PROBLEMS ON POINTWISE CONVERGENCE OF SOLUTIONS TO THE SCHRÖDINGER EQUATION

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Abstract. In this paper we consider several variants of the pointwise convergence problem for the Schrödinger equation, which generalize the previously known results.

1. Introduction

Let us consider the free Schrödinger equation in $\mathbb{R}^d \times \mathbb{R}$, $d \geqslant 1$,

$$i\partial_t u + \Delta_x u = 0,$$

with initial datum f. Then the solution can be formally written as

$$u(x,t) = e^{it\Delta} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - t|\xi|^2)} \widehat{f}(\xi) d\xi$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$. The problem which was first considered by Carleson [7] is to determine the minimal regularity s for which

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \text{ a.e.}$$

whenever $f \in H^s(\mathbb{R}^d)$. Here H^s is the L^2 Sobolev space of order s which is defined by $\|f\|_{H^s}^2 = \int |\widehat{f}(\xi)|^2 (1+|\xi|)^{2s} d\xi$.

For one spatial dimension ($d = 1$) Carleson showed a.e. convergence for $f \in H^s$ when $s \geqslant 1/4$ and the sharpness was later proven by Dahlberg and Kenig [10] who showed that the condition $s \geqslant 1/4$ is necessary (also see [28]). In higher dimensions, $d \geqslant 2$, Sjölin [23] and Vega [28] independently showed convergence for $f \in H^s$, $s > 1/2$ (also see [6, 9] for earlier results). Results under weaker regularity assumptions ($s < 1/2$) had been known for $d = 2$, which were improved along with the progress of Fourier restriction estimates for the paraboloid or the sphere (see [2, 14, 18, 26, 27]). The best known result is convergence for $s > 3/8$ ([14]). For $d \geqslant 3$, progress was very recently achieved by Bourgain [3]. By making use of multilinear estimates for Fourier extension operators [1, 4] he showed convergence for $s > \frac{1}{2} - \frac{1}{4d}$. Surprisingly, he also showed that the condition $s \geqslant \frac{1}{2} - \frac{1}{d}$ is necessary. So this gives a new lower bound for $d \geqslant 5$.

In this note we consider several variants of the pointwise convergence problem. Notwithstanding recent progresses the problem is still open in higher dimensions $d \geqslant 2$. It might be premature to consider its variants in higher dimensions. So, we mainly work with such variants in $\mathbb{R}^1$ (see also [17] for a related problem in the periodic case).

A natural generalization of the pointwise convergence problem is to ask a.e. convergence along a wider approach region instead of vertical lines. One of such problems may be the

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nontangential convergence to the initial data (boundary values). It is natural to expect that more regularity on the initial data is necessary to guarantee a.e. existence of the nontangential limit. Since $\sup_{t,x} |e^{it\Delta} f(x)| \leqslant \|f\|_{H^s}$ if $s > \frac{d}{2}$ by Sobolev imbedding, the nontangential convergence follows by the standard argument if $s > \frac{d}{2}$. However it was shown by Sjölin and Sjoğren [21] that non-tangential convergence fails for $s \leqslant \frac{d}{2}$. They showed that there is an $f \in H^{d/2}$ such that

$$\limsup_{\substack{(y,t)\to(x,0)\\ |x-y|<\gamma(t),\, t>0}} |e^{it\Delta} f(y)| = \infty$$

for all $x \in \mathbb{R}^d$. Here γ is a strictly increasing function with $\gamma(0) = 0$. This raises a question about how the size (or dimension) of the approach region and the regularity which implies pointwise convergence are related. One may also ask a similar question about the relation between the degree of tangency and regularity when (x, t) approaches to x tangentially. To investigate these questions we consider some model problems.

Convergence along restricted directions in $\mathbb{R} \times \mathbb{R}$. Let Θ be a compact set in $\mathbb{R}$. To measure the dimension of Θ, we use a simple notion of dimension. Let $N(\Theta, \delta)$ be the minimal number of δ-intervals which cover Θ and set

$$\beta(\Theta) = \limsup_{\delta\to 0} \frac{\log N(\Theta, \delta)}{-\log \delta},$$

which is called *upper box counting dimension* (for example see [12]). It is useful for the study of some maximal operators of which boundedness depends on the size of parameter set (see [11, 20] for related works).

We consider convergence of $e^{it\Delta} f(y)$ to $f(x)$ as $(y, t) \to (x, 0)$ while $y - x \in t\Theta$. More precisely, we intend to find the optimal regularity s which guarantees

$$\lim_{\substack{(y,t)\to(x,0)\\ y-x\in t\Theta}} e^{it\Delta} f(y) = f(x) \text{ a.e.} \tag{1.1}$$

whenever $f \in H^s$. Following the usual argument, we consider the associated maximal operator which is given by

$$M_\Theta f(x) = \sup_{(t,\theta)\in[0,1]\times\Theta} |e^{it\Delta} f(x + \theta t)|, \qquad x \in \mathbb{R}.$$

Theorem 1.1. *Let Θ be a compact subset of $\mathbb{R}$. If $s > \frac{\beta(\Theta)+1}{4}$, then*

$$\|M_\Theta f\|_{L^2([-1,1])} \leqslant C\|f\|_{H^s}. \tag{1.2}$$

Obviously, by translation and dilation the above estimate holds for any finite interval. Once it is established, we have the following result which can be proved by the usual argument (see [22] for example).

Corollary 1.2. *Let Θ be a compact subset of $\mathbb{R}$. Then* (1.1) *holds whenever $f \in H^s$ with $s > \frac{\beta(\Theta)+1}{4}$.*

This seems sharp because so is it when $\beta(\Theta) = 1$ ([21]) and $\beta(\Theta) = 0$ ([10]) but we don't know whether it is sharp or not when $0 < \beta(\Theta) < 1$.

Convergence along variable curves $\mathbb{R}^d \times \mathbb{R}$. Let γ be a continuous function such that

$$\gamma : \mathbb{R}^d \times [-1,1] \to \mathbb{R}^d, \qquad \gamma(x,0) = x.$$

Now we consider the pointwise convergence problem along the curve $(\gamma(x,t),t)$. That is to say, we want to find the optimal regularity s for which the convergence

$$\lim_{t\to 0} e^{it\Delta} f(\gamma(x,t)) = f(x) \text{ a.e.} \tag{1.3}$$

holds whenever $f \in H^s$. When the curve γ is smooth, precisely, a C^1 function, it was shown in [15] that the boundedness of related maximal operator is essentially equivalent to that of the free Schrödinger operator. However such smoothness condition excludes the curves which approach $(x,0)$ tangentially to the hyperplane $\{(x,t) : t = 0\}$.

Here, we consider a curve which satisfies Hölder condition of order α, $0 < \alpha \leqslant 1$, in t;

$$|\gamma(x,t) - \gamma(x,t')| \leqslant C|t-t'|^{\alpha} \tag{1.4}$$

and is bilipschitz in x,

$$C_1|x-y| \leqslant |\gamma(x,t) - \gamma(y,t)| \leqslant C_2|x-y|. \tag{1.5}$$

A simple example of such curve is $\gamma(x,t) = x - vt^{\alpha}, v \in \mathbb{R}^d$ with $v \neq 0$. When $d = 1$, we can prove the optimal results except for the endpoint cases (see Proposition 1.5).

Let us denote by $B_R(x) \subset \mathbb{R}^d$ the ball (possibly interval) which has center at x with radius r and by $I_T(t)$ the interval which has center at t and length $2T$.

Theorem 1.3. *Let $d = 1$ and $0 < \alpha \leqslant 1$. Suppose that* (1.4) *and* (1.5) *hold for $x, y \in B_R(x_0)$ and $t, t' \in I_T(t_0)$. Then*

$$\| \sup_{t\in I_T(t_0)} |e^{it\Delta} f(\gamma(x,t))\|_{B_R(x_0)} \leqslant C\|f\|_{H^s} \tag{1.6}$$

holds if $s > \max(\frac{1}{2} - \alpha, \frac{1}{4})$.

Obviously, (1.6) holds for any continuous function γ if $s > \frac{1}{2}$ since the maximal inequality is true by Sobolev imbedding. With $\gamma(x,t) = x + 1/\log(1/t)$ and the interval $(0,1)$, one can show that this inequality fails if $s < 1/2$. See Proposition 1.5 below. The following is an immediate consequence of Theorem 1.3.

Corollary 1.4. *Let $d = 1$ and $0 < \alpha \leqslant 1$. Suppose that for every $x_0 \in \mathbb{R}$, there is a neighborhood V of $(x_0, 0)$ such that* (1.4) *holds for $(x,t), (x,t') \in V$ and* (1.5) *holds for all $(x,t), (y,t) \in V$. Then* (1.3) *holds whenever $f \in H^s$ and $s > \max(\frac{1}{2} - \alpha, \frac{1}{4})$.*

Now we discuss on the necessity of the condition on s in Theorem 1.3. It is sharp in the sense that there are curves γ satisfying both (1.4) and (1.5) but (1.6) fails if $s < \max(\frac{1}{2} - \alpha, \frac{1}{4})$. In fact, we will show this with $\gamma(x,t) = x - t^{\alpha}$ (see Proposition 1.5 below). Furthermore, with this particular curves, it also can be shown that for $s < \max(\frac{1}{2} - \alpha, \frac{1}{4})$, there is an $f \in H^s$ for which (1.3) fails. This can be done by making use of Stein's maximal theorem [24].

In order to show the sharpness of Theorem 1.3, we begin by proving the following proposition.

