SHARP NULL FORM ESTIMATES FOR THE WAVE EQUATION

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ABSTRACT. Null form estimates (from $\dot{H}^{\alpha_1} \times \dot{H}^{\alpha_2}$ to $L^q_t(L^r_x)$) for the wave equation in $\mathbb{R}^{n+1}$ are studied. For $n \geq 4$, we obtain the sharp null form estimates except for the endpoints. For $n = 2, 3$ we obtain the estimates under the additional assumption $4/n < q$ when $2 < r$.

1. INTRODUCTION

In this paper we consider null form estimates for the wave equation. Let ϕ, ψ be solutions of the homogeneous wave equation in $\mathbb{R}^{n+1}$, $n \geq 2$;

$$\Box\phi = 0, \quad \Box\psi = 0, \ (\Box = \Delta_x - \partial_t^2, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}).$$

The well known Strichartz's estimates say that

$$\|\phi\|_{L^q_t L^r_x} \leq C[\|\phi(0)\|_{H^s} + \|\partial_t \phi(0)\|_{H^{s-1}}]$$

if $1/q + n/r = n/2 - s$ and $2/q + (n-1)/r \leq (n-1)/2$ with the exception $(q, r) = (2, \infty)$ when $n = 3$ (see [4]). However, bilinear generalization of such estimates makes it possible to introduce additional multiplier weights which compensate interaction between two waves. This gives further available estimates which are not allowed in the linear setting.

Let D_0, D_+, D_- denote the Fourier multiplier operators defined by

$$\widehat{D_0 f}(\xi, \tau) = |\xi| \widehat{f}(\xi, \tau),$$
$$\widehat{D_+ f}(\xi, \tau) = (|\xi| + |\tau|) \widehat{f}(\xi, \tau),$$
$$\widehat{D_- f}(\xi, \tau) = ||\xi| - |\tau|| \widehat{f}(\xi, \tau).$$

Here ξ, τ are the Fourier variables corresponding to x, t respectively. We are mainly concerned with the estimates of the form

$$(1) \qquad \|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\|_{L^q_t L^r_x} \leq C(\|\phi(0)\|_{\dot{H}^{\alpha_1}} + \|\partial_t \phi(0)\|_{\dot{H}^{\alpha_1 - 1}}) \times (\|\psi(0)\|_{\dot{H}^{\alpha_2}} + \|\partial_t \psi(0)\|_{\dot{H}^{\alpha_2 - 1}})$$

where $\dot{H}^\alpha$ is the homogeneous L^2-Sobolev space of order α.

2000 *Mathematics Subject Classification.* Primary 42B15; Secondary 35L05, 35B45.

Key words and phrases. Wave equations, null form estimates.

The research of the first author was supported in part by the Post-doctoral Fellowship Program of Korea Science and Engineering Foundation and the Research Fund of SNU. The research of the second author was supported in part by the EC project HPRN-CT-2001-00273 - HARP, the MEC (Spain) project MTM2004-00678 and the UAM-CM project CCG06.UAM/ESP-0286.

This type of bilinear estimate was studied by M. Beals [1] and by Klainerman and Machedon [5]. They obtained some of the estimates (1) for $q = r = 2$ and non-trivial exponents $(\beta_0, \beta_-, \beta_+, \alpha_1, \alpha_2)$. These estimates played an important role in their study of nonlinear wave equations possessing null form structure. Results for some particular cases were obtained by the same authors in [6], [7] and by Klainerman and Selberg [8].

In [3] D. Foschi and S. Klainerman determined all $(\beta_0, \beta_-, \beta_+, \alpha_1, \alpha_2)$ for which (1) holds for $q = r = 2$ and they also conjectured that for $1 \le q, r \le \infty$ (1) holds if and only if the following conditions on $(q, r, \beta_0, \beta_-, \beta_+, \alpha_1, \alpha_2)$ are satisfied:

- *Scaling invariance:*

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{1}{q} - n(1 - \frac{1}{r}). \tag{2}$$

- *Geometry of the cones:*

$$\frac{1}{q} \le \frac{n+1}{2}(1 - \frac{1}{r}), \quad \frac{1}{q} \le \frac{n+1}{4}. \tag{3}$$

- *Concentration near null directions:*

$$\beta_- \ge \frac{1}{q} - \frac{n-1}{2}(1 - \frac{1}{r}). \tag{4}$$

- *Low frequency interactions* $(++)$*:*

$$\beta_0 \ge \frac{1}{q} - n(1 - \frac{1}{r}), \tag{5}$$

$$\beta_0 \ge \frac{2}{q} - (n+1)(1 - \frac{1}{r}). \tag{6}$$

- *Low frequency interactions* $(+-)$*:*

$$\alpha_1 + \alpha_2 \ge \frac{1}{q}, \tag{7}$$

$$\alpha_1 + \alpha_2 \ge \frac{3}{q} - n(1 - \frac{1}{r}). \tag{8}$$

- *Interaction between high and low frequency:*

$$\alpha_i \le \beta_- + \frac{n}{2}, \tag{9}$$

$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r}), \tag{10}$$

$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}), \tag{11}$$

$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}) + (\frac{1}{2} - \frac{1}{q}). \tag{12}$$

These necessary conditions can be obtained by considering various interactions between two waves. However, a close examination of (1) reveals that further conditions are necessary for the estimate (1)(see Section 5). When $1 \le q \le 2 \le r \le \infty$, the following conditions should be additionally satisfied:

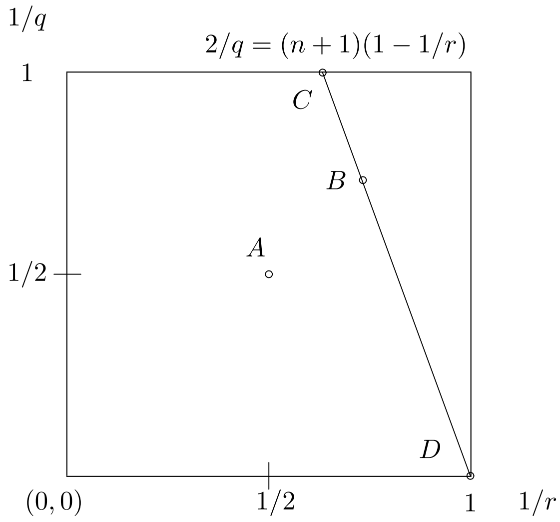

FIGURE 1. The previously know $(1/r, 1/q)$-range of sharp estimates, $n \geq 4$: The sharp estimates at A are due to Klainerman and Foschi [3] and the estimates at B due to Tao [15]. The extension to the line segment (C, D) is due Tataru [20].

- *Interaction between high and low frequency:*

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{1}{2}, \tag{13}$$

$$\alpha_i \leq \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}) + \frac{1}{r} - \frac{1}{q}. \tag{14}$$

- *Low frequency interactions* $(++)$*:*

$$\beta_0 \geq \frac{2}{q} - n(1 - \frac{1}{r}) - \frac{1}{2}. \tag{15}$$

Bilinear inequalities for $q = r < 2$, $n = 2$ were first considered by Bourgain [2] in connection with the cone multiplier problem. He showed the estimates

$$\|\phi\psi\|_{L^q_t L^q_x} \leq C\|\phi(0)\|_{L^2}\|\psi(0)\|_{L^2}$$

for some $q < 2$ under the assumption that the two waves are Fourier supported in the cone $\{(\xi, \tau) : |\xi| = \tau, \tau \sim 1\}$ and their Fourier supports are separated by O(1) angle. These estimates were used to improve the boundedness of the cone multiplier due to Mockenhaupt [11]. Bourgain's result was improved by Tao and Vargas [17]. Finally, Wolff [22] and Tao [15] obtained the optimal result in all dimensions.

Regarding to the estimates (1) for $(q, r) \neq (2, 2)$, Tao and Vargas [18] obtained some partial results for the case $q = r < 2$. Some sharp L^p-estimates were obtained by Tao [15]. Later Tataru [20] extended Tao's results to mixed norms estimates for some hyperbolic equations with rough coefficients (see Figure 1). However, these results only give sharp $L^q_t(L^r_x)$-estimates for $(1/r, 1/q)$ on the line $2/q = (n+1)(1-$

$1/r$). When $(1/r, 1/q)$ is away from it, for most of q, r the sharp estimates have not been known. The following is our main result.

Theorem 1.1. *Suppose* (2) *holds and the inequalities* (3)–(15) *hold with strict inequalities. Then if* $n \ge 4$, (1) *holds for* $1 < r \le \infty, 1 < q \le \infty$ *and when* $n = 2, 3$, (1) *holds for* $1 < r \le 2$, $1 < q \le \infty$ *and for* $2 < r \le \infty$, $4/n < q \le \infty$.

Hence all sharp estimates for $n \ge 4$ are obtained (except for the endpoints) but there remain some gaps for $2 < r \le \infty$, $4/(n+1) \le q \le 4/n$ when $n = 2, 3$. We hope to return to this matter later. It is actually possible to obtain some of the end point estimates but most of them are left open. As it will be seen, these are related to the unresolved endpoint estimates for bilinear restriction estimates [15, 16].

There are other null forms of special interest which are related to the study of nonlinear equations, such as Wave maps and the Yang-Mills equations (see [3, 5, 8, 9]). In particular, the following ones;

$$\begin{aligned} Q_0(\phi, \psi) &= \partial_t\phi\partial_t\psi - \nabla_x\phi \cdot \nabla_x\psi, \\ Q_{0j}(\phi, \psi) &= \partial_t\phi\partial_{x_j}\psi - \partial_{x_j}\phi\partial_t\psi, \\ Q_{ij}(\phi, \psi) &= \partial_{x_i}\phi\partial_{x_j}\psi - \partial_{x_j}\phi\partial_{x_i}\psi. \end{aligned}$$

For the first one, a simple calculation shows that $-2Q_0(\phi, \psi) = D_+D_-(\phi, \psi)$. Hence, (1) is valid if $Q_0(\phi, \psi)$, $\beta_+ - 1$ and $\beta_- - 1$ replace $\phi\psi$, β_+ and β_-, respectively. As it is known from the analysis of the multipliers (see e.g. [3], [18]), for the others it is expected that these null-forms (heuristically) behave like

$$\begin{aligned} Q_{0j}(\phi, \psi) &\sim D_+^{1/2}D_-^{1/2}(D_0^{1/2}\phi D_0^{1/2}\psi), \\ Q_{ij}(\phi, \psi), &\sim D_0 D_+^{-1/2}D_-^{1/2}(D_0^{1/2}\phi D_0^{1/2}\psi). \end{aligned}$$

A slight modification of arguments allows us to obtain the analogous of (1) for these null forms as if the above heuristics was correct (see Section 3.3).

Corollary 1.2. *Suppose* (2) *holds and* (3)–(15) *hold with strict inequalities. Then if* $n \ge 4$, *for* $1 < r \le \infty, 1 < q \le \infty$,

$$\begin{aligned} \|D_0^{\widetilde\beta_0}D_+^{\widetilde\beta_+}D_-^{\widetilde\beta_-}Q(\phi, \psi)\|_{L_t^qL_x^r} \le C(&\|\phi(0)\|_{\dot H^{\alpha_1+1/2}} + \|\partial_t\phi(0)\|_{\dot H^{\alpha_1-1/2}}) \\ &\times (\|\psi(0)\|_{\dot H^{\alpha_2+1/2}} + \|\partial_t\psi(0)\|_{\dot H^{\alpha_2-1/2}}), \end{aligned} \tag{16}$$

holds in each of the following cases;

$$\begin{aligned} Q =&Q_{0,j},\ \widetilde\beta_0 = \beta_0,\ \widetilde\beta_+ = \beta_+ - 1/2,\ \widetilde\beta_- = \beta_- - 1/2, \\ Q =&Q_{i,j},\ \widetilde\beta_0 = \beta_0 - 1,\ \widetilde\beta_+ = \beta_+ + 1/2,\ \widetilde\beta_- = \beta_- - 1/2. \end{aligned}$$

When $n = 2, 3$, (16) *holds for* $1 < r \le 2$, $1 < q \le \infty$ *and* $2 < r \le \infty$, $4/n < q \le \infty$. *Moreover, conditions* (2) *and* (3)–(15) *are necessary for* (16) *to hold.*

It is not difficult to show that conditions (2)–(12) are necessary with a minor modification of known examples ([3]). For the necessity of (13)–(15) see Section 5.

The paper is organized as follows: In Section 2, we state several bilinear (adjoint) restriction estimates which are needed in the proof of Theorem 1.1. Those estimates will be proven in Section 4. Assuming the estimates in Section 2 we prove Theorem 1.1 and Corollary 1.2 in Section 3. Theorem 1.1 is proven first and the proof of Corollary 1.2 is similar; details will be given in Section 3.3. In Section 5 we show the necessity of conditions (13), (14) and (15).

