Iu(x,t)|^2|Iu(y,t)|^2 dxdydt \\
&+ 2\int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |Iu(x,t)|^2\frac{y-x}{|y-x|}\cdot\{\mathcal{N}_{good}, Iu\}(y,t)dxdydt \\
&+ 2\int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |Iu(x,t)|^2\frac{y-x}{|y-x|}\cdot\{\mathcal{N}_{bad}, Iu\}(y,t)dxdydt \\
&\lesssim \|Iu\|^2_{L_t^\infty L_x^2(J_T\times\mathbb{R}^n)}\|Iu\|^2_{L_t^\infty \dot H^{\frac12}(J_T\times\mathbb{R}^n)} \\
&\qquad + \int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathrm{Im}\,(\mathcal{N}_{bad}\overline{Iu}(t,y))\nabla(Iu(t,x))Iu(t,x)|dxdydt.
\end{aligned} \tag{4.3}$$

Since $-\Delta(|x|^{-1}) = 4\pi\delta$ for $n = 3$ and $-\Delta(|x|^{-1}) = c_n|x|^{-(n-3)}$ for $n \geq 4$, the first integral of (4.3) is just

$$c_n(n-1)\int_0^T \int_{\mathbb{R}^n} \Big||\nabla|^{-\frac{n-3}{2}}|Iu(t,x)|^2\Big|^2\, dxdt.$$

We will estimate the error term which is given by

$$\begin{aligned}\text{Error} &= \left|\int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |Iu(x,t)|^2 \frac{y-x}{|y-x|}\cdot \{\mathcal{N}_{bad}, Iu\}(y,t)dxdydt\right| \\ &\quad + \int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathrm{Im}\,(\mathcal{N}_{bad}\overline{Iu}(t,y))\nabla(Iu(t,x))Iu(t,x)|dxdydt.\end{aligned}$$

Since the second term of (4.3) is positive, it follows from interpolation between the estimates $\||\nabla|^{\frac12} f\|_{L_t^\infty L_x^2} \lesssim \|f\|_{L_t^\infty \dot H_x^{\frac12}}$ and $\||\nabla|^{-\frac{n-3}{4}} f\|_{L^4_{t,x}} \lesssim \||\nabla|^{-\frac{n-3}{2}}|f|^2\|^{\frac12}_{L^2_{t,x}}$ (for details see Proposition 4.1 in [3] or Lemma 5.6 in [25]) and Young's inequality that

$$\begin{aligned}&\|Iu\|_{L_t^{2(n-1)} L_x^{\frac{2(n-1)}{n-2}}(J_T\times\mathbb{R}^n)} \\ &\quad\lesssim (\|Iu\|^2_{L_t^\infty L_x^2}\|Iu\|^2_{L_t^\infty \dot H_x^{\frac12}} + \text{ Error})^{\frac{\theta}{4}} \|Iu\|^{1-\theta}_{L_t^\infty \dot H_x^{\frac12}}, \quad \theta = 2/(n-1) \\ &\quad\lesssim \|Iu\|^{\frac{\theta}{2}}_{L_t^\infty L_x^2}\|Iu\|^{1-\frac{\theta}{2}}_{L_t^\infty \dot H_x^{\frac12}} + \|Iu\|^{\frac{4(1-\theta)}{4-\theta}}_{L_t^\infty \dot H_x^{\frac12}} + \text{ Error}.\end{aligned}$$

Now Hölder's inequality in time gives

$$\begin{aligned}&\|Iu\|_{L_t^{\frac{4(n-1)}{n}} L_x^{\frac{2(n-1)}{n-2}}(J_T\times\mathbb{R}^n)} \\ &\quad\lesssim T^{\frac{n-2}{4(n-1)}}\|Iu\|_{L_t^{2(n-1)} L_x^{\frac{2(n-1)}{n-2}}(J_T\times\mathbb{R}^n)} \\ &\quad\lesssim T^{\frac{n-2}{4(n-1)}}\left(\|Iu\|^{\frac{1}{n-1}}_{L_t^\infty L_x^2}\|Iu\|^{\frac{n-2}{n-1}}_{L_t^\infty \dot H_x^{\frac12}} + \|Iu\|^{\frac{2n-6}{2n-3}}_{L_t^\infty \dot H_x^{\frac12}} + \text{ Error}\right).\end{aligned} \tag{4.4}$$

Then (4.2) with $p = 3/2$ is the consequence of the following.

Proposition 4.3. *Let $0 < s < 1$ and $N \gg 1$. Suppose that $u_0 \in C_0^\infty(\mathbb{R}^n)$ and u is the solution to (1.4). Then for any $\epsilon > 0$ and $T > 0$*

$$\text{Error} \lesssim N^{-\frac32+\epsilon}(Z_I(T)^6 + Z_I(T)^{12}).$$

In the whole argument N is assumed to be sufficiently large. So the small frequency part of the solution does not play any significant role. Hence we do not need dyadic decomposition for such portion. Here, we recall that $\widetilde{P}_1 = id - \sum_{N>1} P_N$. For simplicity, abusing notation, we denote $\widetilde{P}_1$ by P_1. Throughout this section $N_1, \dots, N_4$ are dyadic numbers ≥ 1 and $\sum_{N_j\geq 1} P_{N_j} = id$ for $j = 1, \dots, 4$.

4.1. **Preliminary estimates.** We first show the following inhomogeneous estimate (cf. [8]) for the solutions with localized frequency.