Proposition 1.5. *Let I be an interval and $\nu : I \to \mathbb{R}^d$ be a continuous function. Suppose that $\gamma(x,t) = x - \nu(t)$ and that there is a point $t_0 \in I$ and $\epsilon > 0$ such that $(t_0, t_0 + \epsilon) \subset I$ and*

$$\text{diam } \{\nu(\tau) : \ \tau \in [t_0, t]\} \gtrsim |t - t_0|^{\alpha}$$

for all $t \in (t_0, t_0 + \epsilon)$. Then (1.6) *holds only if $s \geqslant \max(\frac{1}{2} - \alpha, 0)$.*

Obviously the above assumption is satisfied with $\nu(t) = (t^\alpha, 0, \dots, 0)$ and $s = 0$.

Proof of Proposition 1.5. Fix $\lambda \gg \epsilon^{-1}$. Let us consider f which is given by

$$\widehat{f}(\xi) = e^{it_0|\xi|^2}\psi(\lambda^{-\frac12}\xi)$$

where $\psi \in C_0^\infty(B(0,1))$. Then by rescaling

$$e^{it\Delta} f(\gamma(x,t)) = (2\pi)^{-d}\lambda^{\frac d2}\int e^{i\lambda(t_0-t)|\xi|^2} e^{i\lambda^{\frac12}\gamma(x,t)\cdot\xi}\psi(\xi)d\xi.$$

So, it follows that

$$|e^{it\Delta} f(\gamma(x,t))| \sim \lambda^{\frac d2}$$

if $|t-t_0| \leqslant \lambda^{-1}$ and $|\lambda^{\frac12}\gamma(x,t)| \leqslant c$ for some sufficiently small $c > 0$. So,

$$\sup_{0\leqslant t\leqslant 1} |e^{it\Delta} f(\gamma(x,t))| \sim \lambda^{\frac d2}$$

if x is contained in $O(\lambda^{-\frac12})$-neighborhood of the set $\{\nu(\tau) : \tau \in [t_0, t_0+\lambda^{-1}]\}$ (of length $\gtrsim \lambda^{-\alpha}$) which has measure $\gtrsim \lambda^{-\frac d2}$ if $\alpha \geqslant \frac12$ and $\lambda^{-\frac{d-1}{2}}\lambda^{-\alpha}$ if $\alpha < \frac12$. Hence the maximal inequality (1.6) implies

$$\lambda^{\frac d2}\lambda^{-\frac{d-1}{4}}\max(\lambda^{-\frac14}, \lambda^{-\frac{\alpha}{2}}) \leqslant C\lambda^{\frac s2}\lambda^{\frac d4}.$$

Now letting $\lambda \to \infty$ we get the desired condition. □

To see the necessity of $s \geqslant \frac14$ for (1.6) let us consider $\gamma(x,t) = x - (t^\alpha, 0, \dots, 0)$ and the function f which is given by

$$\widehat{f}(\xi) = \psi(\lambda^{-\frac12}(\xi - \lambda e_1)).$$

Here ψ is a smooth bump function compactly supported in a small neighborhood of the origin and $\lambda \gg 1$. Then by translation and rescaling it is easy to see that

$$|e^{it\Delta} f(\gamma(x,t))| \sim \lambda^{\frac d2}$$

provided that $|t| \leqslant \lambda^{-1}$ and $|\lambda^{\frac12}(x_1 - t^\alpha + 2\lambda t, \bar x))| \leqslant c$ for some small $c > 0$. Here $x = (x_1, \bar x) \in \mathbb{R}^\times\mathbb{R}^{d-1}$. Hence $\sup_{0\leqslant t\leqslant 1}|e^{it\Delta} f(\gamma(x,t))| \sim \lambda^{\frac d2}$ if $0 \leqslant x_1 \leqslant c/100$ and $|\bar x| \leqslant c\lambda^{-\frac12}/100$. So, the maximal inequality (1.6) implies

$$\lambda^{\frac d2}\lambda^{-\frac{d-1}{4}} \leqslant C\lambda^s\lambda^{\frac d4}.$$

Now letting $\lambda \to \infty$ we get the condition $\frac14 \leqslant s$.

Schrödinger equation with quadratic potentials. Let $\omega = (\omega_1, \dots, \omega_d) \in C^1_{loc}(\mathbb{R})$ and set

$$\mathcal{H}_\omega = \frac12\big(\Delta - \sum_{j=1}^d \omega_j(t)x_j^2\big),\ x = (x_1, x_2, \dots, x_d).$$

We now consider the Schrödinger equation with time dependent potential of the form

$$i\partial_t u + \mathcal{H}_\omega u = 0, \quad u(x,0) = f(x), \tag{1.7}$$

with $f \in H^s(\mathbb{R}^d)$. We denote by $e^{it\mathcal{H}_\omega}f$ the solution of (1.7). Then similarly as before we are interested in the problem of finding the optimal s for which

$$\lim_{t\to 0} e^{it\mathcal{H}_\omega} f(x) = f(x)\ a.e. \tag{1.8}$$

whenever $f \in H^s(\mathbb{R}^d)$. When the potential is time independent, namely, $\omega_1 = \omega_2 = \cdots = \omega_d = 1$ (this gives the Hermite Schrödinger operator), it was shown in [15] that the problem is equivalent to that of the free Schrödinger operator except the endpoint cases. In what follows we show that such equivalence is also valid for $e^{it\mathcal{H}_\omega} f$. In fact, it is a consequence of a more general result that local estimates for $e^{it\mathcal{H}_\omega} f$ and $e^{it\Delta} f$ are essentially equivalent in the mixed norm space $L^q_x L^r_t$. Both operators can be related to each other via generalized Mehler's formula [5, 25], which is also called lens transform (see Lemma 4.1).

Let $\mathfrak{c}(t) = (c_1(t), c_2(t), \cdots, c_d(t))$ defined on the interval $I_T(t_0)$ such that,

$$c'_j(t) > 0 \tag{1.9}$$

for all $j = 1, \dots, d$, and $t \in I_T(t_0)$. We define an auxiliary operator

$$U^{\mathfrak{c}}_\gamma f(x,t) = \frac{1}{(2\pi)^d} \int e^{i(\gamma(x,t)\cdot\xi - \sum_{j=1}^d c_j(t)|\xi_j|^2)} \widehat{f}(\xi) d\xi.$$

If $c_1(t) = c_2(t) = \cdots = c_d(t)$, by a simple change of variables $U^{\mathfrak{c}}_\gamma$ can be transformed $e^{it\Delta} f(\widetilde{\gamma}(x,t))$ for some $\widetilde{\gamma}$. But it does not seem that such transformation is available in general. The following is concerned about equivalence between local estimates for $U^{\mathfrak{c}}_\gamma f$ and $e^{it\Delta} f$.

Theorem 1.6. *Let $q, r \geqslant 2$, $s_0 \in \mathbb{R}$, and $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}$. Suppose that $\gamma \in Lip(B_R(x_0) \times I_T(t_0))$ satisfies* (1.5) *for $x, y \in B_R(x_0)$, $t \in I_T(t_0)$, and $\mathfrak{c} \in C^2(\overline{I_T(t_0)})$ satisfies* (1.9)*. Then*

$$\|e^{it\Delta} f\|_{L^q_x(B_1(0), L^r_t[0,1])} \leqslant C\|f\|_{H^s(\mathbb{R}^d)} \tag{1.10}$$

holds for $s > s_0$ if and only if

$$\|U^{\mathfrak{c}}_\gamma f\|_{L^q_x(B_R(x_0), L^r_t(I_T(t_0)))} \leqslant C\|f\|_{H^s(\mathbb{R}^d)} \tag{1.11}$$

holds for $s > s_0$. If we additionally assume that $\gamma \in C^\infty(\overline{B_R(x_0)} \times \overline{I_T(t_0)})$ and $\mathfrak{c} \in C^\infty(\overline{I_T(t_0)})$, then (1.10) *and* (1.11) *are equivalent except for $r = \infty$.*

When γ is smooth, (1.5) can be replaced by $\det D_x\gamma(x,t) \neq 0$ for all $(x,t) \in B_R(x_0) \times I_T(t_0)$. Such equivalence is also valid for the local estimates in $L^q_t L^r_x$. If the signatures of $c'_1(t), c'_2(t), \cdots, c'_d(t)$ are different, then the equivalence between (1.10) and (1.11) fails. For example, when $d \geqslant 2$ $\|e^{it\Box} f\|_{L^2_x L^\infty_t(B_1\times I)} \leqslant \|f\|_{H^s}$ fails if $s < \frac{1}{2}$ ([19]) but $\|e^{it\Delta} f\|_{L^2_x L^\infty_t(B\times I)} \leqslant \|f\|_{H^s}$ is known to be valid for $s > \frac{1}{2} - \frac{1}{4d}$ ([3, 14]). However, from the proof of Theorem 1.6 it is obvious that the same equivalence remains valid if we replace Δ by $\partial^2_{x_1} + \cdots + \partial^2_{x_m} - \partial^2_{x_{m+1}} - \cdots - \partial^2_{x_d}$ where m is the number of positive $c'_i(t)$ and $c'_i(t) \neq 0$.

If we combine Theorem 1.6 and Lemma 4.1, various estimates ([14, 15]) which hold for $e^{it\Delta} f$ remain valid for $e^{it\mathcal{H}_\omega} f$. In particular, from the equivalence between maximal estimates (see [3, 14]) we have the following.