Throughout the paper, the constant C may vary depending on the dimension n, $1 \le q, r \le \infty$, $\epsilon > 0$ and the exponents α_1, α_2, $\beta_0, \beta_+, \beta_-$ if it is not mentioned otherwise.

Acknowledgement. The authors wish to thank the referee for comments and suggestions which helped to improve the paper.

2. Mixed norm estimates for bilinear extension operators

In this section we state several preliminary estimates which are needed for the proof of the main theorem but that also have interest for themselves. In fact, they are mixed norm generalizations of the bilinear restriction estimates for the cone and the paraboloid (c.f. [10, 16, 22, 15, 21]).

To obtain sharp null form estimates it is essential to get sharp estimates for various interactions between two waves. Through suitable decomposition of the waves (especially, Littewood-Paley and angular Whitney-type decompositions) matters are reduced to obtaining the estimates for high-low and low-low (or high-high) frequency interactions with specified angular separation between two waves (see the reductions at the beginning of Section 3). From the estimates for large angular separation we deduce the sharp estimates for small angular separation using rescaling arguments (see Proposition 2.4, 2.5).

2.1. **Estimates with large angular separation.** For convenience, we define the energy of the wave ϕ by

$$E(\phi) := \|\phi(t)\|_2^2$$

which is independent of t by Plancherel's theorem. For $l, k \ge 0$ let us define

$$\Gamma_0(2^{-l}) = \{(\xi'', \xi_{n-1}, \xi_n, \tau) : \tau = |\xi|,\ \xi_n \sim 1,\ \xi_{n-1} \sim 2^{-l},\ |\xi''| \le 2^{-l}\},$$
$$\Gamma'_k(2^{-l}) = 2^k\{(\xi'', \xi_{n-1}, \xi_n, \tau) : \tau = -|\xi|,\ \xi_n \sim -1,\ \xi_{n-1} \sim 2^{-l},\ |\xi''| \le 2^{-l}\},$$

respectively, where $\xi'' = (\xi_1, \dots, \xi_{n-2})$. Even though the results in this section are stated in terms of the specified conic subsets $\Gamma_0(2^{-l})$ and $\Gamma'_k(2^{-l})$, it should be noted that they obviously remain valid under spatial rotations.

The following is a mixed norm generalization of the bilinear restriction estimate for the cone [15, 22].

Theorem 2.1. *Let ϕ, ψ (or $\bar\psi$) be waves having Fourier supports contained in $\Gamma_0(1)$, $\Gamma'_k(1)$, respectively. Then, for $\epsilon > 0$ and $1 < q, r \le 2$ satisfying $1/q < \min(1, \frac{n+1}{4})$, $1/q < \frac{n+1}{2}(1 - \frac{1}{r})$, there is a constant C (independent of k, ϕ and ψ) such that*

$$\|\phi\psi\|_{L_t^qL_x^r} (or\ \|\phi\bar\psi\|_{L_t^qL_x^r}) \le C2^{(\frac{1}{q} - \frac{1}{2} + \epsilon)k} E(\phi)^{1/2} E(\psi)^{1/2}.$$

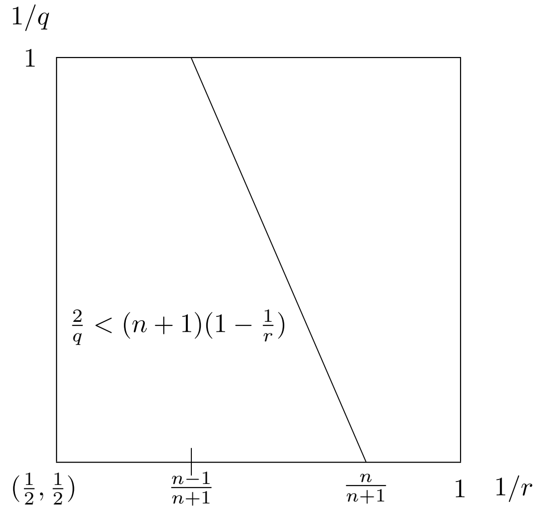

FIGURE 2. The range of (q, r) for Theorem 2.1, $n \geq 3$

The conditions $1/q \leq \min(1, \frac{n+1}{4})$, $1/q \leq \frac{n+1}{2}(1 - \frac{1}{r})$ are necessary. It can be shown using the test functions for (3) given in [3]. It seems highly possible to obtain the estimates for $2/q = (n+1)(1 - \frac{1}{r})$ by adapting the argument in [15] but the $\epsilon > 0$ has to be removed to obtain some of the endpoint estimates in Theorem 1.1.

The estimates (1) for q, r satisfying both (7), (8) are rather special. As it was pointed out in [15], they are closely related to the bilinear restriction to the paraboloid (or to elliptic surfaces). Let us set

$$\theta(\xi') = \sqrt{1 + |\xi'|^2} - 1, \ \xi' = (\xi_1, \xi_2, \dots, \xi_{n-1}). \tag{17}$$

Then for $0 < a \ll 1$ and $1 \leq b \leq 2$ we define extension operators by

$$\widehat{f d\sigma_i}(x, t) = \int_{S_i} e^{i(x'\xi' + x_n \xi_n + t\xi_n \theta(\xi'/\xi_n))} f(\xi) d\xi, \ i = 1, 2 \tag{18}$$

where

$$S_i = \{\xi \in \mathbb{R}^n : b \leq \xi_n \leq b + a, |\xi'/\xi_n + (-1)^i e'_{n-1}| \leq 1/2\}, \ i = 1, 2.$$

Theorem 2.2. *Let $\widehat{f d\sigma_1}$, $\widehat{f d\sigma_2}$ be defined as above. If $0 < a \ll 1, 1 \leq b \leq 2$ and $n \geq 2$, then for q, r satisfying $1/q < \min(1, n/4)$, $2/q < n(1 - 1/r)$, and for any $\epsilon > 0$, there is a constant C, independent of a and b, such that*

$$\| \prod_{i=1}^{2} \widehat{f_i d\sigma_i} \|_{L_t^q L_x^r} \leq C a^{1 - \frac{1}{r} - \epsilon} \|f_1\|_2 \|f_2\|_2. \tag{19}$$

By the argument in Section 4 and the estimates in [10] (c.f. (67)-(70)), it is possible to obtain a mixed norm version of the bilinear restriction estimates for hypersurfaces with non-vanishing Gaussian curvature. We record it here hoping that it could be useful somewhere else.

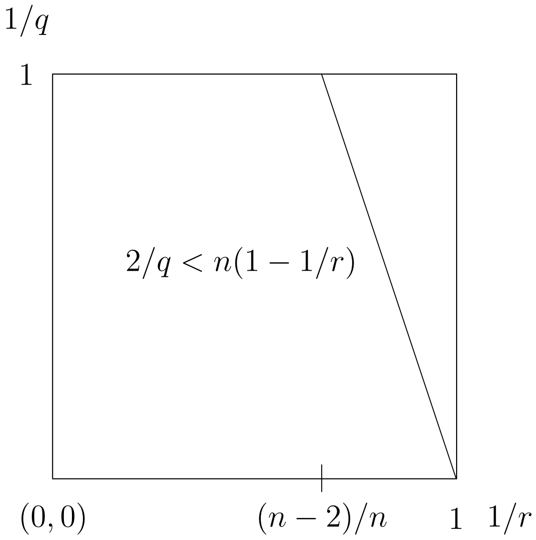

FIGURE 3. The range of (q, r) for Theorem 2.2, $n \geq 4$

Let $n \geq 2$ and ϕ_1, ϕ_2 be smooth functions defined on $[-1,1]^{n-1}$. Define extension operators by

$$E_i f(x', t) = \int_{[-1,1]^{n-1}} e^{i(x'\xi' + t\phi_i(\xi'))} f(\xi') d\xi', \ \ i = 1, 2.$$

Theorem 2.3. *If* $\det H\phi_1$, $\det H\phi_2 \neq 0$ *on* $[-1,1]^{n-1}$ *and for all* $\xi', \zeta' \in [-1,1]^{n-1}$,

$$|\langle H\phi_i^{-1}(\xi')(\nabla\phi_1(\xi') - \nabla\phi_2(\zeta')), \nabla\phi_1(\xi') - \nabla\phi_2(\zeta')\rangle| \geq c > 0, \ \ i = 1, 2,$$

then for $1 < q, r$ *satisfying* $1/q < \min(1, n/4)$, $2/q < n(1 - 1/r)$, *there is a constant* C *such that*

$$\|E_1(f_1)E_2(f_2)\|_{L_t^q L_{x'}^r} \leq C\|f_1\|_2\|f_2\|_2.$$

It is also possible to obtain such mixed norm estimates for bilinear restriction to the conic surfaces studied in [10] which generalize Theorem 2.1.

The condition $2/q \leq n(1 - 1/r)$ is necessary in Theorem 2.3 and when $n = 2$, there is an additional condition[1] $1/r + 2/q \leq 3/2$.

Indeed, for the first one consider $\phi_1(\xi') = |\xi' + 2e_1'|^2$ and $\phi_2(\xi') = |\xi' - 2e_1'|^2$ defined on $[-1,1]^{n-1}$. For $\epsilon \ll 1$, let $A = \{\xi' : |\xi_1| \leq \epsilon^2, |\xi_i| \leq \epsilon, i = 2, \dots n-1\}$ and set $f_1(\xi') = f_2(\xi') = \chi_A(\xi')$. Then, the condition follows from routine argument because $|E_1 f_1(x', t)|$, $|E_2 f_2(x', t)| \geq c\epsilon^n$ provided $|x_1| \leq c\epsilon^{-2}$, $|t| \leq c\epsilon^{-2}$, $|x_i| \leq c\epsilon^{-1}$, $i = 2, \dots n-1$. The second condition can be obtained using $\phi_1(\xi') = (\xi' + 4)^2$, $\phi_2(\xi') = (\xi')^2$, $f_1 = \chi_{\{|\xi'| \leq \epsilon\}}$ and $f_2 = \chi_{\{|\xi'| \leq \epsilon^2\}}$ because $|(E_1 f_1 E_2 f_2)(x', t)| \sim \epsilon^3$ if $|x' + 8t| \leq c\epsilon^{-1}$ and $|t| \leq c\epsilon^{-2}$.

[1]This is less restrictive than the condition $1/q < n/4$, $2/q < n(1 - 1/r)$ in Theorem 2.3 when $n = 2$

2.2. **Estimates with small angular separation.** Now we give small angle versions of Theorem 2.1, 2.2. These will be obtained from rescaling and stability of bilinear estimates with large angle separation. (See Section 4.2, 4.4.)

Proposition 2.4. *Let $n \geq 2$. Suppose that ϕ, ψ (or $\bar\psi$) are free waves with Fourier supports in $\Gamma_0(2^{-l})$, $\Gamma'_k(2^{-l})$, respectively. Then, for $\epsilon > 0$ and $1 < q, r \leq 2$ satisfying $1/q < \min(1, \frac{n+1}{4})$, $1/q < \frac{n+1}{2}(1-\frac{1}{r})$,*

$$\|\phi\psi\|_{L^q_tL^r_x}, (\text{or } \|\phi\bar\psi\|_{L^q_tL^r_x}) \leq C2^{k(\frac{1}{q}-\frac{1}{2}+\epsilon)}2^{l(\frac{2}{q}-(n-1)(1-\frac{1}{r}))}E(\phi)^{\frac{1}{2}}E(\psi)^{\frac{1}{2}}.$$

Let $0 < a \ll 1$ and $1 \leq b \leq 2$ as before. We define $\Lambda(2^{-l})$, $\Lambda'(2^{-l})$ by

$$\Lambda(2^{-l}) = \{(\xi', \xi_n, \tau) \in \Gamma_0(2^{-l}) : b \leq \xi_n \leq a + b\},$$
$$\Lambda'(2^{-l}) = \{(\xi', \xi_n, \tau) \in \Gamma'_0(2^{-l}) :\ b \leq -\xi_n \leq a + b\},$$

respectively. The following is a corollary of Theorem 2.2.

Proposition 2.5. *Let ϕ, ψ (or $\bar\psi$) be waves with Fourier supports in $\Lambda(2^{-l})$, $\Lambda'(2^{-l})$, respectively. Then, for $2/q < n(1 - 1/r)$, $q > \max(1, 4/n)$ and any $\epsilon > 0$,*

$$\|\phi\psi\|_{L^q_tL^r_x} \leq C2^{l(2/q-(n-1)(1-1/r))}a^{1-1/r-\epsilon}E(\phi)^{\frac{1}{2}}E(\psi)^{\frac{1}{2}}.$$

As usual, for the proof of Theorem 1.1 it is convenient to factor the multiplier weight $D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}$ as $(D_0^{\beta_0}D_+^{\beta_+-\beta_-})|\Box|^{\beta_-}$. Here $|\Box| = D_+D_-$. The following is useful in handling the multiplier weight $|\Box|^{\beta_-}$.