Lemma 4.4. *Let N_1, N_2, N_3 be dyadic numbers ≥ 1 and let u be a smooth solution of $iu_t + \Delta u = F$ on $J_T \times \mathbb{R}^n$ with the initial data u_0. Then for $(u_1, u_2, u_3) =$*

$(P_{N_1}u, P_{N_2}u, P_{N_3}u)$ *it holds that for any admissible pair* $(\tilde q, \tilde r)$

$$\sup_{(\eta_1,\eta_2,\eta_3)\in\mathbb{R}^{3n}} \||\nabla|^{2-n}(u_1(\cdot-\eta_1)u_2(\cdot-\eta_2))u_3(\cdot-\eta_3)\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)} \lesssim C(N_1,N_2,N_3)(\|u_0\|_{L^2}^3 + \|F\|^3_{L_t^1L_x^2(J_T\times\mathbb{R}^n)}),$$

where $C(N_1,N_2,N_3)$ *is same as in Proposition 2.6.*

We show this by using Proposition 2.6 which works when all $N_i > 1$. However, due to $P_1(=\widetilde{P}_1)$ which has symbol supported in $B(0,2)$, we need to use Corollary 2.8 when one of N_i is 1. By Proposition 2.6 and Corollary 2.8 we get

$$\|\mathcal{H}(P_{N_1}f, P_{N_2}g, P_{N_3}h)\|_{L_t^{\tilde q'}L_x^{\tilde r'}} \lesssim C(N_1,N_2,N_3)\|P_{N_1}f\|_{L^2}\|P_{N_2}g\|_{L^2}\|P_{N_3}h\|_{L^2} \tag{4.5}$$

for N_1, N_2, N_3 dyadic numbers ≥ 1 and any admissible pair $(\widetilde q, \widetilde r)$.

Proof. By taking P_{N_j} to the equation and using Duhamel's formula, we have

$$\tilde u_j(t) = e^{it\Delta}(\tilde u_{0j} + \tilde{\mathbf{F}}_j(t)),\ j = 1,2,3,$$

where $\tilde u_{0j} = P_{N_j}(u_0(\cdot-\eta_j))$, $\tilde{\mathbf{F}}_j(t,x) = -iP_{N_j}(\int_0^t \tilde F_j(t')\,dt')$ and $\tilde F_j(t') = e^{-it'\Delta}(F(t',\cdot-\eta_j))$. Then we obtain

$$\||\nabla|^{2-n}(\tilde u_1\tilde u_2)\tilde u_3\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)} \lesssim \sum_{k=1}^8 L_k,$$

where

$$\begin{aligned}
L_1 &= \|\mathcal{H}(\tilde u_{01}, \tilde u_{02}, \tilde u_{03})\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}, & L_2 &= \|\mathcal{H}(\tilde{\mathbf{F}}_1, \tilde u_{02}, \tilde u_{03})\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)},\\
L_3 &= \|\mathcal{H}(\tilde u_{01}, \tilde{\mathbf{F}}_2, \tilde u_{03})\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}, & L_4 &= \|\mathcal{H}(\tilde u_{01}, \tilde u_{02}, \tilde{\mathbf{F}}_3)\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)},\\
L_5 &= \|\mathcal{H}(\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2, \tilde u_{03})\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}, & L_6 &= \|\mathcal{H}(\tilde u_{01}, \tilde{\mathbf{F}}_2, \tilde{\mathbf{F}}_3)\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)},\\
L_7 &= \|\mathcal{H}(\tilde{\mathbf{F}}_1, \tilde u_{02}, \tilde{\mathbf{F}}_3)\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}, & L_8 &= \|\mathcal{H}(\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2, \tilde{\mathbf{F}}_3)\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}.
\end{aligned}$$

The first term is easily handled by using (4.5). We only consider L_8. The remaining terms can be treated similarly. By Minkowski's inequality we have

$$L_8 \lesssim \left(\int_{J_T}\left(\int_0^t\int_0^t\int_0^t \|\mathcal{H}(\widetilde F_1(t_1), \widetilde F_2(t_2), \widetilde F_3(t_3))\|_{L^{\tilde r'}}\,dt_1dt_2dt_3\right)^{\tilde q'} dt\right)^{\frac{1}{\tilde q'}}.$$

The right hand side is again bounded by

$$\int_{J_T\times J_T\times J_T} \|\mathcal{H}(\widetilde F_1(t_1), \widetilde F_2(t_2), \widetilde F_3(t_3))\|_{L_t^{\tilde q'}L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}\,dt_1dt_2dt_3.$$

Applying (4.5) again, we get $L_8 \lesssim C(N_1,N_2,N_3)\|F\|^3_{L_t^1L_x^2(J_T\times\mathbb{R}^n)}$. $\square$

Let $\sigma(\xi)$ be an infinitely differentiable function satisfying that for all $\alpha \in N^{3n}$ and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^{3n}$ there is a constant $c(\alpha)$ such that

$$|\partial_\xi^\alpha \sigma(\xi)| \le c(\alpha)(1 + |\xi|)^{-|\alpha|}. \tag{4.6}$$

Let us define a multilinear operator Λ by

$$[\Lambda(f, g, h)](x) = \int e^{ix\cdot\Xi} \sigma(\xi_1, \xi_2, \xi_3)|\xi_2 + \xi_3|^{-(n-2)} \widehat{f}(\xi_1)\widehat{g}(\xi_2)\widehat{h}(\xi_3)\, d\xi_1 d\xi_2 d\xi_3, \tag{4.7}$$

where $\Xi = \sum_{i=1}^3 \xi_i$. We notice that $\Lambda(f, g, h) = cf|\nabla|^{2-n}(gh)$ if $\sigma(\xi_1, \xi_2, \xi_3) = 1$. We now have the following.

Lemma 4.5. *Let Λ be defined as above and let u be a smooth solution of $iu_t + \Delta u = F$ on $J_T \times \mathbb{R}^n$ with the initial data u_0. Then for $(u_1, u_2, u_3) = (P_{N_1}u, P_{N_2}u, P_{N_3}u)$ it holds that for any admissible pair $(\tilde q, \tilde r)$*

$$\|\Lambda(u_1, u_2, u_3)\|_{L_t^{\tilde q'} L_x^{\tilde r'}(J_T\times\mathbb{R}^n)} \lesssim C(N_1, N_2, N_3)(\|u_0\|_{L^2}^3 + \|F\|^3_{L_t^1 L_x^2(J_T\times\mathbb{R}^n)}),$$

where $C(N_1, N_2, N_3)$ is same as in Proposition 2.6.