Corollary 1.7. *Suppose $\omega \in C^1(-1,1)$. Then, then* (1.8) *holds whenever $f \in H^s(\mathbb{R}^d)$ and $s \geqslant \frac{1}{4}$ when $d = 1$, $s > \frac{1}{2} - \frac{1}{4d}$ when $d \geqslant 2$.*

The equivalence of local estimates is related to the fact that the propagation speed of Schrödinger waves is not finite. For the wave equation there is no such equivalence as it can be seen by a simple example. In fact, let $\psi \in C^\infty_0(1,2)$ and let us consider

$$\widehat{f}(\xi) = \lambda^{-d} e^{i|\xi|} \psi(|\xi|/\lambda), \ \gamma(x,t) = (t+1)x,$$

and $\lambda \gg 1$. Then by making use of asymptotic expansion for Bessel function it is easy to see that $\|e^{it\sqrt{-\Delta}}f(\gamma(\cdot,t))\|_{L^r_xL^q_t(B_1(2e_1)\times[0,1])} \sim \lambda^{-\frac{d-1}{2}-\frac{1}{r}}$ and $\|e^{it\sqrt{-\Delta}}f\|_{L^r_xL^q_t(B_1(2e_1)\times[0,1])} \sim \lambda^{-\frac{d-1}{2}-\frac{1}{q}}$. Hence this shows that the equivalence fails unless $q = r$. On the contrary when the order of propagation speed increases one can further relax the Lipschitz condition on γ to Hölder conditions. (See Proposition 4.3.)

The paper is organized as follows. In Section 2 we show a few preliminary lemmas including a temporal localization lemma and in Section 3 the proofs of Theorems 1.1 and 1.3 are given. Finally, in Section 4 we prove Theorem 1.6 and Corollary 1.7. Throughout the paper C denotes constants which may be different from line to line.

2. Preliminaries; a temporal localization Lemma

Let $m \geqslant 2$ and $Q(\cdot,t)$ be a real valued smooth function satisfying that for $|\xi| \gg 1$

$$|\nabla_\xi Q(\xi,t) - \nabla_\xi Q(\xi,t')| \sim |t-t'||\xi|^{m-1}, \quad t,t' \in I_T(t_0), \tag{2.1}$$

$$|\partial_\xi^\beta Q(\xi,t) - \partial_\xi^\beta Q(\xi,t')| \leqslant C|t-t'||\xi|^{m-|\beta|}, \ t,t' \in I_T(t_0). \tag{2.2}$$

For a continuous function γ which is defined on $B_R(x_0)\times I_T(t_0)$ let us set

$$T_\gamma^Q f(x,t) = \frac{1}{(2\pi)^d}\int e^{i(\gamma(x,t)\cdot\xi - Q(\xi,t))}\widehat{f}(\xi)d\xi.$$

The following version of temporal localization is very useful for the proof of the theorems. This was first observed in [14] for $e^{it\Delta}f$. A sharp version without ϵ-loss of bounds was obtained [15] by making use of wave-packet decomposition (for example see [13]). Here we provide a simpler proof based on TT^* method.

Lemma 2.1. *Let $\lambda \geqslant 1$ and $\alpha \in \mathbb{R}$. And let $q, r \geqslant 2$ and $\mathfrak{J} = \{J\}$ be a collection of intervals of length λ^{1-m} such that $J \subset I_T(t_0)$ and $\sum_{J\in\mathfrak{J}}\chi_J \leqslant 4$. Suppose that* (2.1) *and* (2.2) *hold for $|\beta| \leqslant \max(2, d-2\alpha+3)$. Also suppose that*

$$\|T_\gamma^Q f\|_{L^q_x\left(B_R(x_0),\, L^r_t(J)\right)} \leqslant C\lambda^\alpha\|f\|_2 \tag{2.3}$$

with C uniform in $J \in \mathfrak{J}$ provided that $\widehat{f}$ is supported in $\{\xi : |\xi| \sim \lambda\}$. Then, there exists $C = C(B, \|\gamma\|_{L^\infty(B_R(x_0)\times I_T(t_0))})$ such that

$$\|T_\gamma^Q f\|_{L^q_x\left(B_R(x_0),\, L^r_t(\bigcup_{J\in\mathfrak{J}} J)\right)} \leqslant C\lambda^\alpha\|f\|_2 \tag{2.4}$$

whenever $\widehat{f}$ is supported in $\{\xi : |\xi| \sim \lambda\}$.

Proof. For simplicity let us set $T = T_\gamma^Q$. Obviously we may assume that the intervals J are disjoint. Since $\widehat{f}$ is supported in the set $\{\xi : |\xi| \sim \lambda\}$, with appropriate $\psi \in C_0^\infty(\mathbb{R}\setminus\{0\})$, we may write

$$Tf(x,t) = \frac{1}{(2\pi)^d}\int e^{i(\gamma(x,t)\cdot\xi - Q(\xi,t))}\widehat{f}(\xi)\psi(|\xi|/\lambda)d\xi.$$

Let T^* denote the adjoint operator of T and set

$$F_J(x,t) = \chi_J(t)F(x,t).$$

Then by duality it is enough to show that if

$$\|T^*F_J\|_2 \leqslant C\lambda^\alpha\|F\|_{q',r'} \tag{2.5}$$

for $J \in \mathfrak{J}$, then

$$\|\sum_{J\in\mathfrak{J}} T^* F_J\|_2 \leqslant C\lambda^{\alpha}\|F\|_{q',r'} \tag{2.6}$$

Here $\|\cdot\|_{q,r}$ denotes $\|\cdot\|_{L^q_x(B_R(x_0),L^r_t(I_T(t_0)))}$. To show (2.6), we may assume that the intervals $\{J\}$ are disjoint. Then (2.6) follows from

$$\big|\sum_{J,J'\in\mathfrak{J}} \langle T^* F_J, T^* F_{J'}\rangle\big| \leqslant C\lambda^{2\alpha}\|F\|^2_{q',r'}. \tag{2.7}$$

We note that

$$TT^*F = \int\int K(x,y,t,t')F(y,t')dydt',$$

where

$$\begin{aligned} K(x,y,t,t') &= \int e^{i\big((\gamma(x,t)-\gamma(y,t'))\cdot\xi-(Q(\xi,t)-Q(\xi,t'))\big)}\psi^2(|\xi|/\lambda)d\xi \\ &= \lambda^d\int e^{i\lambda^m(t-t')\varphi(\xi)}\psi^2(|\xi|)d\xi. \end{aligned}$$

Here we set

$$\varphi(\xi) = \frac{1}{\lambda^m(t-t')}\big(\lambda(\gamma(x,t)-\gamma(y,t'))\cdot\xi-(Q(\lambda\xi,t)-Q(\lambda\xi,t'))\big).$$

Let us set $\|\gamma\|_\infty = \|\gamma\|_{L^\infty(B_R(x_0)\times I_T(t_0))}$. From (2.1) and (2.2) we have

$$\begin{aligned} |\nabla_\xi\varphi(\xi)| &\geqslant C - 2\lambda^{1-m}|t-t'|^{-1}\|\gamma\|_\infty, \\ |\partial^\beta_\xi\varphi(\xi)| &\leqslant C + 2\lambda^{1-m}|t-t'|^{-1}\|\gamma\|_\infty \end{aligned}$$

for some $C>0$. Hence by routine integration by parts ($\max(2,[d-2\alpha]+3)$ times) we see that if $|t-t'| \geqslant C\lambda^{-m+1}(\|\gamma\|_\infty+1)$ for a sufficiently large C,

$$|K(x,y,t,t')| \leqslant C\lambda^d(1+\lambda^m|t-t'|)^{-\max(2,d-2\alpha+2)}. \tag{2.8}$$

So we get for $\operatorname{dist}(J,J') \geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)$

$$\|\chi_J TT^*(F_{J'})\|_{\infty,\infty} \leqslant C\lambda^d(1+\lambda^m \operatorname{dist}(J,J'))^{-\max(2,d-2\alpha+2)}\|F_{J'}\|_{1,1}.$$

Since F may be assumed to be supported in the closure of $B_R(x_0)\times I_T(t_0)$, it follows that if $\operatorname{dist}(J,J') \geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)$

$$\|\chi_J TT^*(F_{J'})\|_{q,r} \leqslant C\lambda^d(1+\lambda^m \operatorname{dist}(J,J'))^{-\max(2,d-2\alpha+2)}\|F\|_{q',r'}.$$

Since $\langle T^*F_J, T^*F_{J'}\rangle = \langle F_J, \chi_J TT^*F_{J'}\rangle$, by Hölder's inequality and using the above

$$\begin{aligned} &\sum_{J,J'\in\mathfrak{J}:\,\operatorname{dist}(J,J')\geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)} \big|\langle T^*F_J, T^*F_{J'}\rangle\big| \\ \leqslant &\sum_{J,J'\in\mathfrak{J}:\,\operatorname{dist}(J,J')\geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)} C\lambda^d(1+\lambda^m \operatorname{dist}(J,J'))^{-\max(2,d-2\alpha+2)}\|F_J\|_{q',r'}\|F_{J'}\|_{q',r'}. \end{aligned}$$

Since $\operatorname{dist}(J,J') \geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)$, for any J'

$$\sum_{J\in\mathfrak{J}:\,\operatorname{dist}(J,J')\geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)} C\lambda^d(1+\lambda^m \operatorname{dist}(J,J'))^{-\max(2,d-2\alpha+2)} \leqslant C\lambda^{2\alpha}\lambda^{-1}.$$