Lemma 2.6. *Let β_- be a complex number and let $|\Box| = D_+D_-$. Suppose that ϕ, ψ (or $\bar\psi$) are waves with Fourier supports in $\Gamma_0(2^{-l})$, $\Gamma'_k(2^{-l})$, respectively. Then for $1 \leq q, r \leq \infty$, $k, l \geq 0$ and any $N > 0$, there is a constant C, independent of β_-, such that*

$$\||\Box|^{\beta_-}(\phi\psi)\|_{L^q_tL^r_x} \leq C(1 + |\beta_-|)^N 2^{Re(\beta_-)(k-2l)} \times \sum_{\mu=(\mu_1,\mu_2)\in\mathbb{Z}^n\times\mathbb{Z}^n} (1 + |\mu|)^{-N}\|\phi_{\mu_1}\psi_{\mu_2}\|_{L^q_tL^r_x} \tag{20}$$

where ϕ_{μ_1}, ψ_{μ_2} are waves satisfying $E(\phi_{\mu_1}) = E(\phi)$, $E(\psi_{\mu_2}) = E(\psi)$, and ϕ_{μ_1}, ψ_{μ_2} (or $\bar\psi_{\mu_2}$) are Fourier supported in $\Gamma_0(2^{-l})$, $\Gamma'_k(2^{-l})$, respectively.

Here β_- is allowed to be complex. For the most of applications it is not necessary but we use it for complex interpolation (see Section 3.2.1).

Proof. We write $\phi\psi$ as

$$(2\pi)^{-2n}\iint e^{i(x'(\xi'+\eta')+x_n(\xi_n+\eta_n)+t(|\xi|\pm|\eta|))}\widehat{\phi(0)}(\xi)\widehat{\psi(0)}(\eta)d\xi d\eta.$$

Here the sign $+$ stands for the case $\bar\psi$ is supported in $\Gamma'_k(2^{-l})$ and $-$ does for the case ψ is supported in $\Gamma'_k(2^{-l})$. Then by re-scaling $\xi' \to 2^{-l}\xi'$, $(\eta', \eta_n) \to (2^{k-l}\eta', 2^k\eta_n)$, the phase part is transformed to

$$\Phi(x, t, \xi, \eta) = x'(2^{-l}\xi' + 2^{k-l}\eta') + x_n(\xi_n + 2^k\eta_n) + t(|(2^{-l}\xi', \xi_n)| \pm |(2^{k-l}\eta', 2^k\eta_n)|).$$

So, applying $|\Box|^{\beta_-}$ produces additional factor $|w^\pm(\xi,\eta)|^{\beta_-}$ in the above integral where

$$w^\pm(\xi,\eta) = |2^{-l}\xi' + 2^{k-l}\eta'|^2 + (\xi_n + 2^k\eta_n)^2 - (|(2^{-l}\xi',\xi_n)| \pm |(2^{k-l}\eta', 2^k\eta_n)|)^2$$
$$= 22^{k-2l}\frac{2^{-2l}[(\xi'\eta')^2 - |\xi'|^2|\eta'|^2] - \xi_n^2\eta_n^2|\xi'/\xi_n - \eta'/\eta_n|^2}{2^{-2l}\xi'\eta' + \xi_n\eta_n \pm \sqrt{\xi_n^2\eta_n^2 + 2^{-2l}|\xi'|^2\eta_n^2 + 2^{-2l}\xi_n^2|\eta'|^2 + 2^{-4l}|\xi'|^2|\eta'|^2}}$$

and the integration is now taken over the set

$$S^\pm = \{\xi : \xi_n \sim 1, \xi_{n-1} \sim 1, |\xi''| \le 1\} \times \{\eta : \eta_n \sim \pm 1, \eta_{n-1} \sim \mp 1, |\eta''| \le 1\}.$$

Hence it is enough to show that

$$\| \int_{S^\pm} e^{i\Phi(x,t,\xi,\eta)}|2^{-k+2l}w^\pm(\xi,\eta)|^{\beta_-}F(\xi,\eta)d\xi d\eta\|_{L_t^qL_x^r} \le C(1+|\beta_-|)^N$$
$$\times \sum_{\mu\in\mathbb{Z}^n\times\mathbb{Z}^n}(1+|\mu|)^{-N}\|\int_S e^{i\Phi(x,t,\xi,\eta)}e^{i\langle\mu,(\xi,\eta)\rangle}F(\xi,\eta)d\xi d\eta\|_{L_t^qL_x^r}.$$

Because we can get the required by reversing all the changes of variables done so far. If $l \ge 0$, $|2^{-k+2l}w^\pm(\eta,\xi)| \sim 1$ because $|\xi'/\xi_n - \eta'/\eta_n|^2 \sim 1$ on $S^\pm$. So $|2^{-k+2l}w^\pm(\xi,\eta)|^{\beta_-}$ is smooth on $S^\pm$ for any complex number β_-. Then by Fourier series expansion we may write

$$|2^{-k+2l}w^\pm(\xi,\eta)|^{\beta_-} = \sum_{\mu\in\mathbb{Z}^{2n}} C_\mu e^{i\langle\mu,(\xi,\eta)\rangle}$$

on $S^\pm$ and from direct differentiation it is easy to see that $|C_\mu| = O((1+|\beta_-|)^N(1+|\mu|)^{-N})$ for any positive integer N. This gives the required estimate (c.f. Lemma 2.1 in [14]). $\square$

Remark 2.7. *In fact this lemma can be strengthened slightly. Let $\mathcal{Q}$ be a form with associated multiplier m. That is,*

$$\mathcal{Q}(\phi,\psi) = (2\pi)^{-2n}\iint e^{i(x(\xi+\eta)+t(|\xi|\pm|\eta|))}m(\xi,\eta)\widehat{\phi(0)}(\xi)\widehat{\psi(0)}(\eta)d\xi d\eta.$$

Then it is obvious from the proof that Lemma 2.6 remains valid if $\phi\psi$ and $\phi_{\mu_1}\psi_{\mu_2}$ in (20) *are replaced by $\mathcal{Q}(\phi,\psi)$ and $\mathcal{Q}(\phi_{\mu_1},\psi_{\mu_2})$, respectively.*

3. Proof of Theorem 1.1

Assuming Theorems 2.1 and 2.2 and Propositions 2.4 and 2.5 we prove Theorem 1.1. By finite decomposition and symmetry, we may assume that the Fourier transform of ϕ is supported in the forward light cone and ψ is supported in the forward or backward light cone.

By the standard Littlewood-Paley decomposition of the initial data $\phi(0), \psi(0)$, we write

$$\phi = \sum_{j=-\infty}^{\infty}\phi_j, \quad \psi = \sum_{k=-\infty}^{\infty}\psi_k$$

Interactions			**Main devices**
high and low frequency, $k \gg 1$			Proposition 2.4, Lemma 2.6, 3.2
low frequency, $k = O(1)$	$(+-)$		Proposition 2.4, 2.5, Lemma 2.6
low frequency, $k = O(1)$	$(++)$	small angle separation	Proposition 2.4, Lemma 2.6
low frequency, $k = O(1)$	$(++)$	large angle separation	Proposition 2.4, 3.3

Table 1. Interactions and main devices.

where ϕ_j, ψ_k are waves having frequency supports on region $D_0 \sim 2^j$, $D_0 \sim 2^k$, respectively in the light cones. Then, to prove the estimate (1), it is sufficient to show that

$$\|D_0^{\beta} D_+^{\beta_+} D_-^{\beta_-}(\phi_j \psi_k)\|_{L_t^q L_x^r} \le C 2^{-\epsilon|j-k|} (2^{2\alpha_1 j} E(\phi_j))^{\frac{1}{2}} (2^{2\alpha_2 k} E(\phi_k))^{\frac{1}{2}}.$$

Then, the required estimate follows from using Cauchy-Schwarz's inequality and Plancherel's theorem. By symmetry and re-scaling with the condition (2), we may also assume $j = 0 \le k$. Hence we are reduced to showing

$$\|D_0^{\beta} D_+^{\beta_+} D_-^{\beta_-}(\phi_0 \psi_k)\|_{L_t^q L_x^r} \le C 2^{-\epsilon k} (E(\phi_0))^{\frac{1}{2}} (2^{2\alpha_2 k} E(\phi_k))^{\frac{1}{2}}. \tag{21}$$

Fixing k, for each $l \ge 0$ we decompose dyadically the double light cone into finitely overlapping projective sectors Γ, Γ' with angle 2^{-l} such that $\phi_0 = \sum_{\Gamma} \phi_{0,\Gamma}$, $\psi_k = \sum_{\Gamma'} \psi_{k,\Gamma'}$ where $\phi_{0,\Gamma}, \psi_{k,\Gamma'}$ are supported sectors Γ, Γ', respectively. We denote by $\angle(\Gamma, \Gamma')$ the angle between the sectors. Then, by a Whitney type decomposition we can write $\phi_0 \psi_k$ as

$$\phi_0 \psi_k = \sum_{l \ge 0} \sum_{\Gamma, \Gamma'; \angle(\Gamma, \Gamma') \sim 2^{-l}} \phi_{0,\Gamma} \psi_{k,\Gamma'}.$$

This kind of decomposition was frequently used to exploit bilinear estimates obtained under separation condition (see [15, 18, 19, 22]). So, it is enough to show that if $\angle(\Gamma, \Gamma') \sim 2^{-l}$, then for $\epsilon > 0$ and $k, l \ge 0$,

$$\|D_0^{\beta} D_+^{\beta_+} D_-^{\beta_-}(\phi_{0,\Gamma} \psi_{k,\Gamma'})\|_{L_t^q L_x^r} \le C 2^{-\epsilon(k+l)} 2^{\alpha_2 k} E(\phi_{0,\Gamma})^{\frac{1}{2}} E(\psi_{k,\Gamma'})^{\frac{1}{2}}. \tag{22}$$

Then, estimate (21) can be obtained by Cauchy-Schwarz's inequality and Plancherel's theorem.

By rotation we may assume that

$$(+-) \qquad \operatorname{supp} \widehat{\phi_{0,\Gamma}} \subset \Gamma_0(2^{-l}),\ \operatorname{supp} \widehat{\psi_{k,\Gamma'}} \subset \Gamma'_k(2^{-l}),\ \text{or}$$
$$(++) \qquad \operatorname{supp} \widehat{\phi_{0,\Gamma}} \subset \Gamma_0(2^{-l}),\ \operatorname{supp} \widehat{\psi_{k,\Gamma'}} \subset -\Gamma'_k(2^{-l}).$$

We denote the first case by $(+-)$ and the second by $(++)$.

We prove (22) considering the cases $k \gg 1$ and $k = O(1)$, separately. They correspond to high and low, low frequency interactions, respectively. Then, low frequency interaction is handled by dividing the cases $(+-)$ and $(++)$. In the latter case the small angle $2^{-1} \ll 1$ and large angle $2^{-1} \sim 1$ cases are shown separately. We summarize the details in Table 1.

3.1. **Proof of estimate (22) when $k \gg 1$; high and low frequency interaction.** In this case the behavior of D_0, D_+ is simple. To be more precise, the Fourier transform of $\phi_{0,\Gamma}\psi_{k,\Gamma'}$ is contained in the region $D_0, D_+ \sim 2^k$ since $k \gg 1$. On the other hand, by the well known multiplier theorem it is easy to see

$$\|D_0^{\beta_0} D_+^{\beta_+ - \beta_-} F\|_{L_t^q L_x^r} \le C 2^{k(\beta_0+\beta_+-\beta_-)} \|F\|_{L_t^q L_x^r}$$

provided the Fourier transform of F is supported in the region $D_0, D_+ \sim 2^k$. Hence, for (22) it suffices to show that if $\angle(\Gamma, \Gamma') \sim 2^{-l}$, then for $\epsilon > 0$ and $k, l \ge 0$,

$$\||\Box|^{\beta_-}(\phi_{0,\Gamma}\psi_{k,\Gamma'})\|_{L_t^q L_x^r} \le C 2^{-\epsilon(k+l)} 2^{-k(\beta_0+\beta_+-\beta_-)} 2^{\alpha_2 k} E(\phi_{0,\Gamma})^{\frac12} E(\psi_{k,\Gamma'})^{\frac12}.$$

By Lemma 2.6, this further reduces to showing

$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \le C 2^{-\epsilon(k+l)} 2^{k(\alpha_2-\beta_0-\beta_+)} 2^{2l\beta_-} E(\phi_{0,\Gamma})^{\frac12} E(\psi_{k,\Gamma'})^{\frac12}.$$

Now, by (2) and (4) with strict inequality it is enough to show the following;

Lemma 3.1. *Let $k \gg 1$. If $1/q < \min(1, \frac{n+1}{4})$ and $2/q < (n+1)(1-1/r)$, then*

$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \le C 2^{k(\beta_- - \alpha_1 - 1/q + n(1-1/r) - \epsilon)} \times 2^{l(2/q-(n-1)(1-1/r))} E(\phi_{0,\Gamma})^{\frac12} E(\psi_{k,\Gamma'})^{\frac12}. \tag{23}$$

We prove Lemma 3.1 considering the cases $n \ge 3$ and $n = 2$ separately.