Proof. We choose another Littlewood-Paley projections $P_{N_i}^o, i = 1, 2, 3$, such that $P_{N_i}^o P_{N_i} = P_{N_i}$ and the corresponding smooth cut-off multiplier $\psi_o(N_i^{-1}\xi)$ is supported in $A(N_i)$ if $N_i > 1$ and $B(0, 2)$ if $N_i = 1$. Then using Fourier inversion for $\widetilde\sigma$ (see below) the multilinear operator Λ can be rewritten as

$$\begin{aligned}
&[\Lambda(u_1, u_2, u_3)](x)\\
&= \int_{\mathbb{R}^{3n}} e^{ix\cdot\Xi}\widetilde\sigma_{1,2,3}(\xi_1, \xi_2, \xi_3)|\xi_2 + \xi_3|^{-(n-2)}\widehat{u_3}(\xi_1)\widehat{u_2}(\xi_2)\widehat{u_1}(\xi_3)\, d\xi_1 d\xi_2 d\xi_3\\
&= c\int_{\eta=(\eta_1,\eta_2,\eta_3)} \widehat{\widetilde\sigma_{1,2,3}}(\eta)[u_1(\cdot-\eta_1)|\nabla|^{2-n}(u_2(\cdot-\eta_2)u_3(\cdot-\eta_3))](x)\, d\eta,
\end{aligned}$$

where $\widetilde\sigma_{1,2,3}(\xi_1, \xi_2, \xi_3) = \psi_o(N_1^{-1}\xi_1)\psi_o(N_2^{-1}\xi_2)\psi_o(N_3^{-1}\xi_3)\sigma(\xi_1, \xi_2, \xi_3)$. By the condition (4.6), support condition of ψ_o and routine integration by parts, we readily get a uniform bound $\|\widehat{\widetilde\sigma_{1,2,3}}\|_{L^1(\mathbb{R}^{3n})} \le C(\alpha)$ with respect to N_1, N_2, N_3 for sufficiently large $|\alpha|$. By Minkowski's inequality and Lemma 4.4 we get

$$\begin{aligned}
&\|\Lambda(u_1, u_2, u_3)\|_{L_t^{\tilde q'} L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}\\
&\lesssim \|\widehat{\widetilde\sigma_{1,2,3}}\|_{L^1(\mathbb{R}^{3n})} \sup_{\eta\in\mathbb{R}^{3n}} \|u_1(\cdot-\eta_1)|\nabla|^{2-n}(u_2(\cdot-\eta_2)u_3(\cdot-\eta_3))\|_{L_t^{\tilde q'} L_x^{\tilde r'}(J_T\times\mathbb{R}^n)}\\
&\lesssim C(N_2, N_3, N_1)(\|u_0\|^3_{L^2} + \|F\|^3_{L_t^1 L_x^2(J_T\times\mathbb{R}^n)}).
\end{aligned}$$

Since $C(N_2, N_3, N_1) = C(N_1, N_2, N_3)$, we obtain the desired estimate. □

The operator $I\langle\nabla\rangle$ behaves like $N^{1-s}|\xi|^s$ for $|\xi| \gtrsim N$. An application of Littlewood-Paley theory shows that the Leibniz rule holds for $I\langle\nabla\rangle(fg)$. Thus taking $I\langle\nabla\rangle$ to the equation (1.4), we have

$$\begin{aligned}
&\|I\langle\nabla\rangle[V(u)u]\|_{L_t^1 L_x^2(J_T\times\mathbb{R}^n)}\\
&\lesssim \Big(\|u\|^2_{L_t^3 L_x^{\frac{6n}{3n-4}}(J_T\times\mathbb{R}^n)} \|I\langle\nabla\rangle u\|_{L_t^3 L_x^{\frac{6n}{3n-4}}(J_T\times\mathbb{R}^n)}\\
&\qquad + \|u\|^2_{L_t^{\frac{8}{3}} L_x^{\frac{4n}{2n-3}}(J_T\times\mathbb{R}^n)} \|I\langle\nabla\rangle u\|_{L_t^4 L_x^{\frac{2n}{n-1}}(J_T\times\mathbb{R}^n)}\Big) \lesssim Z_I(T)^3.
\end{aligned}$$

Combining this and Lemma 4.5, we obtain the following.

Lemma 4.6. *Let u solve $iu_t + \Delta u = V(u)u$ with the initial data $u_0 \in C_0^\infty$. Then for $(u_1, u_2, u_3) = (P_{N_1}u, P_{N_2}u, P_{N_3}u)$ it holds that for any $0 < s < 1$, $T > 0$ and admissible pair $(\tilde q, \tilde r)$*

$$\|\Lambda(I\langle\nabla\rangle u_1, I\langle\nabla\rangle u_2, I\langle\nabla\rangle u_3)\|_{L_t^{\tilde q'} L_x^{\tilde r'}(J_T\times\mathbb{R}^n)} \lesssim C(N_1,N_2,N_3)(Z_I(T)^3 + Z_I(T)^9),$$

where $C(N_1, N_2, N_3)$ is same as in Proposition 2.6.

Now we are ready to prove the propositions. We first give the proof of Proposition 4.1.

4.2. **Proof of Proposition 4.1.** Differentiating the energy $E(Iu)(t)$ of Iu with respect to time, $\frac{d}{dt}E(Iu)(t) = \operatorname{Re}\int_{\mathbb{R}^n} \overline{\partial_t Iu}[V(Iu)Iu - I(V(u)u)]dx$. Thus we get

$$E(Iu(T)) - E(Iu(0)) = \operatorname{Re}\int_0^T \int_{\mathbb{R}^n} \overline{\partial_t Iu}\,[V(Iu)Iu - I(V(u)u)]\; dx\, dt'.$$