By Schur's test

$$\sum_{J,J'\in\mathfrak{J}:\,\mathrm{dist}(J,J')\geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)} \big|\langle T^*F_J, T^*F_{J'}\rangle\big| \leqslant C\lambda^{2\alpha-1}\big(\sum_J \|F_J\|^2_{q',r'}\big).$$

Since $1\leqslant q',r'\leqslant 2$ and J are disjoint, $\big(\sum_J \|F_J\|^2_{q',r'}\big)\leqslant \|(\sum_J |F_J|^2)^{\frac12}\|^2_{q',r'} = \|\sum_J F_J\|^2_{q',r'}$. Therefore,

$$\sum_{J,J'\in\mathfrak{J}:\,\mathrm{dist}(J,J')\geqslant C\lambda^{1-m}(\|\gamma\|_\infty+1)} \big|\langle T^*F_J, T^*F_{J'}\rangle\big| \leqslant C\lambda^{2\alpha-1}\|\sum_J F_J\|^2_{q',r'}. \tag{2.9}$$

Now, by Hölder's inequality and (2.5) we have $\big|\langle T^*F_J, T^*F_{J'}\rangle\big|\leqslant C\lambda^{2\alpha}\|F_J\|_{q',r'}\|F_{J'}\|_{q',r'}$. Hence,

$$\begin{aligned}
&\sum_{J,J'\in\mathfrak{J}:\,\mathrm{dist}(J,J')< C\lambda^{1-m}(\|\gamma\|_\infty+1)} \big|\langle T^*F_J, T^*F_{J'}\rangle\big| \\
&\leqslant C\lambda^{2\alpha}\sum_{J,J'\in\mathfrak{J}:\,\mathrm{dist}(J,J')< C\lambda^{1-m}(\|\gamma\|_\infty+1)} \|F_J\|_{q',r'}\|F_{J'}\|_{q',r'} \\
&\leqslant C(\|\gamma\|_\infty+1)\lambda^{2\alpha}\big(\sum_J \|F_J\|^2_{q',r'}\big)\leqslant C(\|\gamma\|_\infty+1)\lambda^{2\alpha}\|\sum_J F_J\|^2_{q',r'}.
\end{aligned}$$

Combining this with (2.9), we get the desired inequality (2.7). This completes the proof. □

In general, Lemma 2.1 does not hold with $f\in L^{p,\alpha}$, $p\neq 2$ and it is valid only for local estimates. Lemma 2.1 also provides a simple proof of the local smoothing estimate

$$\|e^{itP}f\|_{L^2_{x,t}\left(B_1(0)\times(0,1)\right)}\leqslant C\|f\|_{H^{-\frac{m-1}{2}}}.$$

(See [8, 23, 28].) Here $e^{itP(D)}f$ is a solution to the dispersive equation (4.7) and P satisfies (4.6). In fact, by Lemma 2.1, Littlewood-Paley decomposition and Plancherel's theorem, it is enough to show that

$$\|e^{itP}f\|_{L^2_{x,t}\left(B_1\times(0,\lambda^{1-m})\right)}\leqslant C\lambda^{-\frac{m-1}{2}}\|f\|_2$$

if $\widehat f$ is supported in $\{\xi : |\xi|\sim\lambda\}$, but this is obvious from Plancherel's theorem and integration in the interval $(0,\lambda^{1-m})$.

Let χ be a smooth function such that $\operatorname{supp}\chi\subset\{|\xi|\sim 1\}$ and $\sum_{k\in\mathbb{Z}}\chi(2^{-k}\cdot)=1$. Let us set $\chi_0=\sum_{k=-\infty}^0\chi(2^{-k}\cdot)=1-\sum_{k=1}^\infty\chi(2^{-k}\cdot)$. As usual, for $k\geqslant 0$, we define the projection operators P_k by

$$\widehat{P_kf}=\chi(2^{-k}\cdot)\widehat f,\ \ k\geqslant 1, \text{ and } \widehat{P_0f}=\chi_0\widehat f.$$

When γ is smooth, it is possible to put together estimates for $U^{\mathfrak{c}}_\gamma P_\lambda f$ without any loss.

Lemma 2.2. *Let γ be a continuous function defined on $B_R(x_0)\times I_{T+\epsilon}(t_0)$, $\epsilon>0$. Suppose that $\partial_t\gamma$ is bounded and smooth in t, that is, $\gamma(x,\cdot)\in C^\infty$ and suppose that $|\partial_tQ(\xi,t)|\sim|\xi|^m$ for $|\xi|\gg 1$, $m>1$. If $1<r<\infty$, then for $N>0$ and $x\in B_R(x_0)$,*

$$\|\sum_{k\geqslant 0}T^Q_\gamma P_kf(x,\cdot)\|_{L^r_t(I_T(t_0))}\leqslant C\|(\sum_{k\geqslant 1}|T^Q_\gamma P_kf(x,\cdot)|^2)^{\frac12}\|_{L^r_t(I_{T+\epsilon}(t_0))}+C_N\|f\|_{H^{-N}(\mathbb{R}^d)}.$$

Proof. Let ψ be a smooth cutoff function which $\psi \equiv 1$ on $I_T(t_0)$ and supported in $I_{T+\epsilon}(t_0)$. For a fixed $x \in B_1$, define $\widetilde{T}_\gamma^Q f$ by

$$\widetilde{T}_\gamma^Q f(t) = \psi(t) T_\gamma^Q f(x,t).$$

Since $\|T_\gamma^Q P_0 f\|_\infty \lesssim \|f\|_{H^{-N}}$ for any N, it is sufficient to show that

$$\|\sum_{k\geqslant 1} \widetilde{T}_\gamma^Q P_k f\|_{L_t^r(I_T(t_0))} \leqslant C\|(\sum_{k\geqslant 1} |\widetilde{T}_\gamma^Q P_k f|^2)^{\frac{1}{2}}\|_{L_t^r(\mathbb{R})} + C_N\|f\|_{H^{-N}(\mathbb{R}^d)}.$$

Let $\widetilde{\psi}$ be a smooth function which $\widetilde{\psi} \equiv 1$ on $\{\tau_0^{-1} \leqslant |\tau| \leqslant \tau_0\}$ and supported on $\{(2\tau_0)^{-1} \leqslant |\tau| \leqslant 2\tau_0\}$ for some $\tau_0 > 0$ and for $k \geqslant 1$, define $\widetilde{P}_k$ by $\widehat{\widetilde{P}_k F} = \widetilde{\psi}(2^{-k}\tau)\widehat{F}(\tau)$. By Minkowski's inequality

$$\begin{aligned}
\|\sum_{k\geqslant 1} \widetilde{T}_\gamma^Q P_k f\|_{L_t^r(I_T(t_0))} &\leqslant \|\sum_{k\geqslant 1} \widetilde{P}_{mk} \widetilde{T}_\gamma^Q P_k f\|_{L_t^r(\mathbb{R})} + \sum_{k\geqslant 1} \|(1-\widetilde{P}_{mk})\widetilde{T}_\gamma^Q P_k f\|_{L_t^r(I_T(t_0))} \\
&= I + II.
\end{aligned}$$

For I, by applying Littlewood-Paley theorem in t, we obtain

$$I \leqslant C\|(\sum_{k\geqslant 1} |\widetilde{T}_\gamma^Q P_k f(x,\cdot)|^2)^{\frac{1}{2}}\|_{L_t^r(\mathbb{R})} \leqslant C\|(\sum_{k\geqslant 1} |T_\gamma^Q P_k f(x,\cdot)|^2)^{\frac{1}{2}}\|_{L_t^r(I_{T+\epsilon}(t_0))}$$

for some $C > 0$. So it suffices to show

$$II \leqslant C_N \|f\|_{H^{-N}(\mathbb{R}^d)}.$$

We now observe that

$$(1-\widetilde{P}_{mk})\widetilde{T}_\gamma^Q P_k f(x,t) = \iint \chi(2^{-k}\xi)(1-\widetilde{\psi}(2^{-mk}\tau))\mathcal{K}(x,\xi,\tau)\widehat{f}(\xi)e^{i\tau t}\,d\tau d\xi, \tag{2.10}$$

where

$$\mathcal{K}(x,\xi,\tau) = \frac{1}{(2\pi)^{d+1}} \int \psi(t) e^{i(\gamma(x,t)\cdot\xi - Q(\xi,t) - t\tau)}\,dt.$$

Since $\tau \leqslant 2^{mk}\tau_0^{-1}$ or $\tau \geqslant 2^{mk}\tau_0$ for $k \geqslant 1$ and $|\xi| \leqslant 2^{k+2}$ and $|\partial_t\gamma| \leqslant C$, we observe that for sufficiently large τ_0, $|\partial_t(\gamma(x,t)\cdot\xi - Q(\xi,t) - t\tau)| \geqslant C\max(2^{mk}, |\tau|)$. By integration by parts, we obtain, for $N > 0$,

$$|\mathcal{K}(x,\xi,\tau)| \leqslant C_N(1 + 2^{mk}|\tau|)^{-N}.$$

Putting this in (2.10) and integrating, we get for any $N > 0$

$$|(1-\widetilde{P}_{mk})\widetilde{T}_\gamma^Q P_k f(t)| \leqslant C_N 2^{-mNk} \int |\chi(2^{-k}\xi)\widehat{f}(\xi)|\,d\xi \leqslant C_{N'} 2^{-2Nk + \frac{d}{2}k}\|P_k f\|_2.$$

Choosing sufficiently large N, by Hölder's inequality and Plancherel's theorem, we see that

$$\|(1-\widetilde{P}_{mk})\widetilde{T}_\gamma^Q P_k f\|_{L^r(I_T(t_0))} \leqslant C_N 2^{-k}\|f\|_{H^{-N}(\mathbb{R}^d)}.$$

Hence we get the desired inequality. □

3. Proofs of Theorems 1.1 and 1.3

In this section we prove Theorems 1.1 and 1.3. The argument here is basically a modification of TT^* argument and it is incorporated with temporal localization (Lemma 2.1) which can be applied after frequency localization.