Proof of Lemma 3.1 when $n \ge 3$. For $n \ge 3$, let us define (see Figure 4) the sets $\Delta_1(n), \Delta_2(n), \dots, \Delta_6(n) \subset [0,1] \times [0,1)$ by setting

$$\Delta_1(n) = \{(\frac1r, \frac1q) \in [0,1]\times[0,1) : \frac2q < (n+1)(1-\frac1r),\ \frac1q \le \frac12 \le \frac1r\},$$

$$\Delta_2(n) = \{(\frac1r, \frac1q) \in [0,1]\times[0,1) : \frac1q \le \frac12,\ \frac1q \le \frac{n-1}{2}(\frac12 - \frac1r)\},$$

$$\Delta_3(n) = \{(\frac1r, \frac1q) \in [0,1]\times[0,1) : \frac12 \le \frac1q,\ \frac1r \le \frac{n-3}{2(n-1)}\},$$

$$\Delta_4(n) = \{(\frac1r, \frac1q) \in [0,1]\times[0,1) : \frac{n-3}{2(n-1)} \le \frac1r \le \frac12, \\ \frac{n-1}{2}(\frac12 - \frac1r) \le \frac1q \le \frac1r + \frac{n+1}{2}(\frac12 - \frac1r)\},$$

$$\Delta_5(n) = \{(\frac1r, \frac1q) \in [0,1]\times[0,1) : \frac1r + \frac{n+1}{2}(\frac12 - \frac1r) \le \frac1q,\ \frac1r \le \frac12\},$$

$$\Delta_6(n) = \{(\frac1r, \frac1q) \in [0,1]\times[0,1) : \frac12 \le \frac1r,\ \frac12 \le \frac1q < \frac{n+1}{2}(1 - \frac1r)\}.$$

Then comparing the conditions[2] (9)–(12), (13) and (14), we see that among those conditions (9) is the strongest on the set $\Delta_2(n)$, (13) on $\Delta_3(n)$, (10) on $\Delta_4(n)$, (14) on $\Delta_5(n)$, (11) on $\Delta_1(n)$, and (12) on $\Delta_6(n)$. Hence, for (23) we need to show the following estimates:

[2]Conditions for high and low frequency interaction

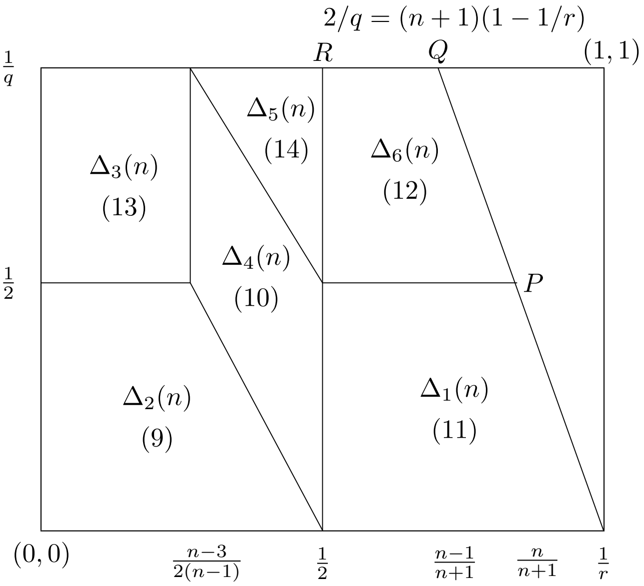

FIGURE 4. For $n \geq 3$, the sets $\Delta_1(n), \Delta_2(n), \dots, \Delta_6(n)$; comparing the conditions (9)–(12), (13) and (14)

Assuming $E(\phi_{0,\Gamma})^{\frac{1}{2}} = E(\psi_{k,\Gamma'})^{\frac{1}{2}} = 1$, for $\epsilon > 0$,

(24)

$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{l(2/q-(n-1)(1-1/r))} \text{ if } (1/r, 1/q) \in \Delta_1(n),$$
$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k(-1/q+n(1/2-1/r)+\epsilon)} 2^{l(2/q-(n-1)(1-1/r))} \text{ if } (1/r, 1/q) \in \Delta_2(n),$$
$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k((n-1)/2-n/r)+\epsilon)} 2^{l(2/q-(n-1)(1-1/r))} \text{ if } (1/r, 1/q) \in \Delta_3(n),$$
$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k\frac{n+1}{2}(1/2-1/r)+\epsilon)} 2^{l(2/q-(n-1)(1-1/r))} \text{ if } (1/r, 1/q) \in \Delta_4(n),$$
$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k(1/q-1/r)+\epsilon)} 2^{l(2/q-(n-1)(1-1/r))} \text{ if } (1/r, 1/q) \in \Delta_5(n),$$
$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k(1/q-1/2+\epsilon)} 2^{l(2/q-(n-1)(1-1/r))} \text{ if } (1/r, 1/q) \in \Delta_6(n).$$

As it can be easily verified from the necessary conditions and rescaling, all these estimates are sharp up to ϵ-loss. In view of interpolation, to prove this we only need to show the corresponding estimates for

$$\left(\frac{1}{r}, \frac{1}{q}\right) = (1,0)\,,\ \left(\frac{1}{2}, 0\right),\ (0,0)\,,$$
$$P = \left(\frac{n}{n+1}, \frac{1}{2}\right),\ \left(\frac{1}{2}, \frac{1}{2}\right),\ \left(\frac{n-3}{2\,(n-1)}, \frac{1}{2}\right),\ \left(0, \frac{1}{2}\right),$$
$$Q = \left(\frac{n-1}{n+1}, 1\right),\ R = \left(\frac{1}{2}, 1\right),\ \left(\frac{n-3}{2\,(n-1)}, 1\right),\ (0,1)\,,$$

or for some $(1/r, 1/p)$ which is arbitrarily close to these points.

For P, Q, R, we are not able to obtain the corresponding end point estimates. However, it is possible to prove estimates for some $(1/r, 1/q)$ arbitrarily close to the points P, Q, R. These estimates follow from Proposition 2.4.

The estimates for $(1,0), (\frac{1}{2}, 0), (0,0)$, $(\frac{n-3}{2(n-1)}, \frac{1}{2}), (0, \frac{1}{2})$, $(\frac{n-3}{2(n-1)}, 1), (0,1)$ can be proven from Strichartz's estimates and re-scaling. We need the following:

Lemma 3.2. *If ψ is a wave with its Fourier support contained in the set*

$$\{(\xi, \pm|\xi|) : \xi_n \sim 2^k, |\xi'| \lesssim 2^{k-l}\},$$

then, for $q \geq 2$, $2/q \leq (n-1)(1/2 - 1/r)$ and $l \geq 0$,

$$\|\psi\|_{L_t^q L_x^r} \leq C 2^{k(n(1/2-1/r)-1/q)} 2^{l(2/q+(n-1)(1/r-1/2))} \|\psi(0)\|_{L^2}$$

with the exception $(q, r) = (2, \infty)$ when $n = 3$.

Proof. We may assume that $l \gg 1$ by the usual Strichartz's estimates for the wave equation [4] and a mild rescaling. We write ψ as

$$\psi(x,t) = \iint e^{i(x'\xi' + x_n\xi_n \pm t|\xi|)} \widehat{\psi(0)}(\xi) d\xi.$$

Then, by changing variables $x_n \to x_n - t$ we may replace the phase part by $x'\xi' + x_n\xi_n \pm t(|\xi| - \xi_n)$. Re-scaling $(\xi', \xi_n) \to (2^{k-l}\xi', 2^k\xi_n)$ transforms the phase to

$$2^{k-l}x'\xi' + 2^k x_n\xi_n \pm t(\sqrt{2^{2k-2l}|\xi'|^2 + 2^{2k}\xi_n^2} - 2^k\xi_n).$$

By $(x', x_n, t) \to (2^{-k+l}x', 2^{-k}x_n, 2^{-k+2l}t)$ it is further changed to

$$\phi(x,t,\xi) = x'\xi' + x_n\xi_n \pm t2^{-k+2l}(\sqrt{2^{2k-2l}|\xi'|^2 + 2^{2k}\xi_n^2} - 2^k\xi_n).$$

Let us set

$$Uf(x,t) = \int_{\xi_n \sim 1, |\xi'| \lesssim 1} e^{i\phi(x,t,\xi)} \widehat{f}(\xi) d\xi.$$

Since $2^{-k+2l}(\sqrt{2^{2k-2l}|\xi'|^2 + 2^{2k}\xi_n^2} - 2^k\xi_n) = |\xi'|^2/\xi_n + O(2^{-l}|\xi'|^3/\xi_n^2)$, we can use the well known argument for Strichartz's estimates to obtain $\|Uf\|_{L_t^q L_x^r} \leq C\|f\|_2$ for $q \geq 2$ and $2/q \leq (n-1)(1/2 - 1/r)$ (see [4]). Reversing rescaling gives the required. $\square$

From Lemma 3.2 it follows that for $\frac{1}{q} \leq \frac{n-1}{2}(1 - \frac{1}{r})$ (with the exception of $(q, r) = (1, \infty)$ when $n = 3$),

$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k(\frac{n}{2}(1-1/r)-1/(2q))} 2^{l(2/q-(n-1)(1-1/r))}, \tag{25}$$

for $q \geq 2$, $\frac{1}{q} \leq \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r})$ (with the exception of $(q, r) = (2, \infty)$ when $n = 3$),

$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k(n(1/2-1/r)-1/q)} 2^{l(2/q-(n-1)(1-1/r))} \tag{26}$$

and for $\frac{1}{q} \leq (n-1)(\frac{1}{2} - \frac{1}{r})$, (with the exception of $q = 1$ when $n = 3$),

$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq C 2^{k(n(1/2-1/r)-1/2q)} 2^{l(2/q-(n-1)(1-1/r))}. \tag{27}$$

For (25) use $\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq \|\phi_{0,\Gamma}\|_{2q,2r}\|\psi_{k,\Gamma'}\|_{2q,2r}$ and apply Lemma 3.2 with exponents $(2q, 2r)$ to each term. (26) follows from $\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L_t^q L_x^r} \leq \|\phi_{0,\Gamma}\|_{\infty,\infty}\|\psi_{k,\Gamma'}\|_{L_t^q L_x^r}$

and Lemma 3.2. For (27) use $\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L^q_tL^r_x} \le \|\phi_{0,\Gamma}\|_{2q,\infty}\|\psi_{k,\Gamma'}\|_{2q,r}$ and apply Lemma 3.2 with exponents $(2q,\infty)$ and $(2q,r)$.

Now, for $n \ge 4$, the estimates for $(\frac{1}{r},\frac{1}{q}) = (1,0)$, $(0,1)$ are particular cases of (25), the estimates for $(\frac{1}{2},0)$, $(0,0)$, $(\frac{n-3}{2(n-1)},\frac{1}{2})$ and $(0,\frac{1}{2})$ are particular cases of (26) and the estimate for $(\frac{n-3}{2(n-1)},1)$ is a particular case of (27). At $(1/2,1/2)$, we need to show

$$\|\phi_{0,\Gamma}\psi_{k,\Gamma'}\|_{L^2_tL^2_x} \le C2^{l(3-n)/2}. \tag{28}$$

Let $d\sigma_0$, $d\sigma_k$ be the induced Lebesgue measures on Γ_0, Γ'_k, respectively. It is easy to see $\|d\sigma_0 * d\sigma_k\|_\infty \le C2^{l(3-n)}$. Then the required estimate follows from Plancherel's Theorem and interpolation between L^∞ and L^1 estimates. (It also can be shown from (42) by rescaling.)

When $n = 3$, the set Δ_3 shrinks to the line segment $[(0,1/2),(0,1)]$. Even though, the required estimates can be obtained by the same argument except for $(1/r,1/q) = (0,1)$, $(0,\frac{1}{2})$ because the estimate for $(1/r,1/q) = (0,1/2)$ is not allowed in Lemma 3.2. However it is possible to obtain the corresponding estimates for some $(1/r,1/q)$ arbitrary close to $(0,1)$ and $(0,\frac{1}{2})$ by Lemma 3.2 and Hölder's inequality. $\square$

Now we turn to the case $n = 2$.