We apply the Parseval formula to the right hand side and use the equation (1.4) to get

$$|E(Iu(T)) - E(Iu(0))| \lesssim E_a + E_b,$$

where

$$\begin{aligned}
E_a \;\equiv\; &\Bigg|\int_0^T\!\!\int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right) |\xi_2+\xi_3|^{-(n-2)}\\
&\qquad \times \widehat{\Delta\overline{Iu}}(\xi_1)\,\widehat{\overline{Iu}}(\xi_2)\,\widehat{Iu}(\xi_3)\,\widehat{Iu}(\xi_4)\; d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4\, dt'\Bigg|,
\end{aligned}$$

$$\begin{aligned}
E_b \;\equiv\; &\Bigg|\int_0^T\!\!\int_{\sum_{j=1}^4 \xi_j = 0} \left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right) |\xi_2+\xi_3|^{-(n-2)}\\
&\qquad \times (\widehat{\overline{I(V(u)u)}})(\xi_1)\,\widehat{\overline{Iu}}(\xi_2)\,\widehat{Iu}(\xi_3)\,\widehat{Iu}(\xi_4)\; d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4\, dt'\Bigg|.
\end{aligned}$$

For both E_a and E_b, we break u into $u_i \equiv P_{N_i}u$ $(i = 1, 2, 3, 4)$ and exploit the interactions among Schrödinger waves of different frequency levels using Proposition 2.6.

For E_a we show that for all $T > 0$ and $\epsilon > 0$

$$E_a \lesssim N^{-\frac{3}{2}+\epsilon}(Z_I(T)^4 + Z_I(T)^{10}). \tag{4.8}$$

It was shown in [3] with N^{-1+}. The improvement is actually due to the interaction gain of $(N_j/N_k)^{\frac{1}{2}}$ in Lemma 4.6. Let us set

$$B = B(N_2, N_3, N_4) \equiv \sup_{|\xi_2|\sim N_2, |\xi_3|\sim N_3, |\xi_4|\sim N_4} \left|1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right|.$$

By dyadic decomposition and factoring out $B(N_2, N_3, N_4)$ from the integral in E_a, we get

$$\begin{aligned} E_a &\lesssim \sum_{N_1,N_2,N_3,N_4} B\left|\int_0^T \int_{\mathbb{R}^n} \mathcal{F}^{-1}[\Lambda(\Delta\overline{Iu_1}, \overline{Iu_2}, Iu_3)](\xi_4)\mathcal{F}(Iu_4)(\xi_4)d\xi_4 dt'\right| \\ &= \sum_{N_1,N_2,N_3,N_4} B\left|\int_0^T \int_{\mathbb{R}^n} [\Lambda(\Delta\overline{Iu_1}, \overline{Iu_2}, Iu_3)](x) Iu_4(x)\, dx dt'\right|, \end{aligned}$$

where Λ is the multiplier operator as defined by (4.7) with the symbol

$$\sigma(\xi_1, \xi_2, \xi_3) = \frac{1}{B(N_2, N_3, N_4)}\left(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right).$$

Note that σ satisfies the condition (4.6). Then for (4.8) we need to show

$$\sum_{N_1,N_2,N_3,N_4} B\frac{N_1}{\langle N_2\rangle\langle N_3\rangle\langle N_4\rangle}\mathcal{E}_a(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) \lesssim N^{-\frac{3}{2}+\epsilon}(Z_I(T)^4 + Z_I(T)^{10}),$$

where $\widetilde{u}_1 = N_1^{-1}\Delta\langle\nabla\rangle^{-1}u_1$ and $\widetilde{u}_j = N_j\langle\nabla\rangle^{-1}u_j, j = 2, 3, 4$, and

$$\mathcal{E}_a(w_1, w_2, w_3, w_4) = \left|\int_0^T \int_{\mathbb{R}^n} [\Lambda(I\langle\nabla\rangle\overline{w_1}, I\langle\nabla\rangle\overline{w_2}, I\langle\nabla\rangle w_3)](x) I\langle\nabla\rangle w_4(x)\, dx dt'\right|.$$

We now note that $\mathcal{E}_a(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) = \mathcal{E}_a(\overline{\widetilde{u}_4}, \widetilde{u}_2, \widetilde{u}_3, \overline{\widetilde{u}_1})$. Hence by Hölder's inequality

$$\mathcal{E}_a(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) \leq \|\Lambda(I\langle\nabla\rangle\widetilde{u}_4, I\langle\nabla\rangle\widetilde{u}_2, I\langle\nabla\rangle\widetilde{u}_3)\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}(J_T\times\mathbb{R}^n)}\|I\langle\nabla\rangle\overline{\widetilde{u}_1}\|_{L_t^{\tilde{q}}L_x^{\tilde{r}}(J_T\times\mathbb{R}^n)}.$$

Taking admissible $(\tilde{q}, \tilde{r})$, we apply Lemma 4.6 together with Sobolev imbedding (or Bernstein's inequality) and Hörmander-Mikhlin theorem to get

$$\mathcal{E}_a(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) \lesssim C(N_4, N_2, N_3)(Z_I(T)^4 + Z_I(T)^{10}).$$

For simplicity we also set

$$a(N_1, N_2, N_3, N_4) \equiv \frac{B(N_2, N_3, N_4)N_1}{\langle N_2\rangle\langle N_3\rangle\langle N_4\rangle} \times C(N_4, N_2, N_3).$$

Then for (4.8) it is sufficient to show that for all $t \in J_T$ and $\epsilon > 0$

$$\sum_{N_1,N_2,N_3,N_4 \geq 1} a(N_1,N_2,N_3,N_4) \lesssim N^{-\frac{3}{2}+\epsilon}. \tag{4.9}$$

Proof of (4.9). Since B and $C(N_2,N_3,N_4)$ are symmetric on the permutation of N_2,N_3,N_4, we may assume

$$N_2 \geq N_3 \geq N_4.$$

Then for the proof we consider the sums of the three cases $N \gg N_2$, $N_2 \gtrsim N \gg N_3 \geq N_4$, $N_3 \gtrsim N$, separately.