Proof of Theorem 1.1. Let P_λ be the projection operator which is given by $P_\lambda f = (\psi(\cdot/\lambda)\widehat{f}\,)^\vee$ with $\psi \in C_0^\infty((-2,-\frac12)\cup(\frac12,2))$. In order to prove Theorem 1.1 it is enough to show that

$$\|M_\Theta P_\lambda f\|_{L^2[-1,1]} \leqslant C\sqrt{N(\Theta,\lambda^{-1/2})}\,\lambda^{\frac14}\|f\|_2. \tag{3.1}$$

In fact, from the definition of $\beta(\Theta)$ it follows that $N(\Theta,\lambda^{-1/2}) \lesssim \lambda^{\frac{\beta(\Theta)}{2}+\epsilon}$ for any $\epsilon > 0$. Hence we have

$$\|M_\Theta P_\lambda f\|_{B_1} \leqslant C\lambda^{\frac{\beta(\Theta)+1}{4}+\epsilon}\|f\|_2.$$

By Littlewood-Paley decomposition, triangle inequality and direct summation we get (1.2) whenever $s > \frac{\beta(\Theta)+1}{4}$.

It remains to show (3.1). Let Ω_1, Ω_2,... denote $N(\Theta,\lambda^{-1/2})$ intervals of length $\lambda^{-\frac12}$ which covers Θ. Then by Cauchy–Schwarz's inequality it follows that

$$\begin{aligned} M_\Theta P_\lambda f(x) &= \sup_{(t,\theta)\in I\times\Theta} |e^{it\Delta}P_\lambda f(x+\theta t)| \\ &\leqslant \Big(\sum_{1\leqslant k\leqslant N(\Theta,\lambda^{-1/2})}\ \sup_{(t,\theta)\in I\times\Omega_k} |e^{it\Delta}P_\lambda f(x+\theta t)|^2\Big)^{\frac12}. \end{aligned}$$

Hence, to get (3.1) it is sufficient to show the following.

Lemma 3.1. *If Ω is an interval of length $\lambda^{-\frac12}$, then*

$$\|M_\Omega P_\lambda f\|_{L^2[-1,1]} \leqslant C\lambda^{\frac14}\|f\|_2.$$

Proof of Lemma 3.1. Let us set

$$\chi(x,t,\theta) = \chi_{[-1,1]\times[0,1]\times\Omega}(x,t,\theta)$$

and

$$Tf(x,\theta,t) = \chi(x,t,\theta)\int e^{i(-t|\xi|^2+(x+t\theta)\cdot\xi)}\psi(\xi/\lambda)f(\xi)d\xi.$$

By Plancherel's Theorem the estimate is equivalent to

$$\|Tf\|_{L^2_xL^\infty_{\theta,t}} \leqslant C\lambda^{\frac14}\|f\|_2.$$

We consider the adjoint operator of T which is given by

$$T^*F(\xi) = \psi(\xi/\lambda)\iiint e^{i(t'|\xi|^2-(y+t'\vartheta)\cdot\xi)}\chi(y,t',\vartheta)F(y,t',\vartheta)dydt'd\vartheta.$$

Then by duality (TT^* argument) it is enough to show that

$$\|TT^*\|_{L^2_xL^\infty_{\theta,t}} \leqslant C\lambda^{\frac12}\|F\|_{L^2_xL^1_{\theta,t}}. \tag{3.2}$$

We now note that

$$TT^*F(x,t,\theta) = \chi(x,t,\theta)\iiint K_\lambda(t,t',x,y,\theta,\vartheta)\chi(y,t',\vartheta)F(y,t',\vartheta)dydt'd\vartheta$$

where

$$K_\lambda(t,t',x,y,\theta,\vartheta) = \chi(x,t,\theta)\chi(y,t',\vartheta)\,\lambda\int e^{i\left(\lambda^2(t'-t)|\xi|^2+\lambda(x-y+\theta t-\vartheta t')\cdot\xi\right)}\psi^2(\xi)d\xi.$$

Since $|x|,|y|,t,t',\theta,\vartheta \lesssim 1$, $|\nabla_\xi\big(\lambda^2(t'-t)|\xi|^2+\lambda(x-y+\theta t-\vartheta t')\cdot\xi\big)| \geqslant C\lambda^2|t-t'|$ if $|t-t'| \geqslant C\lambda^{-1}$ for some large C. Hence, by integration by parts it follows that

$$|K_\lambda(t,t',x,y,\theta,\vartheta)| \leqslant C\lambda^{-N}(1+\lambda|t-t'|)^{-N}$$

if $|t-t'| \geqslant C\lambda^{-1}$. So, the operator is localized at scale of λ^{-1} in time. By a standard localization argument it is enough to show that

$$\|TT^*F\|_{L^2_x L^\infty_\theta L^\infty_t(J)} \leqslant C\lambda^{\frac{1}{2}}\|F\|_{L^2_x L^1_\theta L^1_t(J)}.$$

Here $J \subset \Omega$ is an interval of length $\sim \lambda^{-1}$. (For example see the proof of Lemma 2.1). Let us set

$$\widetilde{\chi}(x,t,\theta) = \chi_J(t)\chi(x,t,\theta).$$

and

$$\widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta) = \widetilde{\chi}(x,t,\theta)\widetilde{\chi}(y,t',\vartheta)\,\lambda \int e^{i(\lambda^2(t'-t)|\xi|^2+\lambda(x-y+\theta t-\vartheta t')\cdot\xi)}\psi^2(\xi)d\xi.$$

Then we are reduced to show that

$$\Big\| \int \widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta)F(y,t',\vartheta)dydt'\Big\|_{L^2_x L^\infty_{\theta,t}} \leqslant C\lambda^{\frac{1}{2}}\|F\|_{L^2_x L^1_{\theta,t}}.$$

This follows from Schur's test and the estimates

$$\int \sup_{\theta,t,\vartheta,s} |\widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta)|dy,\ \int \sup_{\theta,t,\vartheta,s} |\widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta)|dx \leqslant C\lambda^{\frac{1}{2}}. \tag{3.3}$$

We now claim that

$$|\widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta)| \lesssim \lambda(1+\lambda|x-y|)^{-\frac{1}{2}} \tag{3.4}$$

provided that $|x-y| \geqslant C\lambda^{-\frac{1}{2}}$ for some large constant $C>0$. Since $t,t' \in J \subset [0,1]$ and $\theta,\vartheta \in \Omega$ we have $t\theta - t'\vartheta = O(\lambda^{-\frac{1}{2}})$ because J, Ω are intervals of length $\sim \lambda^{-1}$, $\lambda^{-\frac{1}{2}}$, respectively. So, if $|x-y| \geqslant C\lambda^{-\frac{1}{2}}$, then $|x-y+\theta t-\vartheta t'| \sim |x-y|$. On the other hand, if $|\lambda^2(t-t')| \gg \lambda|x-y|$ or $|\lambda^2(t-t')| \ll \lambda|x-y|$, by integration by parts it follows that $|\widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta)| \leqslant \lambda(1+\lambda|x-y|)^{-N}$. Hence we may assume that $|\lambda^2(t-t')| \sim \lambda|x-y|$. Then by Van der Corput's lemma $|\widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta)| \leqslant C\lambda(1+\lambda^2|t'-t|)^{-\frac{1}{2}}$. So, we get (3.4).

Since $|\widetilde{K}_\lambda| \lesssim \lambda$, by (3.4)

$$\int \widetilde{K}_\lambda(t,t',x,y,\theta,\vartheta)|dx \leqslant C\lambda(\int_0^{C\lambda^{-\frac{1}{2}}} dx + \lambda^{-\frac{1}{2}}\int_0^1 |x-y|^{-\frac{1}{2}}dx) \lesssim \lambda^{\frac{1}{2}}.$$

Hence we get the desired estimates (3.3). This completes the proof. □

Proof of Theorem 1.3. By changing variables $(x,t) \to (x_0+Rx, t_0-T+2Tt)$, we may assume that $B_R(x_0) = [-1,1]$ and $I_T(t_0) = [0,1]$. We set

$$U_\gamma f(x,t) = e^{it\Delta}f(\gamma(x,t)),$$

and

$$U^*_\gamma f(x) = \sup_{0\leqslant t\leqslant 1} |e^{it\Delta}f(\gamma(x,t))|.$$