Proof of Lemma 3.1 when $n = 2$. For $n = 2$ the picture is slightly different (see Figure 5). Recalling the condition (3), let us define the sets $\Delta_1(2)$, $\Delta_2(2)$, $\Delta_4(2)$, $\Delta_5(2)$, $\Delta_6(2) \subset [0,1]\times[0,3/4)$ by setting

$$\begin{aligned}
\Delta_1(2) &= \{(\frac{1}{r},\frac{1}{q}) \in [0,1]\times[0,3/4) : \frac{2}{q} < 3(1-\frac{1}{r}),\ \frac{1}{q} \le \frac{1}{2} \le \frac{1}{r}\},\\
\Delta_2(2) &= \{(\frac{1}{r},\frac{1}{q}) \in [0,1]\times[0,3/4) : \frac{1}{q} \le \frac{1}{2},\ \frac{1}{q} \le \frac{1}{2}(\frac{1}{2}-\frac{1}{r})\},\\
\Delta_4(2) &= \{(\frac{1}{r},\frac{1}{q}) \in [0,1]\times[0,3/4) : \frac{1}{2}(\frac{1}{2}-\frac{1}{r}) \le \frac{1}{q} \le \frac{1}{r} + \frac{3}{2}(\frac{1}{2}-\frac{1}{r})\},\\
\Delta_5(2) &= \{(\frac{1}{r},\frac{1}{q}) \in [0,1]\times[0,3/4) : \frac{1}{r} + \frac{3}{2}(\frac{1}{2}-\frac{1}{r}) \le \frac{1}{q},\ \frac{1}{r} \le \frac{1}{2}\},\\
\Delta_6(2) &= \{(\frac{1}{r},\frac{1}{q}) \in [0,1]\times[0,3/4) : \frac{1}{2} \le \frac{1}{r},\ \frac{1}{2} \le \frac{1}{q} < \frac{3}{2}(1-\frac{1}{r})\}.
\end{aligned}$$

To obtain (23) we need to show (24) with $n = 2$ but for Theorem 1.1 it is enough to do it for $(1/r,1/q) = (1,0),(1/2,0),(0,0)$, $(1/2,1/2),(1/2,3/4), P = (2/3,1/2)$, $(0,1/4)$ and $(0,1/2)$ or arbitrarily close to them. The estimates for $(1,0)$ and $(0,1/2)$ follow from (25) and for $(0,0)$, $(1/2,0)$ and $(0,1/4)$ follow from (26). For $(1/2,1/2)$ we use (28) and for $P = (2/3,1/2)$ and $(1/2,3/4)$ we use Proposition 2.4. $\square$

3.2. **Proof of estimate (22) when $k = O(1)$; low frequency interactions.** Obviously, (22) is equivalent to

$$\|D_0^\beta D_+^{\beta_+-\beta_-}|\Box|^{\beta_-}(\phi_{0,\Gamma}\psi_{k,\Gamma'})\|_{L^q_tL^r_x} \le C2^{-\epsilon l}E(\phi_{0,\Gamma})^{\frac{1}{2}}E(\psi_{k,\Gamma'})^{\frac{1}{2}} \tag{29}$$

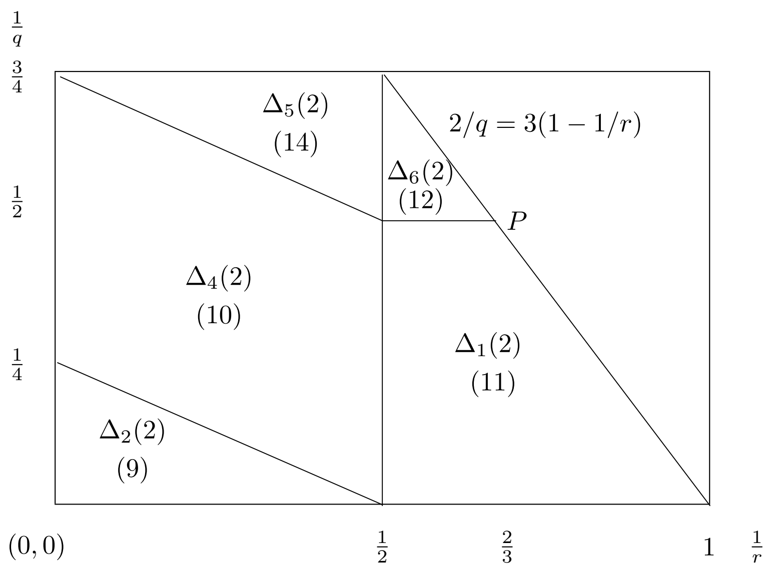

FIGURE 5. $n = 2$, $\Delta_1(2), \Delta_2(2), \Delta_4(2), \Delta_5(2), \Delta_6(2)$; comparing (9)–(12), (14)

for some $\epsilon > 0$. The symbol $D_0^{\beta} D_+^{\beta_+ - \beta_-}$ no longer behaves so nicely as in high and low frequency interactions. We decompose the waves further to handle $D_0^{\beta} D_+^{\beta_+ - \beta_-}$ near the origin. We will prove (29) by considering two cases, $(++)$ and $(+-)$, separately.

3.2.1. $(+-)$ *case.* First we show (29) when $(1/r, 1/q)$ is close enough to the line $2/q - (n+1)(1-1/r) = 0$ and next we will show (29) for $2/q < n(1-1/r)$ (see Figure 6). Then interpolation between these two kind of estimates gives (29) for all r, q, $2/q < (n+1)(1-1/r) < 0$. Indeed, it is possible to obtain estimates (29) even if β_0, β_+, β_- are complex number as long as $Re(\beta_0)$, $Re(\beta_+)$, $Re(\beta_-)$ satisfy the conditions (2)-(15) with strict inequality instead of β_0, β_+, β_-. And it is easy to see the constant C in (29) is $O(1 + |\beta_0| + |\beta_+| + |\beta_-|)^N$ from the argument below (c.f. Lemma 2.6). Hence we can use complex interpolation along β_0, β_+, β_-.

To begin with, we make a further decomposition to handle $D_0^{\beta} D_+^{\beta_+ - \beta_-}$. For each m, $0 \le m \le l$, we decompose dyadically along the direction of light ray to break each of the surfaces $\Gamma_0(2^{-l})$, $\Gamma'_k(2^{-l})$ into sectors $\Lambda_{m,j}, \Lambda'_{m,i}$ of size $2^{-l} \times \cdots \times 2^{-l} \times 2^{m-l}$ essentially (the longest direction is parallel to to the light ray). So, for each fixed m, we have

$$\Gamma_0(2^{-l}) = \bigcup_j \Lambda_{m,j}\,, \qquad \Gamma'_k(2^{-l}) = \bigcup_i \Lambda'_{m,i}.$$

Then, we may write

$$\Gamma_0(2^{-l}) \times \Gamma'_k(2^{-l}) = \bigcup_{0 \le m \le l} \ \bigcup_{\text{dist } (\Lambda_{m,j}, -\Lambda'_{m,i}) \sim 2^{m-l}} \Lambda_{m,j} \times \Lambda'_{m,i}.$$

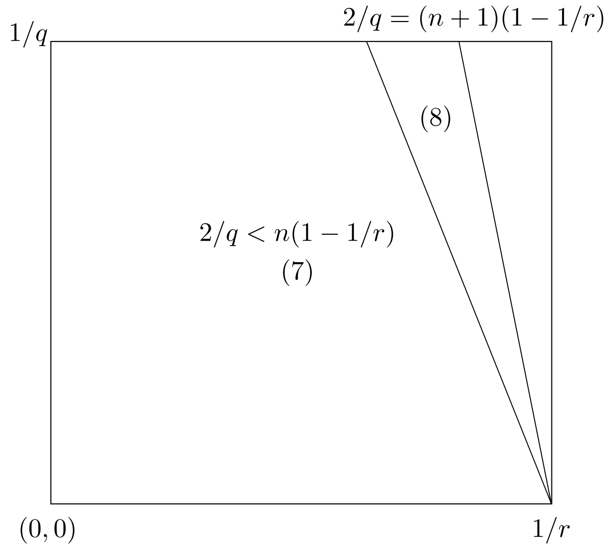

FIGURE 6. Comparing the conditions (7) and (8)

Note that dist $(\Lambda_{m,j}, -\Lambda'_{m,i}) \gtrsim 2^{-l}$ (in the case $m = 0$ this follows from the angular separation). Hence, we have

$$\phi_{0,\Gamma}\psi_{k,\Gamma'} = \sum_{0\le m\le l}\ \sum_{\text{dist } (\Lambda_{m,j},-\Lambda'_{m,i})\sim 2^{m-l}} \phi_{\Lambda_{m,j}}\psi_{\Lambda'_{m,i}}$$

where $\phi_{\Lambda_{m,j}}$, $\psi_{\Lambda'_{m,i}}$ are waves with their Fourier supports contained in $\Lambda_{m,j}$, $\Lambda'_{m,i}$, respectively.

It is easy to see that if the Fourier transform of F is supported on $\Lambda_{m,j} + \Lambda'_{m,i}$

$$\|D_0^{\beta_0} D_+^{\beta_+-\beta_-} F\|_{L^q_t L^r_x} \le C2^{(m-l)(\beta_0+\beta_+-\beta_-)}\|F\|_{L^q_t L^r_x}$$

because dist $(\Lambda_{m,j}, -\Lambda'_{m,i}) \sim 2^{m-l}$. Hence, using this and Lemma 2.6, we have

$$\begin{aligned}(30)\qquad &\|D_0^{\beta_0} D_+^{\beta_+-\beta_-}|\Box|^{\beta_-}(\phi_{\Lambda_{m,j}}\psi_{\Lambda'_{m,i}})\|_{L^q_t L^r_x}\\ &\qquad \le C2^{m(\beta_0+\beta_+-\beta_-)}2^{-l(\beta_0+\beta_++\beta_-)}\|\phi_{\Lambda_{m,j}}\psi_{\Lambda'_{m,i}}\|_{L^q_t L^r_x}.\end{aligned}$$

When $(1/q, 1/r)$ *is near the line* $2/q = (n+1)(1-1/r)$. From the above (30) and Proposition 2.4 we have

$$\begin{aligned}\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi_{\Lambda_{m,j}}\psi_{\Lambda'_{m,i}})\|_{L^q_t L^r_x} &\le C2^{(m-l)(\beta_0+\beta_+-\beta_-)}\\ &\times 2^{-l(2\beta_--2/q+(n-1)(1-1/r))}E(\phi_{\Lambda_{m,j}})^{\frac12}E(\psi_{\Lambda'_{m,i}})^{\frac12}.\end{aligned}$$

Hence, if $\beta_0+\beta_+-\beta_- \ge 0$, then (29) follows from summation in m, Cauchy-Schwarz's inequality and Plancherel's theorem because $2\beta_- - 2/q + (n-1)(1-1/r) > 0$ by (4) with strict inequality.

On the other hand, if $\beta_0 + \beta_+ - \beta_- < 0$, then by using the above estimates and direct summation in m (also by Schwarz's inequality and Plancherel's theorem) we get

$$\sum_{0\le m\le l}\ \sum_{\text{dist }(\Lambda_{m,j},-\Lambda'_{m,i})\sim 2^{m-l}} \|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}(\phi_{\Lambda_{m,j}}\psi_{\Lambda'_{m,i}})\|_{L_t^qL_x^r}$$
$$\le C2^{-l(\beta_0+\beta_++\beta_--2/q+(n-1)(1-1/r))}E(\phi_{0,\Gamma})^{\frac12}E(\psi_{k,\Gamma'})^{\frac12}.$$

From (2), the exponent of 2^{-l} equals to $\alpha_1+\alpha_2-1/q-(1-1/r)$. Since (8) is valid with strict inequality, to say, $\alpha_1+\alpha_2-3/q+n(1-1/r)=\delta$ for some $\delta>0$. Obviously $\alpha_1+\alpha_2-1/q-(1-1/r)>0$ if $\delta/2>2/q-(n+1)(1-1/r)>-\delta/2$. Hence we get the required estimates (29) as long as $(1/r,1/q)$ is close enough to the line $2/q-(n+1)(1-1/r)=0$.

When $2/q<n(1-1/r)$. In this case, to obtain the sharp estimates it is not enough to apply Proposition 2.4 to (30). Instead, we use Proposition 2.5 to get better bounds. Note that the conditions[3] (7), (8) coincide along the line $2/q=n(1-1/r)$ and (7) is stronger when $2/q<n(1-1/r)$. (See figure 6.)

Using (30) and Proposition 2.5, we have

$$\|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}(\phi_{\Lambda_{m,j}}\psi_{\Lambda'_{m,i}})\|_{L_t^qL_x^r}\le C2^{(m-l)(\beta_0+\beta_+-\beta_-+1-1/r-\epsilon)}$$
$$\times\, 2^{-l(2\beta_--2/q+(n-1)(1-1/r))}E(\phi_{\Lambda_{m,j}})^{\frac12}E(\psi_{\Lambda'_{m,i}})^{\frac12}.$$

If $\beta_0+\beta_+-\beta_-+1-1/r>0$, choosing sufficiently small $\epsilon>0$ such that $\beta_0+\beta_+-\beta_-+1-1/r-\epsilon>0$ we get (29) by summation in m and the condition (4) with strict inequality.