Case 1. $N \gg N_2$. We have $m(\xi_i) = 1$, $i = 2,3,4$, and $m(\xi_1) = 1$ since $\sum_{i=1}^4 \xi_i = 0$. So, the symbol $(1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}) = 0$. Hence

$$B(N_2,N_3,N_4) = 0.$$

Case 2. $N_2 \gtrsim N \gg N_3 \geq N_4$. Since $\sum_{i=1}^4 \xi_i = 0$, we have $N_1 \sim N_2$. By the mean value theorem,

$$\Big|1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}\Big| = \Big|\frac{m(\xi_2) - m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)}\Big| \lesssim \frac{|\nabla m(\xi_2)\cdot(\xi_3+\xi_4)|}{m(\xi_2)} \lesssim \frac{N_3}{N_2}.$$

Since $C(N_2,N_3,N_4) = (N_4/N_2)^{\frac{1}{2}}$, we thus have

$$a \lesssim \frac{1}{N_2N_4}\left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

Taking sum in the order of N_2,N_3,N_4, we get

$$\sum_{N_1\sim N_2\geq N\gg N_3\geq N_4} a \lesssim N^{-\frac{3}{2}}\ln N.$$

Case 3. $N_3 \gtrsim N$. For this we need only to consider two subcases $N_1 \sim N_2$ and $N_2 \gg N_1$ since the case $N_1 \gg N_2$ cannot happen by the condition $\sum_i \xi_i = 0$.

Case 3-1. $N_3 \gtrsim N$, $N_1 \sim N_2$. In this case, we have the bound

$$\Big|1 - \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)}\Big|\frac{N_1}{N_2N_3N_4} \lesssim \frac{1}{N_3m(\xi_3)N_4m(\xi_4)}$$

since $0 < m(\xi_i) \leq 1$. Then we consider two possible cases $N_1 \sim N_2 \geq N_3 \geq N_4 \gtrsim N$, $N_1 \sim N_2 \geq N_3 \geq N \gg N_4$, separately. When $N_1 \sim N_2 \geq N_3 \geq N_4 \gtrsim N$, we have

$$a \lesssim \sup_{|\xi_3|\sim N_3,|\xi_4|\sim N_4} \frac{1}{N_3m(\xi_3)N_4m(\xi_4)}\left(\frac{N_4}{N_2}\right)^{\frac{1}{2}} \lesssim \frac{1}{N^{2-2s}N_3^sN_4^s}\left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

Hence, summation in the order of N_2,N_3,N_4 gives

$$\sum_{N_1\sim N_2\geq N_3\geq N_4\gtrsim N} a \lesssim N^{-2}.$$

For the case $N_1 \sim N_2 \geq N_3 \geq N \gg N_4$, it follows from the fact $m(\xi_4) = 1$ that

$$a \lesssim \sup_{|\xi_3|\sim N_3, |\xi_4|\sim N_4} \frac{1}{N_3 m(\xi_3) N_4} \left(\frac{N_4}{N_2}\right)^{\frac{1}{2}} \lesssim \frac{1}{N^{1-s} N_3^s N_4} \left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

Summing N_2, N_3, N_4, successively, we have acceptable bound

$$\sum_{N_1\sim N_2\geq N_3 \geq N_4 \gtrsim N} a \lesssim N^{-\frac{3}{2}}.$$

Case 3-2. $N_3 \gtrsim N$, $N_2 \gg N_1$. We have $N_2 \sim N_3$ from $\sum_{i=1}^4 \xi_i = 0$. Since $m(\xi_1) \geq m(\xi_2)$, we get

$$\left|1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\right| \frac{N_1}{N_2 N_3 N_4} \lesssim \frac{m(\xi_1)}{m(\xi_2)^2 m(\xi_4)} \frac{N_1}{N_2^2 N_4}. \tag{4.10}$$

We handle the cases $N_1 \gtrsim N$ and $N_1 \leq N$, separately.

If $N_1 \gtrsim N$, we have three possible cases; $N_2 \sim N_3 \geq N_4 \geq N_1 \geq N$, $N_2 \sim N_3 \geq N_1 \geq N_4 \geq N$, $N_2 \sim N_3 \gg N_1 \gtrsim N \geq N_4$. When $N_2 \sim N_3 \geq N_4 \geq N_1 \geq N$, using (4.10) we get

$$a \lesssim \frac{N_1^s}{N^{2-2s} N_2^{2s} N_4^s} \left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

Summing in the order of N_2, N_4, N_1, we have $\sum_{N_2\sim N_3 \geq N_4 \geq N_1 \geq N} a \lesssim N^{-2}$, which is acceptable. Similarly, when $N_2 \sim N_3 \geq N_1 \geq N_4 \geq N$, it follows that

$$a \lesssim \frac{N_1^s}{N^{2-2s} N_2^{2s} N_4^s} \left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

Then we get $\sum_{N_2\sim N_3 \geq N_1 \geq N_4 \geq N} a \lesssim N^{-2}$ by summing in N_2, N_1, N_4, successively. Finally, when $N_2 \sim N_3 \gg N_1 \gtrsim N \geq N_4$, we have

$$a \lesssim \frac{N_1^s}{N^{1-s} N_2^{2s} N_4} \left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

So summation gives $\sum_{N_2\sim N_3 \gg N_1 \gtrsim N \geq N_4} a \lesssim N^{-\frac{3}{2}}$.

Now we turn to the case $N_1 \leq N$. We again have three possible cases; $N_2 \sim N_3 \geq N_4 \geq N \geq N_1$, $N_2 \sim N_3 \geq N \geq N_4 \geq N_1$, $N_2 \sim N_3 \geq N \geq N_1 \geq N_4$. Firstly, when $N_2 \sim N_3 \geq N_4 \geq N \geq N_1$, we have

$$a \lesssim \frac{N_1}{N^{3-3s} N_2^{2s} N_4^s} \left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}$$

which gives acceptable bound $\sum_{N_2\sim N_3 \geq N_4 \geq N \geq N_1} a \lesssim N^{-2}$. For the case $N_2 \sim N_3 \geq N \geq N_4 \geq N_1$ it follows that

$$a \lesssim \frac{N_1}{N^{2-2s} N_2^{2s} N_4} \left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

So, we have $\sum_{N_2\sim N_3\geq N\geq N_4\geq N_1} a \lesssim N^{-2}$. Finally when $N_2 \sim N_3 \geq N \geq N_1 \geq N_4$, we have

$$a \lesssim \frac{N_1}{N^{2-2s}N_2^{2s}N_4}\left(\frac{N_4}{N_2}\right)^{\frac{1}{2}}.$$