By Littlewood-Paley decomposition it is sufficient to show that for $s \geqslant \max(\frac{1}{2}-\alpha, \frac{1}{4})$

$$\|U^*_\gamma P_\lambda f\|_{L^2_x L^\infty_t([-1,1]\times[0,1])} \leqslant C\lambda^s\|f\|_2,$$

where as before P_λ is the projection operator to the set $\{|\xi| \sim \lambda\}$. Let J be an interval of length λ^{-1} contained in $[0,1]$. By Lemma 2.1, it is enough to show

$$\|U^*_\gamma P_\lambda f\|_{L^2_x L^\infty_t([-1,1]\times J)} \leqslant C\lambda^s\|f\|_2$$

with C independent of J. By TT^* argument it suffices to show that

$$\|\int K(x,y,t,t')F(y,t')dydt'\|_{L^2_xL^\infty_t([-1,1]\times J)} \leqslant C\lambda^{2s}\|F\|_{L^2_xL^1_t([-1,1]\times J)} \tag{3.5}$$

where

$$K(x,y,t,t') = \int e^{i\big((\gamma(x,t)-\gamma(y,t'))\cdot\xi+(t'-t)|\xi|^2\big)}\psi^2(\xi/\lambda)d\xi$$

and $\psi \in C_0^\infty((-2,-\frac{1}{2})\cup(\frac{1}{2},2))$. Changing of variables $\xi \to \lambda\xi$, we have

$$K(x,y,t,t') = \lambda\int e^{i\big(\lambda(\gamma(x,t)-\gamma(y,t'))\cdot\xi+\lambda^2(t'-t)|\xi|^2\big)}\psi^2(\xi)d\xi.$$

Lemma 3.2. *Let $J \subset [0,1]$ be an interval of sidelength λ^{-1}. Suppose that $\gamma : [0,1] \to \mathbb{R}$ satisfies* (1.4) *and* (1.5)*. Then, if $|x-y| \geqslant C\lambda^{-\alpha}$ for some large C, then for $t,t' \in J$*

$$|K(x,y,t,t')| \leqslant C\lambda(1+\lambda|x-y|)^{-\frac{1}{2}}. \tag{3.6}$$

Proof. Let us set

$$\varphi(\xi) = \lambda(\gamma(x,t)-\gamma(y,t'))\cdot\xi+\lambda^2(t'-t)|\xi|^2.$$

Since $t,t' \in J$, from the conditions (1.4) and (1.5) we observe that

$$\begin{aligned}\gamma(x,t)-\gamma(y,t') &= (\gamma(x,t)-\gamma(x,t'))+(\gamma(x,t')-\gamma(y,t')),\\ |\gamma(x,t)-\gamma(x,t')| &\lesssim |t-t'|^\alpha = O(\lambda^{-\alpha}), \qquad (3.7)\\ |\gamma(x,t')-\gamma(y,t')| &\sim |x-y|.\end{aligned}$$

So, we separately consider three cases:

$$|x-y| \gg \lambda|t-t'|,\ |x-y| \ll \lambda|t-t'|,\ |x-y| \sim \lambda|t-t'|.$$

For the first case $|x-y| \gg \lambda|t-t'|$, we have $|\frac{d}{d\xi}\varphi| \gtrsim \lambda|x-y|$ because $|x-y| \geqslant C\lambda^{-\alpha}$. Hence, by non stationary phase method (integration by parts), we get for any N

$$|K(x,y,t,t')| \leqslant C\frac{\lambda}{(1+\lambda|x-y|)^N}.$$

If $|x-y| \ll \lambda|t-t'|$, then we see that $|\frac{d}{d\xi}\varphi(\xi)| \gtrsim \lambda^2|t-t'|$ because ψ is supported away from zero. Integration by parts gives the bound

$$|K(x,y,t,t')| \leqslant C\frac{\lambda}{(1+\lambda^2|t-t'|)^N} \leqslant C\frac{\lambda}{(1+\lambda|x-y|)^N}.$$

For the last case $|x-y| \sim \lambda|t-t'|$, $|\frac{d^2}{d\xi^2}\varphi| \gtrsim \lambda^2|t-t'|$. Hence by van der Corput's lemma we obtain that

$$|K(x,y,t,t')| \leqslant C\lambda\frac{1}{(1+\lambda^2|t-t'|)^{1/2}} \sim \frac{\lambda}{(1+\lambda|x-y|)^{1/2}}.$$

Combining these three cases we get the desired (3.6). ☐

Lemma 3.3. *Assume that $J \subset [0,1]$ be an interval of sidelength $\lambda^{-1}$and $\gamma : [0,1] \to \mathbb{R}$ satisfies* (1.4) *and* (1.5)*. Then, for $t,t' \in J$*

$$|K(x,y,t,t')| \leqslant C\max(\frac{\lambda^{1/2}}{|x-y|^{1/2}}, |x-y|^{-\frac{1}{2\alpha}}). \tag{3.8}$$

Proof. Here we use the same notation as in the proof of Lemma 3.2. We first consider two cases $|x-y| \lesssim |t-t'|^\alpha$, $|x-y| \gg |t-t'|^\alpha$, separately. If $|x-y| \lesssim |t-t'|^\alpha$, we use the fact that $|\frac{d^2}{d\xi^2}\varphi| \gtrsim \lambda^2|t-t'|$ and van der Corput's lemma to obtain

$$|K(x,y,t,t')| \lesssim \lambda \frac{1}{(1+\lambda^2|t-t'|)^{1/2}} \leqslant |t-t'|^{-1/2} \lesssim |x-y|^{-1/2\alpha}.$$

Now we consider the case $|x-y| \gg |t-t'|^\alpha$. Then, recalling (3.7), we see that

$$|\frac{d}{d\xi}\varphi(\xi)| = |\lambda(\gamma(x,t)-\gamma(y,t')) + 2\lambda^2(t'-t)\xi| \gtrsim \lambda|x-y| - O(\lambda^2|t-t'|).$$

Thus if $|x-y| \gg \lambda|t-t'|$, then $|\frac{d}{d\xi}\varphi| \geqslant \lambda|x-y|$. So it follows from integration by parts that $|K(x,y,t,t')| \leqslant \frac{\lambda}{(1+\lambda|x-y|)^N}$. And if $\lambda|t-t'| \gg |x-y|$, then by van der Corput's lemma again we have

$$|K(x,y,t,t')| \leqslant \frac{\lambda^{1/2}}{|x-y|^{1/2}}.$$

Hence we have the desired bounds. $\square$

Now we prove (3.5). We break the interval $[-1,1]$ into essentially intervals of side length $C\lambda^{-\alpha}$ so that $[-1,1] = \bigcup I_k$. So we bound the square of the left hand side of (3.5) by

$$\sum_k \| \sum_{k'} \int \chi_{I_k}(x)K(x,y,t,t')\chi_{I_{k'}}(y)F(y,t')dydt'\|^2_{L^2_xL^\infty_t(I_k\times J)},$$

which is again bounded by the sum of

$$2\sum_k \| \int \chi_{I_k}(x)K(x,y,t,t')\chi_{\widetilde{I}_k}(y)F(y,t')dydt'\|^2_{L^2_xL^\infty_t(I_k\times J)}\,, \tag{3.9}$$

$$2\sum_k \| \sum_{k'\not\sim k} \int \chi_{I_k}(x)K(x,y,t,t')\chi_{I_{k'}}(y)F(y,t')dydt'\|^2_{L^2_xL^\infty_t(I_k\times J)}, \tag{3.10}$$

where we say $k \not\sim k'$ if the distance between the two intervals I_k and $I_{k'}$ is bigger than $4C\lambda^{-\alpha}$ and $\widetilde{I}_k$ is an interval containing I_k and of length slightly bigger than $2C\lambda^{-\alpha}$.

We handle the case $\frac{1}{2} \leqslant \alpha \leqslant 1$ first. In this case we need to show (3.5) with $s=\frac{1}{4}$. We first deal with (3.9), which is easier. Note that $|K| \leqslant C\lambda$ and the length of interval $I_k \sim \lambda^{-\alpha}$. Hence it follows that

$$\int \sup_{t,t'\in J} |\chi_{I_k}(x)K(x,y,t,t')\chi_{\widetilde{I}_k}(y)|dx, \int \sup_{t,t'\in J} |\chi_{I_k}(x)K(x,y,t,t')\chi_{\widetilde{I}_k}(y)|dy \leqslant C\lambda^{1-\alpha}.$$

Schur's test gives the bound

$$\Big\| \int \chi_{I_k}(x)K(x,y,t,t')\chi_{\widetilde{I}_k}(y)F(y,t')dydt'\Big\|^2_{L^2_xL^\infty_t(I_k\times J)} \leqslant C\lambda^{2(1-\alpha)}\|F_k\|^2_{L^2_xL^1_t},$$

where $F_k(x,t) = \chi_{\widetilde{I}_k}(x)F(x,t)$. Now using the disjoint of the supports, we get

$$(3.9) \leqslant C\lambda^{2(1-\alpha)} \sum_k \|F_k\|^2_{L^2_xL^1_t} \leqslant C\lambda^{2(1-\alpha)}\|F\|^2_{L^2_xL^1_t} \leqslant C\lambda\|F\|^2_{L^2_xL^1_t}.$$

Now we consider (3.10). Since $\mathrm{dist}(I_k, I_{k'}) \geqslant C\lambda^{-\alpha}$ if $k \neq k'$, from Lemma 3.2 we see that

$$\begin{aligned}
&|\sum_{k' \not\sim k} \int \chi_{I_k}(x) K(x,y,t,t') \chi_{I_{k'}}(y) F(y,t') dy dt'| \\
&\qquad \leqslant \chi_{I_k}(x) \int |K(x,y,t,t')| |\sum_{k' \not\sim k} \chi_{I_{k'}}(y) F(y,t')| dy dt' \\
&\qquad \leqslant C\chi_{I_k}(x) \int \lambda (1+\lambda|x-y|)^{-\frac12} |\sum_{k' \not\sim k} \chi_{I_{k'}}(y) F(y,t')| dy dt' \\
&\qquad \leqslant C\chi_{I_k}(x) \int \lambda (1+\lambda|x-y|)^{-\frac12} |F(y,t')| dy dt'.
\end{aligned}$$

Hence (3.10) is bounded by

$$\| \int \lambda (1+\lambda|x-y|)^{-\frac12} |F(y,t')| dy dt' \|^2_{L^2_x L^\infty_t([-1,1]\times J)}.$$

Since $\|\lambda(1+\lambda|\cdot|)^{-\frac12}\|_{L^1[-2,2]} \leqslant C\lambda^{\frac12}$. Hence by Schur's test again we get

$$(3.10) \leqslant C\lambda \|F\|^2_{L^2_x L^1_t}.$$

Combining the above two estimate for (3.9) and (3.10), we get (3.5) with $s = \frac14$.