On the other hand, if $\beta_0+\beta_+-\beta_-+1-1/r\le 0$, from summation in m and the condition (3) we see

$$\sum_{0\le m\le l}\ \sum_{\text{dist }(\Lambda_{m,j},-\Lambda'_{m,i})\sim 2^{m-l}} \|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}(\phi_{\Lambda_{m,j}}\psi_{\Lambda'_{m,i}})\|_{L_t^qL_x^r}$$
$$\le C2^{-l(\alpha_1+\alpha_2-1/q-\epsilon)}E(\phi_{0,\Gamma})^{\frac12}E(\psi_{k,\Gamma'})^{\frac12}.$$

Hence, if (7) holds with strict inequality, we get (29) with sufficiently small $\epsilon>0$.

It should be noted that when $n=2,3$, the additional assumption $4/n<q$ was used for $r>2$ (see Proposition 2.5).

3.2.2. *Case* $(++)$. We consider the two cases, $2^{-l}\sim 1$ and $2^{-l}\ll 1$, separately.

Case $2^{-l}\ll 1$. In this case the multipliers for D_0, D_+ are smooth and bounded away from zero on the Fourier support of $(\phi_{0,\Gamma}\psi_{k,\Gamma'})$. So, the harmless $D_0^{\beta_0}$, $D_+^{\beta_+-\beta_-}$ can be discarded. Hence it is enough to show that

$$\||\Box|^{\beta_-}(\phi_{0,\Gamma}\psi_{k,\Gamma'})\|_{L_t^qL_x^r}\le C2^{-\epsilon l}E(\phi_{0,\Gamma})^{\frac12}E(\psi_{k,\Gamma'})^{\frac12}.$$

From Lemma 2.6 and Proposition 2.4, we see that the left hand side of the above inequality is bounded by

$$2^{-l(2\beta_--2/q+(n-1)(1-1/r))}E(\phi_{0,\Gamma})^{\frac12}E(\psi_{k,\Gamma'})^{\frac12}.$$

[3]conditions for low frequency interaction $(+-)$

Therefore from the condition (4) we get the required estimate (29).

Case $2^{-l} \sim 1$. If the angular separation $2^{-l} \sim 1$, on the Fourier support of $\phi_{0,\Gamma}\psi_{k,\Gamma'}$ $D_+^{\beta_+}$, $D_-^{\beta_-}$ are a harmless smooth operator because the multipliers are bounded away from zero. So we may discard them as before.

For $m \geq 10$, let P_m be the usual Littlewood-Paley projection in ξ to the set
$$\{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R} : |\xi| \sim 2^{-m}\}.$$
Then it is enough to show that for $\epsilon > 0$,
$$\|P_m D_0^\beta(\phi_{0,\Gamma}\psi_{k,\Gamma'})\|_{L_t^q L_x^r} \leq C2^{-\epsilon m} E(\phi_{0,\Gamma})^{\frac12} E(\psi_{k,\Gamma'})^{\frac12}.$$
Equivalently, it suffices to show
$$\|P_m(\phi_{0,\Gamma}\psi_{k,\Gamma'})\|_{L_t^q L_x^r} \leq C2^{m\beta_0-\epsilon m} E(\phi_{0,\Gamma})^{\frac12} E(\psi_{k,\Gamma'})^{\frac12}.$$
We decompose $\Gamma_0(2^{-l})$ and $\Gamma_k'(2^{-l})$ into disjoint surfaces Λ, Λ' of diameter $\sim 2^{-m}$ so that
$$\phi_{0,\Gamma} = \sum_\Lambda \phi_\Lambda, \ \psi_{k,\Gamma'} = \sum_{\Lambda'} \psi_{\Lambda'},$$
where ϕ_Λ and $\psi_{\Lambda'}$ are waves Fourier supported in Λ, Λ', respectively. Then by Cauchy–Schwarz's inequality and Plancherel's theorem we are reduced to showing

$$\|P_m(\phi_\Lambda\psi_{\Lambda'})\|_{L_t^q L_x^r} \leq C2^{m\beta_0-\epsilon m} E(\phi_\Lambda)^{\frac12} E(\psi_{\Lambda'})^{\frac12} \tag{31}$$

for some $\epsilon > 0$. Note that $P_m(\phi_\Lambda\psi_{\Lambda'}) \neq 0$ only if $\Lambda + \Lambda' \in \{(\xi,\tau) : |\xi| \leq C2^{-m}\}$. So, by rescaling and rotation we may assume that

$$\Lambda = \{(\xi,|\xi|) : \xi \in B(e_n, C2^{-m})\}, \ \Lambda' = \{(\xi,|\xi|) : \xi \in B(-e_n, C2^{-m})\}. \tag{32}$$

Let us set $I = [0,1]$
$$\Delta_7 = \{(\frac1r,\frac1q) \in I\times I : \frac1q \leq 1-\frac1r, \ \frac1q \leq \frac12\},$$
$$\Delta_8 = \{(\frac1r,\frac1q) \in I\times I : \min(1,\frac{n+1}{4}) > \frac1q \geq \frac12, \ \frac1r \leq \frac12\},$$
$$\Delta_9 = \{(\frac1r,\frac1q) \in I\times I : \min(1,\frac{n+1}{4}) > \frac1q \geq 1-\frac1r,$$
$$\frac1r \geq \frac12, \frac2q < (n+1)(1-\frac1r)\}.$$
Hence comparing the conditions[4] (5), (6) and (15), for (31) it is sufficient to show the following estimates:

For any $\epsilon > 0$,

$$\begin{aligned} &\|\phi_\Lambda\psi_{\Lambda'}\|_{L_t^q L_x^r} \leq C2^{m(1/q-n(1-1/r)+\epsilon)} E(\phi_\Lambda)^{\frac12} E(\psi_{\Lambda'})^{\frac12} \text{ if } (1/r,1/q) \in \Delta_7, \\ &\|\phi_\Lambda\psi_{\Lambda'}\|_{L_t^q L_x^r} \leq C2^{m(2/q-n(1-1/r)-1/2+\epsilon)} E(\phi_\Lambda)^{\frac12} E(\psi_{\Lambda'})^{\frac12} \text{ if } (1/r,1/q) \in \Delta_8, \\ &\|\phi_\Lambda\psi_{\Lambda'}\|_{L_t^q L_x^r} \leq C2^{m(2/q-(n+1)(1-1/r)+\epsilon)} E(\phi_\Lambda)^{\frac12} E(\psi_{\Lambda'})^{\frac12} \text{ if } (1/r,1/q) \in \Delta_9. \end{aligned} \tag{33}$$

[4]Conditions for low frequency interaction (++)

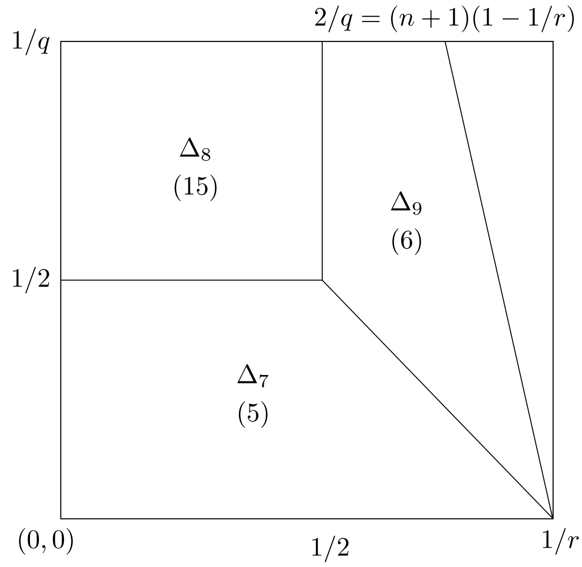

FIGURE 7. Comparing (5), (6) and (15), $n \geq 3$

For $n \geq 3$ it is enough to show the estimates corresponding to the points

$$\left(\frac{1}{r}, \frac{1}{q}\right) = (1,0)\,, (0,0)\,, \left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}\right), \left(\frac{n-1}{n+1}, 1\right), \left(\frac{1}{2}, 1\right), (0,1)\,,$$

or $(1/r, 1/q)$ arbitrary close to these points (see Figure 7). If $n = 2$, we need estimates corresponding to the points

$$\left(\frac{1}{r}, \frac{1}{q}\right) = (1,0)\,, (0,0)\,, \left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(0, \frac{3}{4}\right).$$

Estimates at $(1,0)$ and $(0,0)$ are trivial from Hölder's inequality and Plancherel's theorem. Since Λ and Λ' transversal to each other, the estimate at $(1/2, 1/2)$ can be proven by the same way as (28). Now note that the Fourier supports of $(\phi_\Lambda \psi_{\Lambda'})(\cdot, t)$ are contained in a set of diameter $C2^{-m}$. So, the estimate at $(0, 1/2)$ can be obtained by applying Bernstein's inequality to the estimate at $(1/2, 1/2)$.

When $n = 2$, the $L_t^q L_x^r$ estimates at $(1/r, 1/q)$ arbitrarily close to $(1/2, 3/4)$ is contained in Proposition 2.4 because angular separation between the sets Λ and Λ' is about 1. Finally the remaining estimates at $(0, 3/4)$ follow from the estimates at $(1/2, 3/4)$ by Bernstein's inequality.

Now we turn to the case $n \geq 3$. Note that the $L_t^q L_x^r$ estimates at $(1/r, 1/q)$ arbitrarily close to $(\frac{n-1}{n+1}, 1)$ are particular cases of Proposition 2.4. As before the estimates at $(1/r, 1/q)$ arbitrarily close to $(0,1)$ follow from the estimates at $(1/2, 1)$ by Bernstein's inequality. Hence, by Lemma 4.4 (Globalization Lemma) it is enough to show

Proposition 3.3. *Let $n \ge 3$ and let Λ and Λ' be the sets given by (32). If ϕ_Λ, $\psi_{\Lambda'}$ are the waves having Fourier supports in Λ, Λ', respectively. Then, for any $\alpha > 0$*

$$\|\phi_\Lambda \psi_{\Lambda'}\|_{L^1_t L^2_x(Q(R))} \le 2^{(3/2-n/2)m} R^\alpha \|\phi_\Lambda(0)\|_2 \|\psi_{\Lambda'}(0)\|_2, R \gg 1 \tag{34}$$

where $Q(R)$ is the cube with side length R centered at the origin.

Indeed, by rescaling and stationary phase method, one can see that ϕ_Λ, $\phi_{\Lambda'}$ are expressed as extension operators given by measures $d\sigma_1$, $d\sigma_2$, respectively, which satisfy $|\widehat{d\sigma_i}(x,t)| \le C2^{mc}(1+|(x,t)|)^{-(n-1)/2}$ for some c. Then from Lemma 4.4 we obtain $L^q_t L^r_x$ estimates for $(1/q, 1/r)$ arbitrarily close to $(1, 1/2)$ at loss of $2^{\epsilon m}$. See Remark 4.5.

Proof of Proposition 3.3. Recall that Λ, Λ' are parts of the cone of diameter $C2^{-m}$-balls in centered at $(e_n, 1)$, $(-e_n, 1)$, respectively. We start with the induction assumption that (34) is valid for some $\alpha > 0$. Following the induction on scales argument [22], we will show that (34) implies that for $0 < \delta, \epsilon \ll 1$ and $R \gg 1$,

$$\|\phi_\Lambda \psi_{\Lambda'}\|_{L^1_t L^2_x(Q(R))} \le CR^\epsilon (R^{\alpha(1-\delta)} + R^{c\delta}) 2^{(3/2-n/2)m} \|\phi_\Lambda(0)\|_2 \|\psi_{\Lambda'}(0)\|_2 \tag{35}$$

with c independent of ϵ, δ, R. Iterative use of this implication gives (34) for any $\alpha > 0$.

By the same argument used for (28) we have

$$\|\phi_\Lambda \psi_{\Lambda'}\|_{L^2_{t,x}(Q(R))} \le C2^{-m(n-1)/2} \|\phi_\Lambda(0)\|_2 \|\psi_{\Lambda'}(0)\|_2.$$

Hence if $R \le 2^{2m}$ it gives (34) with $\alpha = 0$ by Hölder's inequality. So, we may assume $R \gg 2^{2m}$. Fixing $R \gg 2^{2m}$, by wave packet decomposition at the scale of R we can write

$$\phi_\Lambda = \sum_{T \in \mathcal{T}} C_T P_T, \quad \psi_{\Lambda'} = \sum_{T' \in \mathcal{T}'} C_{T'} P_{T'}$$

on Q_R. Here $\mathcal{T}, \mathcal{T}'$ are the collections of tubes associated to wave packet decomposition for ϕ_Λ, $\psi_{\Lambda'}$ at scale R, respectively. P_T, $P_{T'}$ are functions, essentially supported on T, T', respectively. Their spatial Fourier transforms are contained in R^{-1} neighborhood of the Fourier supports of ϕ_Λ, $\psi_{\Lambda'}$ and $\|P_T(0)\|_{L^2} = \|P_{T'}(0)\|_{L^2} = 1$. Moreover $\|\phi_\Lambda(0)\|^2_{L^2} \approx \sum_T C_T^2$ and $\|\psi_{\Lambda'}(0)\|^2_{L^2} \approx \sum_{T'} C_{T'}^2$. See [10] for more details (see also [16]).