Summation gives the bound $\sum_{N_2\sim N_3\geq N\geq N_1\geq N_4} a \lesssim N^{-\frac{3}{2}}$. Thus we conclude the proof of (4.9). □

Now we turn to E_b and claim

$$E_b \lesssim N^{-\frac{3}{2}+\epsilon}(Z_I(T)^6 + Z_I(T)^{12}). \tag{4.11}$$

Decomposing the integral for E_b dyadically, factoring out $B(N_2, N_3, N_4)$ and using Plancherel's formula as before, we see that

$$\begin{aligned}E_b &\leq \sum_{N_1,N_2,N_3,N_4} B\left|\int_0^T\int_{\mathbb{R}^n}\mathcal{F}^{-1}[\Lambda(\overline{Iu_2}, Iu_3, Iu_4)](\xi_1)\mathcal{F}(P_{N_1}I(V(u)\overline{u}))(\xi_1)\,d\xi_1dt'\right|\\ &= \sum_{N_1,N_2,N_3,N_4} B\left|\int_0^T\int_{\mathbb{R}^n}[\Lambda(I\overline{u_2}, Iu_3, Iu_4)](x)P_{N_1}I(V(u)\overline{u})(x)\,dxdt'\right|.\end{aligned}$$

Hence, we get

$$E_b \lesssim \sum_{N_1,N_2,N_3,N_4}\frac{B}{\langle N_2\rangle\langle N_3\rangle\langle N_4\rangle}\times\mathcal{E}_b(u,\tilde{u}_2,\tilde{u}_3,\tilde{u}_4), \tag{4.12}$$

where $\mathcal{E}_b$ is defined by

$$\mathcal{E}_b(u,w_2,w_3,w_4) \equiv \left|\int_0^T\int_{\mathbb{R}^n}[\Lambda(I\langle\nabla\rangle\overline{w_2}, I\langle\nabla\rangle w_3, I\langle\nabla\rangle w_4)](x)P_{N_1}I(V(u)\overline{u})(x)\,dxdt'\right|.$$

Here $\tilde{u}_j$ are defined by the same way as for $\mathcal{E}_a$. We need the following lemma to get a control of $\mathcal{E}_b$.

Lemma 4.7. *Let u be a smooth solution of $iu_t + \Delta u = V(u)u$ with initial data u_0 on $J_T \times \mathbb{R}^n$. Then, it holds that*

$$\mathcal{E}_b \lesssim C(N_2, N_3, N_4)N_1(Z_I(T)^6 + Z_I(T)^{12}).$$

Proof. For any admissible pair (q, r), the Hölder's inequality yields

$$\mathcal{E}_b \leq \|\Lambda(I\langle\nabla\rangle\overline{\tilde{u}_2}, I\langle\nabla\rangle\tilde{u}_3, I\langle\nabla\rangle\tilde{u}_4)\|_{L_t^{q'}L_x^{r'}(J_T\times\mathbb{R}^n)}\|P_{N_1}I(V(u)\overline{u})\|_{L_t^qL_x^r(J_T\times\mathbb{R}^n)}.$$

Applying Lemma 4.6 and Hörmander-Mikhlin theorem we have

$$\mathcal{E}_b \lesssim C(N_2,N_3,N_4)(Z_I(T)^3 + Z_I(T)^9)\|P_{N_1}I(V(u)\overline{u})\|_{L_t^qL_x^r(J_T\times\mathbb{R}^n)}.$$

Then Lemma 4.7 is the consequence of the estimate

$$\|P_{N_1}I(V(u)\overline{u})\|_{L_t^qL_x^r(J_T\times\mathbb{R}^n)} \lesssim N_1(Z_I(T))^3 \tag{4.13}$$

for admissible (q, r) with $2 \leq q \leq 4$ if $n = 3$ and $2 \leq q \leq \infty$ if $n \geq 4$.

In fact, using Bernstein's inequality and Hörmander-Mikhlin theorem, we see that for $r \ge \tilde r$

$$\|P_{N_1} I(V(u)\overline{u})\|_{L^r_x} \lesssim N_1^{\frac{n}{\tilde r}-\frac{n}{r}} \|P_{N_1} I(V(u)\overline{u})\|_{L^{\tilde r}_x} \lesssim N_1^{\frac{n}{\tilde r}-\frac{n}{r}-1} \|I\langle\nabla\rangle(V(u)\overline{u})\|_{L^{\tilde r}_x}. \tag{4.14}$$

From Leibniz rule for the operator $I\langle\nabla\rangle$ and Hölder's inequality with $1/\tilde r = 1/r_1 + 1/r_2$ we bound $\|I\langle\nabla\rangle(V(u)\overline{u})\|_{L^{\tilde r}_x}$ by

$$\||x|^{-2} * (I\langle\nabla\rangle|u|^2)\|_{L^{r_1}_x} \|u\|_{L^{r_2}_x} + \||x|^{-2} * |u|^2\|_{L^{r_1}_x} \|I\langle\nabla\rangle u\|_{L^{r_2}_x}.$$

It follows from the fractional integration that $\|I\langle\nabla\rangle(V(u)u)\|_{L^{\tilde r}_x} \lesssim \|I\langle\nabla\rangle u\|^3_{L^{r_2}_x}$ for $1/r_1 = 2/r_2 - (n-2)/n$ and $\frac{n}{n-1} < r_2 < \frac{2n}{n-2}$. Since $\tilde r \ge 1$, the equation $1/\tilde r = 3/r_2 - 1 + 2/n$ also implies $r_2 \ge 3n/(2n-2)$. Combining this with (4.14) we get

$$\|P_{N_1} I(V(u)u)\|_{L^q_t L^r_x(J_T\times\mathbb{R}^n)} \lesssim N_1^{n(\frac{3}{r_2}-\frac{1}{r}-1+\frac{2}{n})-1} \|I\langle\nabla\rangle u\|^3_{L^{3q}_t L^{r_2}_x(J_T\times\mathbb{R}^n)}.$$