Now we consider the case $0 < \alpha < 1/2$. Note that $\int_{-1}^1 \min(|x-y|^{-\frac{1}{2\alpha}}, \lambda) \lesssim \lambda^{1-2\alpha}$ when $0 < \alpha < 1/2$. Hence by (3.8) and the fact that $|K| \leqslant C\lambda$ implies

$$\int_{-1}^1 \sup_{t,t'\in J} |K(x,y,t,t')| dx, \int_{-1}^1 \sup_{t,t'\in J} |K(x,y,t,t')| dy \lesssim \max(\lambda^{\frac12}, \lambda^{1-2\alpha}).$$

Using Schur's test again, we obtain

$$\| \int K(x,y,t,t') F(y,t') dy dt' \|_{L^2_x L^\infty_t([-1,1]\times J)} \leqslant C \max(\lambda^{\frac12}, \lambda^{1-2\alpha}) \|F\|_{L^2_x L^\infty_t([-1,1]\times J)}.$$

This completes the proof of Theorem 1.3.

4. Time dependent quadratic potentials; proof of Theorem 1.6

We begin by recalling the following result in [25] (Lemma 1 in [25] and also see Lemma 3.3 in [5]) which generalizes Mehler's formula to the equation (1.7) with $\omega \in C^1_{loc}$. [1)] If $\omega \in C^1_{loc}$, there exists $T > 0$ such that for $t \in (-T, T)$

$$e^{it\mathcal{H}_\omega} f = \Big(\prod_{j=1}^d 2\pi i \tau_j(t) \Big)^{-\frac12} \int_{\mathbb{R}^d} e^{\frac{i}{2}\phi(x,y,t)} f(y) dy,$$

where

$$\phi(x,y,t) = \sum_{j=1}^d (a_j(t) x_j^2 - 2b_j(t) x_j y_j + d_j(t) y_j^2),$$

1) In fact, it remains valid for locally Lipschitz continuous function ω. See Lemma 3.3 in [5].

and τ_j, a_j, b_j, and d_j are given by

$$\tau_j'' + \omega_j(t)\tau_j = 0; \quad \tau_j(0) = 0, \quad \tau_j'(0) = 1, \tag{4.1}$$

$$a_j = \tau_j'/\tau_j, \quad b_j = 1/\tau_j,$$

$$d_j = \tau_j^{-1}\Big((\tau_j')^{-1} - \int_0^t \omega_j(t')(\tau_j'(t'))^{-2}dt'\Big). \tag{4.2}$$

Using this we have the following lemma which relates $e^{it\mathcal{H}_\omega}f$ to $e^{it\Delta}f$.

Lemma 4.1. *Let $\omega \in C^1_{loc}$. There is a $T > 0$ such that*

$$e^{itH_\omega}f(x,t) = \varepsilon(x,t)U^{\mathfrak{c}}_\gamma f(x,t), \ (x,t) \in \mathbb{R}^d \times [-T,T],$$

$|\varepsilon(x,t)| \sim 1$, and $\mathfrak{c} \in C^2[-T,T]$ satisfies $\mathfrak{c}(t) = (t,t,\dots,t,t) + O(t^2)$ and

$$\gamma(x,t) = (\gamma_1(t)x_1, \gamma_2(t)x_2, \cdots, \gamma_d(t)x_d), \ \ \gamma_j(t) = 1 + O(t).$$

Proof. By completing square and using the fundamental solution to the free Schrödinger equation we see

$$\begin{aligned} e^{it\mathcal{H}_\omega} &= \frac{e^{\frac{i}{2}\sum_{j=1}^d (a_j - \frac{b_j^2}{d_j^2})x_j^2}}{\prod_{j=1}^d (2\pi i\tau_j(t))^{1/2}} \int_{\mathbb{R}^d} e^{\frac{i}{2}\sum_{j=1}^d d_j(t)(y_j - \frac{b_j(t)}{d_j(t)}x_j)^2} f(y)dy \\ &= \frac{e^{\frac{i}{2}\sum_{j=1}^d (a_j - \frac{b_j^2}{d_j^2})x_j^2}}{\prod_{j=1}^d (\tau_j(t)d_j(t))^{1/2}} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\sum_{j=1}^d \frac{b_j(t)}{d_j(t)}x_j\xi_j - \sum_{j=1}^d \frac{1}{2d_j(t)}|\xi_j|^2)} \widehat{f}(\xi)d\xi. \end{aligned}$$

Now we set

$$\begin{aligned} \varepsilon(x,t) &= e^{\frac{i}{2}\sum_{j=1}^d (a_j - \frac{b_j^2}{d_j^2})x_j^2} \prod_{j=1}^d (\tau_j(t)d_j(t))^{-\frac{1}{2}}, \\ c_j(t) &= \frac{1}{2d_j(t)} = \frac{1}{2}\tau_j\Big((\tau_j')^{-1} - \int_0^t \omega_j(t')(\tau_j'(t'))^{-2}dt'\Big)^{-1}, \\ \gamma_j(t) &= b_j(t)(d_j(t))^{-1} = \Big((\tau_j')^{-1} - \int_0^t \omega_j(t')(\tau_j'(t'))^{-2}dt'\Big)^{-1}. \end{aligned}$$

Since $\omega \in C^1_{loc}$, it follows from (4.1) that $\tau_j \in C^3$ locally. Using the second equation above, we see that $c_j \in C^2$ locally. From (4.1), $\tau_j(t) = t + O(t^2)$ and $\tau_j'(t) = 1 + O(t)$. Hence it is easy to see $|\varepsilon(x,t)| \sim 1$ because $\tau_j(t)d_j(t) \sim 1$ by (4.2) if t is sufficiently small. The other properties are easy to check. So, we omit the details. □

Proof of Corollary 1.7. Now assuming Theorem 1.6 we prove Corollary 1.7. By Lemma 4.1 it is sufficient to show

$$\|\sup_{0\leqslant t\leqslant T} U^{\mathfrak{c}}_\gamma f\|_{L^2(B_1(x_0))} \leqslant C\|f\|_{H^s}$$

for any $x_0 \in \mathbb{R}^d$. Now it is easy to see that γ, $\mathfrak{c}$ in Lemma 4.1 satisfy the assumptions in Theorem 1.6. Hence the above estimate holds for $s > s_0$ if $\|\sup_{0\leqslant t\leqslant T} e^{it\Delta}f\|_{L^2(B_1(0))} \leqslant C\|f\|_{H^{s_0}}$ which is valid for $s_0 \geqslant \frac{1}{4}$ when $d = 1$, $s_0 > \frac{1}{2} - \frac{1}{4d}$ when $d \geqslant 2$ (see [3,14]). This proves Corollary 1.7 except the endpoint case $s = \frac{1}{4}$ when $d = 1$, which can be proven by following the argument in [22]. The details are omitted.

Proof of Theorem 1.6. We only prove that (1.10) implies (1.11). The converse also can be shown similarly. To begin with, we first establish the following equivalence of the estimates over intervals of length $\sim \lambda^{-1}$, which will be combined to get the desired estimate by making use of Lemma 2.1.

Lemma 4.2. *Let $\lambda \gg 1$ and $q, r \geqslant 1$. Suppose that γ and $\mathfrak{c}$ satisfy the assumptions in Theorem 1.6. Let $J \subset I_T(t_0)$ be an interval of length $\sim \lambda^{-1}$ and assume that $\widehat{f}$ is supported in $\{\xi : |\xi| \sim \lambda\}$. Then the followings are equivalent :*

$$\|e^{it\Delta} f\|_{L^q_x(B_1(0), L^r_t[0,\lambda^{-1}])} \leqslant C\lambda^s \|f\|_{L^2}, \tag{4.3}$$

$$\|U^{\mathfrak{c}}_\gamma f\|_{L^q_x(B_R(x_0), L^r_t(J))} \leqslant C\lambda^s \|f\|_{L^2}. \tag{4.4}$$

Proof. First we prove the implication from (4.3) to (4.4). Let $t_* \in J$. For simplicity let us set $\Xi = (|\xi_1|^2, \dots, |\xi_d|^2)$. By rescaling $\xi \to \lambda\xi$ we have

$$\begin{aligned} U^{\mathfrak{c}}_\gamma f(x,t) &= \lambda^d \int_{\frac{1}{2} \leqslant |\xi| < 2} e^{i(\lambda\gamma(x,t)\cdot\xi - \lambda^2 \mathfrak{c}(t)\cdot\Xi)} \widehat{f}(\lambda\xi) d\xi \\ &= \lambda^d \int_{\frac{1}{2} \leqslant |\xi| < 2} e^{i\Psi(x,t,\xi)} e^{i(\lambda\gamma(x,t_*)\cdot\xi - (t-t_*)\lambda^2 \mathfrak{c}'(t_*)\cdot\Xi)} e^{-i\lambda^2 \mathfrak{c}(t_*)\cdot\Xi} \widehat{f}(\lambda\xi) d\xi, \end{aligned}$$

where

$$\Psi(x,t,\xi) = \lambda\big(\gamma(x,t) - \gamma(x,t_*)\big)\cdot\xi - \lambda^2\big(\mathfrak{c}(t) - \mathfrak{c}(t_*) - (t-t_*)\mathfrak{c}'(t_*)\big)\cdot\Xi$$