By the standard argument involving pigeonholing we are reduced to showing that if $R \gg 2^{2m}$, $0 < \delta \ll 1$, then

$$\|\sum_{T,T'} P_T P_{T'}\|_{L^1_t L^2_x(Q(R))} \le C2^{(3/2-n/2)m} (R^{\alpha(1-\delta)} + R^{c\delta}) \#(\mathcal{T})^{1/2} \#(\mathcal{T}')^{1/2}.$$

Then partition $Q(R)$ into $R^{1-\delta}$-cubes and we denote by $\{b\}$ these cubes. The left hand side of the above is bounded by

$$\sum_b \|\sum_{T,T'} P_T P_{T'}\|_{L^1_t L^2_x(b)} \le \sum_b \|\sum_{T\sim b, T'\sim b} P_T P_{T'}\|_{L^1_t L^2_x(b)} + \sum_b \|\sum_{T\not\sim b \text{ or } T'\not\sim b} P_T P_{T'}\|_{L^1_t L^2_x(b)}$$

where $\sim$ is the usual relation between T, T' and b defined in [16] (see also [10]).

For the first term we can use the induction assumption to bound it by

$$C2^{(3/2-n/2)m}R^{\alpha(1-\delta)}\#(\mathcal{T})^{1/2}\#(\mathcal{T}')^{1/2}.$$

For the second term,

$$\begin{aligned}\|\sum_{T\not\sim b \text{ or } T'\not\sim b} P_T P_{T'}\|_{L^1_t L^2_x(b)} &\le CR^{1/2}\|\sum_{T\not\sim b \text{ or } T'\not\sim b} P_T P_{T'}\|_{L^2_t L^2_x(b)} \\ &\le CR^{c\delta}R^{1/2}R^{-(n-1)/4}\#(\mathcal{T})^{1/2}\#(\mathcal{T}')^{1/2} \\ &\le CR^{c\delta}2^{(3-n)m/2}\#(\mathcal{T})^{1/2}\#(\mathcal{T}')^{1/2}\end{aligned}$$

because $R \ge 2^{2m}$. The second inequality is a consequence of estimate (2.12) in [10], which is in the same spirit as inequality (23) in Tao's [16]. □

3.3. **Remarks for the proof of Corollary 1.2; Sufficiency part.** The sufficiency part can be proven following the same way as before. But we need to handle the additional multiplier weights for $Q_{0,j}$ $Q_{i,j}$ at some stages of the argument. It should be noted that the null forms $Q_{0,j}$, $Q_{i,j}$ are not completely invariant under spatial rotation. In other words, we may not be able to assume that the angularly decomposed waves $\phi_{0,\Gamma}, \psi_{k,\Gamma'}$ are supported in the specified conic subsets $\Gamma_0(2^{-l})$ and $\Gamma'_k(2^{-l})$ as before. However it does not cause any problem since we can work in an adapted coordinates frame.

After dyadic decomposition, angular Whitney-type decompositions and rotation if necessary, to each pair of waves we can apply Lemma 2.6 and the following.

Lemma 3.4. *Let $Q = Q_{0,j}$, or $Q = Q_{i,j}$ and set $\Lambda_k(2^{-l}) = 2^k\{(\xi, |\xi|) : |\xi| \sim 1, |\xi/|\xi| - v| \le 2^{-l}\}$ for some $|v| = 1$. Suppose ϕ, ψ (or $\bar\psi$) are waves with Fourier supports in $\Lambda_0(2^{-l})$, $\Lambda_k(2^{-l})$, respectively. Then for $1 \le q, r \le \infty$, $k, l \ge 0$ and any $N > 0$, there is a constant C, independent of v, such that*

$$\|Q(\phi,\psi)\|_{L^q_t L^r_x} \le C2^{k-l}\sum_{\mu=(\mu_1,\mu_2)\in\mathbb{Z}^n\times\mathbb{Z}^n}(1+|\mu|)^{-N}\|\phi_{\mu_1}\psi_{\mu_2}\|_{L^q_t L^r_x}$$

where ϕ_{μ_1}, ψ_{μ_2} are waves satisfying $E(\phi_{\mu_1}) = E(\phi)$, $E(\psi_{\mu_2}) = E(\psi)$, and ϕ_{μ_1}, ψ_{μ_2} (or $\bar\psi_{\mu_2}$) are Fourier supported in $\Lambda_0(2^{-l})$, $\Lambda_k(2^{-l})$, respectively.

This lemma with Remark 2.7 enables us to trade off the multiplier $Q = Q_{0,j}$, $Q = Q_{i,j}$ for sharp bounds in terms of 2^k, 2^l. Using Lemma 3.4 one can easily check that the arguments for the proof Theorem 1.1 in Sections 3.1, 3.2 work except for the case $2^{-l} \sim 1$[5] in section 3.2.2 ((++) in low frequency interaction). For the remaining case one can use the following:

Lemma 3.5. *Let $\Lambda = \{(\xi, |\xi|) : |\xi - v| \le C2^{-m}\}$, $\Lambda' = \{(\xi, |\xi|) : |\xi + v| \le C2^{-m}\}$ for some v, $|v| = 1$. Suppose ϕ, ψ (or $\bar\psi$) are waves with Fourier supports in Λ, Λ', respectively. Then for $1 \le q, r \le \infty$, $k, l \ge 0$ and any $N > 0$, there is a constant C,*

[5]We did not use Lemma 2.6 in this case.

independent of v, such that

$$\|Q_{0,j}(\phi,\psi)\|_{L^q_tL^r_x} \le C \sum_{\mu=(\mu_1,\mu_2)\in\mathbb{Z}^n\times\mathbb{Z}^n} (1+|\mu|)^{-N}\|\phi_{\mu_1}\psi_{\mu_2}\|_{L^q_tL^r_x}, \tag{36}$$

$$\|Q_{i,j}(\phi,\psi)\|_{L^q_tL^r_x} \le C2^{-m} \sum_{\mu=(\mu_1,\mu_2)\in\mathbb{Z}^n\times\mathbb{Z}^n} (1+|\mu|)^{-N}\|\phi_{\mu_1}\psi_{\mu_2}\|_{L^q_tL^r_x} \tag{37}$$

where ϕ_{μ_1}, ψ_{μ_2} are waves satisfying $E(\phi_{\mu_1}) = E(\phi)$, $E(\psi_{\mu_2}) = E(\psi)$, and ϕ_{μ_1}, ψ_{μ_2} (or $\bar\psi_{\mu_2}$) are Fourier supported in Λ, Λ', respectively.

After decomposing $\phi_{0,\Gamma}, \psi_{k,\Gamma'}$ into disjoint waves ϕ_Λ, $\psi_{\Lambda'}$ as in section 3.2.2, using this lemma we can drop the additional weight. Then the last of argument in Section 3.2.2 works without modification.

Now we close this section with the proof of Lemma 3.4, 3.5. The proof is similar to that of Lemma 2.6. We shall be brief.

Proof of Lemma 3.4. To begin with, we write $Q(\phi,\psi)$ as

$$(2\pi)^{-2n}\iint e^{i(x(\xi+\eta)+t(|\xi|\pm|\eta|))}m(\xi,\eta)\widehat{\phi(0)}(\xi)\widehat{\psi(0)}(\eta)d\xi d\eta.$$

Here, $m(\xi,\eta) = \xi_j|\eta| \mp \eta_j|\xi|$, or $\xi_j\eta_i - \xi_i\eta_j$ and $\widehat{\phi(0)}, \widehat{\psi(0)}$ are supported in

$$S(2^{-l}) = \{\xi : |\xi| \sim 1, |\xi/|\xi| - v| \le 2^{-l}\},\ 2^kS(2^{-l}),$$

respectively. The $-$ sign in the integral stands for the case that $\bar\psi$ is Fourier supported in $\Lambda_k(2^{-l})$. Obviously, we may assume $k = 0$ by the change of $\eta \to 2^k\eta$.

Following the proof Lemma 2.6, we use translation and rescaling to transform the multiplier weight m to $2^{-l}\widetilde{m}$ so that $\widetilde{m}$ has uniformly bounded derivatives on the set of integration. Then, we can expand out $\widetilde{m}$ using Fourier series and we get the desired by reversing the change of the variables. Hence, it is enough to show that such transformation is possible for m.

First, we show it with $m = \xi_j|\eta| \mp \eta_j|\xi|$. We only consider $m = \xi_j|\eta| - \eta_j|\xi|$ because the other one can be handled similarly.

Let (c'', c_{n-1}, c_n) be the coordinates of ξ with respect to the (ordered) orthonormal basis $\{v_1, \dots, v_{n-1}, v\}$. Similarly, let (d'', d_{n-1}, d_n) be the coordinates of η with respect to the same basis. Then, we may assume

$$S(2^{-l}) = \{(c'', c_{n-1}, c_n) : c_n \sim 1, |(c'', c_{n-1})| \le 2^{-l}\}.$$

Without loss of generality we may also assume that ξ_j-axis is contained the two plane spanned by v_{n-1}, v. Then, $\xi_j = \cos\theta c_{n-1} + \sin\theta c_n$ for some θ depending the angle between ξ_j axis and v. Hence in the new coordinates we can write

$$m(c,d) = (\cos\theta c_{n-1} + \sin\theta c_n)|(d'', d_{n-1}, d_n)| - (\cos\theta d_{n-1} + \sin\theta d_n)|(c'', c_{n-1}, c_n)|.$$

Rescaling $(c'', c_{n-1}) \to 2^{-l}(c'', c_{n-1})$ and $(d'', d_{n-1}) \to 2^{-l}(d'', d_{n-1})$ transforms $m(c,d)$ to $2^{-l}\widetilde{m}(c,d)$ where

$$\widetilde{m}(c,d) = \cos\theta(c_{n-1}d_n - c_nd_{n-1}) + O(2^{-l})$$

because $|(c'', c_{n-1}, c_n)| = c_n + |(c'', c_{n-1})|^2/2c_n + O(|(c'', c_{n-1})|^4)$. Note that the integration is now taken over $S(1)$. Obviously, $\widetilde{m}$ is a smooth function with bounded derivatives on $S(1)$.

Now we turn to the case $m(\xi, \eta) = \xi_j \eta_i - \xi_i \eta_j$. Here we may assume ξ_i axis is contained in the span of v_{n-2}, v_{n-1}, v. Hence, $\xi_i = \alpha c_{n-2} + \beta c_{n-1} + \gamma c_n$ with $|(\alpha, \beta, \gamma)| = 1$. Therefore $m(c, d)$ is equal to

$$(\alpha c_{n-2}+\beta c_{n-1}+\gamma c_n)(\cos\theta d_{n-1}+\sin\theta d_n)-(\alpha d_{n-2}+\beta d_{n-1}+\gamma d_n)(\cos\theta c_{n-1}+\sin\theta c_n).$$

After rescaling $(c'', c_{n-1}) \to 2^{-l}(c'', c_{n-1})$ and $(d'', d_{n-1}) \to 2^{-l}(d'', d_{n-1})$, $m(c, d)$ is changed to $2^{-l}\widetilde{m}$ and the integration is taken over $S(1)$ where $\widetilde{m}$ has bounded derivatives. □

Proof of Lemma 3.5. We follow the same argument in the above proof. The first (36) is obvious because the associated multiplier $\xi_j|\eta| \pm \eta_j|\xi|$ smooth if $|\xi| \sim 1, |\eta| \sim 1$. For (37), after translation $\xi \to \xi - v$, $\eta \to \eta + v$ and rescaling $(\xi, \eta) \to 2^{-m}(\xi, \eta)$, the associated multiplier is given by

$$\mu(\xi, \eta) = \det \begin{pmatrix} 2^{-m}\xi_i - a & 2^{-m}\xi_j - b \\ 2^{-m}\eta_i + a & 2^{-m}\eta_j + b \end{pmatrix}$$

for some a, b, $|a|, |b| \le C$. Hence it is obvious that $2^m \mu(\xi, \eta)$ is a smooth function with bounded derivatives in $B(0, C)$ where the integration is taken. □

4. Proofs of the bilinear extension estimates

As it was already seen in the proof of Proposition 3.3, the proofs of Theorem 2.1 and 2.2 are based on the so called *induction on scale* argument which was used to obtain sharp bilinear restriction estimates for the cone and the paraboloid [22, 15, 16]. We also use the crucial L^2 estimates in [15, 16] to get various mixed norm generalizations. After obtaining almost sharp local estimates on a cube of side length R, we use Lemma 4.4 to get estimates on the whole space.