If (q, r) and $(3q, r_2)$ are admissible, then $n(\frac{3}{r_2} - \frac{1}{r} - 1 + \frac{2}{n}) - 1 = 1$. The admissibility and range of r_2 ensure that $2 \le q \le 4$ if $n = 3$ and $2 \le q \le \infty$ if $n \ge 4$. This proves (4.13). ☐

Using (4.12), Lemma 4.7 and the Hörmander-Mikhlin theorem, we have

$$E_b \lesssim \sum_{N_1,N_2,N_3,N_4} B \frac{\langle N_1\rangle}{\langle N_2\rangle\langle N_3\rangle\langle N_4\rangle} C(N_2, N_3, N_4)(Z_I(T)^6 + Z_I(T)^{12}).$$

Then from (4.9) we see $\sum_{N_1,N_2,N_3,N_4} B \frac{\langle N_1\rangle}{\langle N_2\rangle\langle N_3\rangle\langle N_4\rangle} C \lesssim N^{-\frac{3}{2}+\epsilon}$. Therefore we get the desired estimate (4.11). This completes the proof of Proposition 4.1.

4.3. **Proof of Proposition 4.3.** We recall that $\mathcal{N}_{bad} = I(V(u)u) - V(Iu)Iu$. Then by Hölder's inequality we get

$$\begin{aligned}
\text{Error} &= \left| \int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |Iu(x,t)|^2 \frac{y-x}{|y-x|} \cdot \big(\mathcal{N}_{bad}\nabla\overline{Iu} - Iu\nabla\overline{\mathcal{N}_{bad}}\big)(y,t)\,dxdydt \right| \\
&\quad + \int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathrm{Im}\,(\mathcal{N}_{bad}\overline{Iu}(t,y))\nabla(Iu(t,x))Iu(t,x)|\,dxdydt \\
&\le \left(\int_0^T \int_{\mathbb{R}^n} (|\mathcal{N}_{bad}| \cdot |\nabla Iu| + |\nabla\mathcal{N}_{bad}| \cdot |Iu|)\,dydt \right) \|Iu\|^2_{L^\infty_t L^2_x(J_T\times\mathbb{R}^n)} \\
&\quad + \|\mathcal{N}_{bad}\|_{L^{\tilde q'}_t L^{\tilde r'}_x(J_T\times\mathbb{R}^n)} \|Iu\|_{L^{\tilde q}_t L^{\tilde r}_x} \|\langle\nabla\rangle Iu\|^2_{L^\infty_t L^2_x(J_T\times\mathbb{R}^n)} \\
&\lesssim \|\langle\nabla\rangle\mathcal{N}_{bad}\|_{L^{\tilde q'}_t L^{\tilde r'}_x(J_T\times\mathbb{R}^n)} \|\langle\nabla\rangle Iu\|_{L^{\tilde q}_t L^{\tilde r}_x} \|\langle\nabla\rangle Iu\|^2_{L^\infty_t L^2_x(J_T\times\mathbb{R}^n)} \\
&\lesssim \|\langle\nabla\rangle\big[I(V(u)u) - V(Iu)Iu\big]\|_{L^{\tilde q'}_t L^{\tilde r'}_x(J_T\times\mathbb{R}^n)} (Z_I(T))^3.
\end{aligned}$$

Hence the proof of Proposition 4.3 is reduced to showing that

$$\|\langle\nabla\rangle\big[I(V(u)u) - V(Iu)Iu\big]\|_{L^{\tilde q'}_t L^{\tilde r'}_x(J_T\times\mathbb{R}^n)} \lesssim N^{-\frac{3}{2}}(Z_I(T)^3 + Z_I(T)^9).$$

For any fixed $\psi \in L_t^{\tilde q} L_x^{\tilde r}(J_T \times \mathbb{R}^n)$ we set

$$E_c = \left| \int_0^T \int_{\mathbb{R}^n} \langle \nabla \rangle \big[I(V(u)u) - V(Iu)Iu \big] \overline{\psi}\, dxdt \right|.$$

Then by duality it suffices to show that for $\epsilon > 0$

$$E_c \lesssim N^{-\frac{3}{2}+\epsilon} (Z_I(T)^3 + Z_I(T)^9) \|\psi\|_{L_t^{\tilde q} L_x^{\tilde r}(J_T \times \mathbb{R}^n)}. \tag{4.15}$$

We now follow the similar lines of argument as in the proof of the Proposition 4.1. By Plancherel's theorem we have

$$E_c \sim \left| \int_0^T \int_{\sum_{j=1}^4 \xi_i = 0} \widetilde{\sigma}(\xi_2, \xi_3, \xi_4) |\xi_2 + \xi_3|^{-(n-2)} \widehat{\overline{Iu}}(\xi_2) \widehat{Iu}(\xi_3) \widehat{Iu}(\xi_4) \widehat{\overline{\psi}}(\xi_1) d\xi dt \right|,$$

where

$$\widetilde{\sigma}(\xi_2, \xi_3, \xi_4) = \langle \xi_2 + \xi_3 + \xi_4 \rangle \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2) m(\xi_3) m(\xi_4)} \right).$$

We decompose u_1, u_2, u_3 and ψ into the sum of dyadic pieces $u_j = P_{N_j} u (j = 1, 2, 3, 4)$ and $\psi_1 = P_{N_1} \psi$. Let us define the maximum of $|\widetilde{\sigma}|$ on each dyadic piece by

$$\widetilde{B} = \widetilde{B}(N_2, N_3, N_4) \equiv \sup_{|\xi_2| \sim N_2, |\xi_3| \sim N_3, |\xi_4| \sim N_4} |\widetilde{\sigma}(\xi_2, \xi_3, \xi_4)|.$$