Since $t_* \in J$ and length of J is $O(\lambda^{-1})$, it is easy to see that $|\partial^\beta_\xi \Psi| = O(1)$ uniformly in x, t because $\gamma \in Lip(B_R(x_0) \times I_T(t_0))$ and $\mathfrak{c} \in C^2(\overline{I_T(t_0)})$. So we may expand $e^{i\Psi(x,t,\xi)}$ into Fourier series on $[-\pi,\pi]^d$ so that $e^{i\Psi(x,t,\xi)} = \sum_{k\in\mathbb{Z}^d} C_k(x,t) e^{ik\cdot\xi}$ with $|C_k(x,t)| \leqslant C(|k|+1)^{-N}$ for large N. Hence we have

$$\begin{aligned} U^{\mathfrak{c}}_\gamma f(x,t) &= \lambda^d \sum_{k\in\mathbb{Z}^d} C_k(x,t) \int_{\frac{1}{2}\leqslant|\xi|<2} e^{ik\cdot\xi} e^{i(\lambda\gamma(x,t_*)\cdot\xi - (t-t_*)\lambda^2\mathfrak{c}'(t_*)\cdot\Xi)} e^{-i\lambda^2\mathfrak{c}(t_*)\cdot\Xi} \widehat{f}(\lambda\xi) d\xi \\ &= \sum_{k\in\mathbb{Z}^d} C_k(x,t) \int_{\lambda/2\leqslant|\xi|<2\lambda} e^{i(\gamma(x,t_*)\cdot\xi - t\mathfrak{c}'(t_*)\cdot\Xi)} \widehat{f_{t_*,\lambda,k}}(\xi) d\xi \end{aligned}$$

with $\|f_{t_*,\lambda,k}\|_2 = \|f\|_2$. Now, recalling (1.9), we make change of variables $\xi_i \to |c'_i(t_*)|^{-\frac{1}{2}}\xi_i$ to get

$$\|U^{\mathfrak{c}}_\gamma f\|_{L^q_x(B_R(x_0), L^r_t(J))} \leqslant \sum_k C(|k|+1)^{-N} \|e^{it\Delta} \widetilde{f}_k(\gamma(x,t_*))\|_{L^q_x(B_R(x_0), L^r_t(J))}$$

with $\widetilde{f}_k$ which is fourier supported in $\{\xi : |\xi| \sim \lambda\}$ and $\|\widetilde{f}_k\|_2 \sim \|f\|_2$. Since $\gamma(x,t_*)$ is independent of t and bilipschitz in x, changing variables in x we get

$$\|U^{\mathfrak{c}}_\gamma f\|_{L^q_x(B_R(x_0), L^r_t(J))} \quad\leqslant\quad \sum_k C(|k|+1)^{-N} \|e^{it\Delta} \widetilde{f}_k\|_{L^q_x(B_{CR}(\gamma(x_0,t_*), L^r_t(J))}.$$

We now use the assumption (4.3) which is translation invariant. So, by (1.10), translation and mild dilation it follows that

$$\|U^{\mathfrak{c}}_\gamma f\|_{L^q_x(B_R(x_0), L^r_t(J))} \leqslant \sum_k C(|k|+1)^{-N} \lambda^s \|\widetilde{f}^k\|_2 \leqslant C\lambda^s \|f\|_2.$$

This completes the proof of the implication (4.3) $\to$ (4.4). The converse can be proven similarly. We omit the details. $\square$

Let $\lambda \gg T^{-1}$. We split the interval $I_T(t_0)$ into a union of disjoint intervals J of length $\sim \lambda^{-1}$. Trivially (1.10) implies (4.3). Hence, by Lemma 4.2 we get for each J

$$\|U_\gamma^{\mathfrak{c}} f\|_{L_x^q(B_R(x_0), L_t^r(J))} \leqslant C\lambda^s \|f\|_2$$

provided that $\widehat{f} \in \{\xi : |\xi| \sim \lambda\}$. By Lemma 2.1, it follows that

$$\|U_\gamma^c f\|_{L_x^q(B_R(x_0), L_t^r(I_T(t_0)))} \leqslant C\lambda^s \|f\|_2 \tag{4.5}$$

if $\widehat{f} \in \{\xi : |\xi| \sim \lambda\}$. Since (1.10) holds for $s > s_0$, so does (4.5). Also note that (1.11) is trivial when $\widehat{f}$ is supported in $\{\xi : |\xi| \lesssim 1\}$. Hence, summation along dyadic pieces gives (1.11).

If we additionally have smoothness for γ and $\mathfrak{c}$, we may use Lemma 2.2. In fact, since we are assuming that $\gamma \in C^\infty(\overline{B_R(x_0)} \times \overline{I_T(t_0)})$ and $\mathfrak{c} \in C^\infty(\overline{I_T(t_0)})$, we may replace $I_T(t_0)$ with a slightly extended region $I_{T+\epsilon}(t_0)$ for some $\epsilon > 0$. So we may assume that (4.5) on $B_R(x_0) \times I_{T+\epsilon}(t_0)$ holds. By Lemma 2.2, for $1 < r < \infty$, $x \in B_R(x_0)$

$$\|U_\gamma^{\mathfrak{c}} f(x, \cdot)\|_{L_t^r(I_T(t_0))} \leqslant C\|(\sum_{k \geqslant 1} |U_\gamma^{\mathfrak{c}} P_k f(x, \cdot)|^2)^{\frac{1}{2}}\|_{L_t^r(I_{T+\epsilon}(t_0))} + C\|f\|_{H^{-N}(\mathbb{R}^d)}.$$

Since q, $r \geqslant 2$, by Minkowski's inequality and (4.5) (with $I_{T+\epsilon}(t_0)$)

$$\begin{aligned} \|U_\gamma^c f\|_{L_x^q(B_R(x_0), L_t^r(I_T(t_0)))} &\leqslant C\Big(\sum_{k \geqslant 1} \|U_\gamma^{\mathfrak{c}} P_k f\|^2_{L_x^q(B_R(x_0), L_t^r(I_{T+\epsilon}(t_0)))}\Big)^{\frac{1}{2}} + C\|f\|_{H^{-N}(\mathbb{R}^d)} \\ &\leqslant C\Big(\sum_{k \geqslant 1} 2^{2sk} \|P_k f\|_2^2)^{\frac{1}{2}} + C\|f\|_{H^{-N}(\mathbb{R}^d)} \leqslant C\|f\|_{H^s}. \end{aligned}$$

This completes the proof of the implication (1.10) $\to$ (1.11).

Higher order dispersive equation. Let $m \geqslant 2$ and P satisfy that for $|\xi| \gg 1$

$$|\partial_\xi^\beta P(\xi)| \leqslant C|\xi|^{m-|\beta|}, \quad |\nabla P(\xi)| \sim |\xi|^{m-1}. \tag{4.6}$$

Let $e^{itP} f$ be the solution of the equation

$$i\partial_t u + P(D)u = 0, \quad u(\cdot, 0) = f. \tag{4.7}$$

When $m > 2$, we can relax Lipschitz condition in t to Hölder condition.

Proposition 4.3. *Let $\gamma : B_r(x_0) \times I_T(t_0) \to \mathbb{R}^d$ satisfy* (1.5) *for $x, y \in B_r(x_0)$, $t \in I_T(t_0)$ and*

$$|\gamma(x, t) - \gamma(x, t')| \leqslant C|t - t'|^{\frac{1}{m-1}} \tag{4.8}$$

all $x \in B(x_0, r)$, $t, t' \in I_T(t_0)$. Let $2 \leqslant q, r \leqslant \infty$, and $s_0 \in \mathbb{R}$. Then, for $s > s_0$

$$\|e^{itP} f\|_{L_x^q(B_1(0), L_t^r[0,1])} \leqslant c\, \|f\|_{H^s(\mathbb{R}^d)} \tag{4.9}$$

if only if for $s > s_0$

$$\|e^{itP} f(\gamma(x, t))\|_{L_x^q(B_R(x_0)), L_t^r(I_T(t_0))} \leqslant C\, \|f\|_{H^s(\mathbb{R}^d)}. \tag{4.10}$$

As before, if γ is smooth in t, using Lemma 2.2 we can show the equivalence of (4.9) and (4.10) except $r = \infty$. However, we don't know whether the equivalence fails if the exponent $\frac{1}{m-1}$ in (4.8) is replaced by a smaller number. It seems interesting to find the exact order of Hölder condition which guarantees the equivalence.

Proposition 4.3 can be proven similarly as Theorem 1.6. In fact, since $|\nabla P(\xi)| \sim |\xi|^{m-1}$, by Lemma 2.1 we are reduced to showing equivalence on an interval of length λ^{1-m}. By recalling the proof of Theorem 1.6, it is not difficult to see that the equivalence follows if we show that $\lambda(\gamma(x,t)-\gamma(x,t')) = O(1)$ when $|t-t'| \lesssim \lambda^{1-m}$. This is obvious from (4.8).

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