We now set $\sigma(\xi_2, \xi_3, \xi_4) = \widetilde{B}^{-1} \widetilde{\sigma}(\xi_2, \xi_3, \xi_4)$ and define the multilinear operator Λ to be as in (4.7) with the symbol σ. Then

$$E_c \lesssim \sum_{N_1, N_2, N_3, N_4} \left| \int_0^T \int_{\mathbb{R}^n} [\Lambda(\overline{Iu_2}, Iu_3, Iu_4)](x) \overline{\psi}_1(x)\, dxdt \right|.$$

Hence, as before we see

$$E_c \lesssim \sum_{N_1, N_2, N_3, N_4} \frac{\widetilde{B}}{\langle N_2 \rangle \langle N_3 \rangle \langle N_4 \rangle} \times \mathcal{E}_c(u, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4), \tag{4.16}$$

where $\widetilde{u}_j = \langle N_j \rangle \langle \nabla \rangle^{-1} u_j$, $j = 2, 3, 4$ and $\mathcal{E}_c$ is defined by

$$\mathcal{E}_c(\psi_1, w_2, w_3, w_4) = \left| \int_0^T \int_{\mathbb{R}^n} [\Lambda(I \langle \nabla \rangle \overline{w_2}, I \langle \nabla \rangle w_3, I \langle \nabla \rangle w_4)](x) \overline{\psi}_1(x)\, dxdt' \right|.$$

Using Hölder's inequality and Lemma 4.6 as before, we get

$$\mathcal{E}_c \lesssim C(N_2, N_3, N_4) (Z_I(T)^3 + Z_I(T)^9) \|\psi\|_{L_t^{\tilde q} L_x^{\tilde r}(J_T \times \mathbb{R}^n)}.$$

Then by this and (4.16) the proof of (4.15) is reduced to showing that for $\epsilon > 0$

$$\sum_{N_1, N_2, N_3, N_4} \frac{\widetilde{B}(N_2, N_3, N_4)}{\langle N_2 \rangle \langle N_3 \rangle \langle N_4 \rangle} \times C(N_2, N_3, N_4) \lesssim N^{-\frac{3}{2}+\epsilon}.$$

Finally notice that $\widetilde{B} \sim BN_1$, where B is the same upper bound appearing in the estimates of E_a and E_b. Then we get the desired bound from (4.9). This completes the proof of Proposition 4.3.

Appendix; wave packet decomposition

For a fixed $\lambda \gg 1$, let us define the spatial and frequency grids $\mathcal{Y}$, $\mathcal{V}$, by

$$\mathcal{Y} = \lambda^{1/2}\mathbb{Z}^n, \ \mathcal{V} = \lambda^{-1/2}\mathbb{Z}^n \cap Q(2),$$

respectively. For each $(y, v) \in \mathcal{Y} \times \mathcal{V}$, we associate a tube $T_{y,v}$ given by

$$T_{y,v} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |t| \le 4\lambda, \ |x - (y + 2tv)| \le \lambda^{1/2}\}.$$

Obviously $T_{y,v}$ meets $(y, 0)$ and its major direction is parallel to $(2v, 1)$. Let us denote by $\mathcal{T}(\lambda)$ the collection of these cubes. Conversely for a given $T = T_{y,v} \in \mathcal{T}(\lambda)$, we set

$$y_T = y, \ v_T = v.$$

Let η be the function satisfying supp $\widehat{\eta} \subset Q(2)$ and $\sum_{k\in\mathbb{Z}^n} \eta(\cdot - k) = 1$. Let $\psi \in C_0^\infty(B(0,1))$ with $\sum_{k\in\mathbb{Z}^n} \psi(\cdot - k) = 1$. For $T \in \mathcal{T}(\lambda)$ we also set

$$f_T(x) = \eta(\frac{x - y_T}{\lambda^{1/2}})\mathcal{F}^{-1}[\widehat{f}\psi(\lambda^{1/2}(\cdot - v_T))].$$

Then it is obvious that

$$e^{it\Delta}f = \sum_{T\in\mathcal{T}(\lambda)} e^{it\Delta}f_T$$

provided $\widehat{f}$ is supported in $Q(1)$. Then by routine integration by parts one can see that $e^{it\Delta}f_T$ is essentially supported in T. More precisely, for any $\delta > 0$ there is a $C = C(M, \delta)$ such that

$$|e^{it\Delta}f(x)| \le C\lambda^{-M}\|f\|_{L^2} \text{ if } (x,t) \notin \lambda^\delta T.$$

For the details of the wave packet decomposition see [22] (also see [17]). For the proof of Proposition 2.1, we use the following estimates due to Tao [22].

Lemma 4.8 (Relation $\sim$ between wave packets and b)**.** *Let $1 \ll \lambda$, $0 < \delta \ll 1$ and $\{b\}$ be the collection of the cubes b of side length $\sim \lambda^{1-\delta}$ partitioning $Q(\lambda) \times (-\lambda, \lambda)$. Suppose that $f, g \in L^2$ with $\widehat{f}, \widehat{g}$ supported in $Q(3/2)$ and they are decomposed at scale λ such that*

$$f = \sum_{T\in\mathcal{T}(\lambda)} f_T, \quad g = \sum_{T\in\mathcal{T}(\lambda)} g_T.$$

Then if dist(supp $\widehat{f}$*, supp* $\widehat{g}$*)* ~ 1*, there is a relation* $\sim$ *between tubes* $T \in \mathcal{T}(\lambda)$ *and cubes* $b \in \{b\}$ *such that for any* $\epsilon > 0$*,*

$$\sum_b \| \sum_{T\sim b} f_T \|_{L^2}^2 \le C\lambda^\epsilon \|f\|_{L^2}^2, \;\; \sum_b \| \sum_{T\sim b} g_T \|_{L^2}^2 \le C\lambda^\epsilon \|g\|_{L^2}^2, \tag{4.17}$$

and for any b *and* $\epsilon > 0$*,*

$$\| \sum_{T\not\sim b \text{ or } T'\not\sim b} e^{it\Delta} f_T e^{it\Delta} g_{T'} \|_{L^2(b)} \le C\lambda^\epsilon \lambda^{c\delta-(n-1)/4} \|f\|_{L^2} \|g\|_{L^2} \tag{4.18}$$

with c *independent of* δ, ϵ*.*

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