4.1. **Proof of Theorem 2.1.** The proof is actually an adaptation of Tao's proof of sharp bilinear restriction for the cone [15]. The argument here is easier since we are not trying to obtain the endpoint estimates. Various mixed norm estimates are obtained using the basic estimates in Proposition 4.1. We begin by quoting several notions and a proposition we need here.

Let S be a finite set. A function $\Phi : \mathbb{R}^{n+1} \to \ell_2(S)$ is called a red wave with frequency 2^k if it takes values in $\ell_2(S)$ and its space-time Fourier transform is supported in the set

$$\Gamma_k^{red} = \{(\xi, |\xi|) : \angle(\xi, e_1) \le \pi/8, |\xi| \sim 2^k\}.$$

A function $\Psi : \mathbb{R}^{n+1} \to \ell_2(S)$ is called a blue wave of frequency 2^k if its space-time Fourier transform is supported in the set

$$\Gamma_k^{blue} = \{(\xi, -|\xi|) : \angle(\xi, e_1) \le \pi/8, |\xi| \sim 2^k\}.$$

Let $R \gg 2^k$ and $Q \subset \mathbb{R}^{n+1}$ be the cube of side length R centered at the origin. For each $0 \le j$, partition dyadically Q into cubes of side length $2^{-j}R$ and denote by $\mathcal{Q}_j(Q)$ the collection of these cubes. Then for $0 < c \ll 1$ set

$$I^{c,C_0}(Q) = \bigcup_{q \in \mathcal{Q}_{C_0}(Q)} (1-c)q.$$

Here $(1-c)q$ is the cube having the same center as q and side length of $(1-c)$ times as long as that of q. A red (or blue) wave table ϕ on Q with depth j is defined to be any red (or blue) wave having the form

$$\phi =: (\phi^{(q)})_{q \in \mathcal{Q}_j(Q)}.$$

If ϕ is a wave table of depth j over Q, then for $1 \le l \le j$ and $q \in \mathcal{Q}_l(Q)$, we set $\phi^{(q)} = (\phi^{(q')})_{q' \in \mathcal{Q}_j(Q), q' \subset q}$. For $0 \le l \le j$, we also define the l-quilt $[\phi]_l$ is defined by

$$[\phi]_l = \sum_{q \in \mathcal{Q}_l(Q)} \chi_q |\phi^{(q)}|.$$

Then the pointwise estimate

$$[\phi]_j \le [\phi]_{j-1} \le \cdots \le [\phi]_1 \le |\phi| \chi_Q \tag{38}$$

follows. For a red wave Φ of frequency 2^k, the margin of Φ is defined by

$$\mathrm{margin}(\Phi) := 2^{-k} \mathrm{dist}(\mathrm{supp}\widehat{\Phi}, \partial \Gamma_k^{red}),$$

and the margin of a blue wave is similarly defined.

Proposition 4.1 (Proposition 15.1 p.245, [15]). *Let m, m' be integers and let $R \gg 2^{-m}$, $2^{m-2m'}$, $0 < c < 2^{-C_0}$. Let Φ be a red wave of frequency 2^m with margin $(\Phi) \ge (2^m R)^{-1/2}$ and Ψ be a blue wave of frequency $2^{m'}$. Then for a sufficiently large number C_0 depending only on n, there is a red wave having values in $l^2(\mathcal{Q}_{C_0}(Q))$*

$$\widetilde{\Phi} = \widetilde{\Phi}_c(\Phi, \Psi; \mathcal{Q}_{C_0}(Q)) = (\widetilde{\Phi}_c(\Phi, \Psi)^{(q)})_{q \in \mathcal{Q}_{C_0}(Q)}$$

of frequency 2^m with margin$(\widetilde{\Phi}) \ge$ margin$(\Phi) - C(2^m R)^{-1/2}$ such that

$$E(\widetilde{\Phi}) \le (1 + Cc) E(\Phi), \tag{39}$$

$$\|(|\Phi| - [\widetilde{\Phi}]_{C_0})\Psi\|_{L^2(I^{c,C_0}(Q))} \le c^{-C} (2^{-m}R)^{-(n-1)/4} E(\Phi)^{1/2} E(\Psi)^{1/2} \tag{40}$$

hold with C independent of c.

Suppose that ϕ and ψ are free waves Fourier supported in subsets of the cone with height 1,2^k and $O(1)$-angular separation. By a finite decomposition, a mild Lorentz transformation, we may assume that ϕ and ψ are supported in the set Γ_0^{red} and $(-\Gamma_k^{blue})$, respectively. Since $\|\phi\psi\|_{L_t^q L_x^r} = \|\phi\bar{\psi}\|_{L_t^q L_x^r}$, taking conjugate and changing variables by reflection, we may assume that ϕ and ψ are supported in the sets Γ_0^{red} and Γ_k^{blue}, respectively and they have margin $\ge 1/100$. And we assume

$$E(\phi), E(\psi) \le 1.$$

To show Theorem 2.1 by interpolation and using Lemma 4.4 below, we need to show that for any $\epsilon > 0$ and for any Q cube of sidelenth $R \ge C2^k$,

$$\|\phi\psi\|_{L_t^q L_x^r(Q)} \le C R^\epsilon 2^{k(1/q - 1/2 + \epsilon)}. \tag{41}$$

Recall that

$$\|\phi\psi\|_2 \le CE(\phi)^{1/2}E(\psi)^{1/2} \tag{42}$$

provided ψ is a blue wave of frequency 2^k and ϕ is a red wave of frequency of 1. Therefore, it is enough to obtain (41) for $(q,r) = (2, \frac{n+1}{n})$, $(1,2)$, $(1, \frac{n+1}{n-1})$ when $n \ge 3$ and for $(q,r) = (\frac{4}{3}, 2)$, $(2, \frac{3}{2})$ when $n = 2$. First we show it when $n \ge 3$.

4.1.1. *Proof* (41) *for* $(q,r) = (2, \frac{n+1}{n})$. For $0 \le j \le k$ which are multiple of C_0, let ϕ_j be the sequence of wave tables of depth j on Q defined which are recursively defined by

$$\phi_0^{(Q)} = \phi,$$
$$\phi_{j+C_0}^{(q)} = \Phi_{c2^{-(k-j)/N}}(\phi_j^{(q)}, \psi; \mathcal{Q}_{C_0}(q)) \text{ for all } q \in \mathcal{Q}_j(Q)$$

using Proposition 4.1. (Here k is also assumed to be a multiple of constant C_0; this can be achieved by a simple re-scaling.) Note that the wave ϕ_{j+C_0} has value in $l^2(\mathcal{Q}_{j+C_0}(Q))$. By induction, $\mathrm{margin}(\phi_j) \ge 1/100 - C(2^{-j}R)^{-1/2}$. So the margin condition is satisfied for each j. Then from (39) we have

$$E(\phi_{j+C_0}) \le (1 + Cc2^{-(k-j)/N})E(\phi_j). \tag{43}$$

Hence $E(\phi_j) \le C$ for all $j \le k$. From (40) we have for $q \in \mathcal{Q}_j(Q)$

$$\begin{aligned} &\|([\phi_j]_j - [\phi_{j+C_0}]_{j+C_0})\psi\|_{L^2(I^{C,c2^{-(k-j)/N}}(q))} \\ &\qquad \le c^{-C}2^{C(k-j)/N}(2^{-j}R)^{-\frac{n-1}{4}}E(\phi_j^{(q)})^{1/2} \end{aligned} \tag{44}$$

applying Proposition 4.1 with $\Phi = \phi_j^{(q)}$, $\Psi = \psi$ and $Q = q$.

Now, let us set

$$X(Q) = \bigcap_{j=C_0}^{k} I^{c2^{-(k-j)/N},j}(Q).$$

Since $\|[\phi_j]_j\|_{L^2(X(Q))} \le 2^{CC_0}2^{-j/2}R^{1/2}$ by the trace lemma and (43), obviously

$$\|([\phi_j]_j - [\phi_{j+C_0}]_{j+C_0})\psi\|_{L_t^2L_x^1(X(Q))} \le 2^{CC_0}2^{-j/2}R^{1/2}. \tag{45}$$

Taking squares and summing (44) over $q \in \mathcal{Q}_j(Q)$ gives

$$\|([\phi_j]_j - [\phi_{j+C_0}]_{j+C_0})\psi\|_{L^2(X(Q))} \le c^{-C}2^{C(k-j)/N}(2^{-j}R)^{-\frac{n-1}{4}}. \tag{46}$$

(This is the estimate (79) in [15].) Then interpolation between (45) and (46) gives

$$\|([\phi_j]_j - [\phi_{j+C_0}]_{j+C_0})\psi\|_{L_t^2L_x^{\frac{n+1}{n}}(X(Q))} \le C2^{C(k-j)/N}. \tag{47}$$

Applying Proposition 4.1 with $\Phi = \psi$ and $\Psi = \phi_k$, we get $\widetilde{\Psi} = \Phi_c(\psi, \phi_k; \mathcal{Q}_{C_0}(Q))$ which satisfies

$$E(\widetilde{\Psi}) \le (1 + Cc)E(\Psi), \tag{48}$$

$$\|[\phi_k]_k(|\psi| - [\widetilde{\Psi}]_{C_0})\|_{L^2(X(Q))} \le c^{-C}(2^{-k}R)^{-\frac{n-1}{4}}. \tag{49}$$

Again by trace lemma, (43) and (48) it is easy to see

$$\|[\phi_k]_k(|\psi| - [\widetilde{\Psi}]_{C_0})\|_{L_t^2L_x^1(X(Q))} \le 2^{CC_0}2^{-k/2}R^{1/2}. \tag{50}$$

Interpolation between the above two estimates gives

$$\|[\phi_k]_k(|\psi| - [\widetilde{\Psi}]_{C_0})\|_{L^2_t L_x^{\frac{n+1}{n}}(X(Q))} \le c^{-C} 2^{CC_0}. \tag{51}$$

The following is a simple modification of the averaging lemma in [15] (Lemma 6.1).

Lemma 4.2. *Let $R > 0$ and $0 < 2^{-C_0}$, F be a smooth function. For $1 \le q, r < \infty$ and any R-cube $\widetilde{Q}$, there is a CR-cube $Q \subset C^2\widetilde{Q}$ such that*

$$\|F\|_{L^q_t L^r_x(\widetilde{Q})} \le (1 + C_1 c)\|F\|_{L^q_t L^r_x(X(Q))}$$

with C_1 depending on c, N, q, r, n.

Proof. From Lemma 6.1 in [15], for any positive $G \in L^1$ there is a CR-cube Q such that

$$\int_{\widetilde{Q}} G(x,t)dxdt \le (1 + C_1 c) \int_{X(Q)} G(x,t)dxdt.$$

We may assume $\|F\|_{L^q_t L^r_x(\widetilde{Q})} = 1$. Then there are functions g and h such that

$$1 = \int_{\widetilde{Q}} F(x,t)g(x,t)h(t)dxdt$$

with $\|g(\cdot,t)\|_{r'} = \|h\|_{q'} = 1$. Hence from the above inequality, there is a cube Q such that

$$1 \le (1 + C_1 c) \int_{X(Q)} |F(x,t)g(x,t)h(t)|dxdt \le (1 + C_1 c)\|F\|_{L^q_t L^r_x(X(Q))}.$$

This gives the required inequality by duality. □

Let $(Blue)_k$ ($(Red)_k$ resp.) be the set of blue (red resp.) waves having support in Γ_k^{blue} (Γ_k^{red} resp.) with energy ≤ 1. For a cube R-cube Q, let us define

$$A_{q,r}(R) = \inf\{C : \|\phi\psi\|_{L^q_t L^r_x(Q)} \le C,\ \phi \in (Blue)_0,\ \psi \in (Red)_k\}.$$

By translation invariance, it is easy to see that $A_{q,r}$ does not depend on a particular choice of Q.

Let $\widetilde{Q}$ be a R-cube and Q be such a CR-cube as in Lemma 4.2. Since Q is now a CR cube, we have to replace R by CR in (43)-(51) but the inequalities remain valid only with different constant C. Then, telescoping summation of (47) and using (51) gives

$$\|\phi\psi\|_{L^2_t L_x^{\frac{n+1}{n}}(\widetilde{Q})} \le (1 + Cc)\|[\phi_k]_k[\widetilde{\Psi}]_{C_0}\|_{L^2_t L_x^{\frac{n+1}{n}}(X(Q))} + (1 + Cc)(2^{CC_0} + 2^{Ck/N}). \tag{52}$$

By triangle inequality and (38) we see

$$\|[\phi_k]_k[\widetilde{\Psi}]_{C_0}\|_{L^q_t L^r_x(X(Q))} \le \|[\Phi]_{C_0}[\widetilde{\Psi}]_{C_0}\|_{L^q_t L^r_x(X(Q))} \le \sum_{q \in \mathcal{Q}_{C_0}} \|\Phi^{(q)}\Psi^{(q)}\|_{L^q_t L^r_x(q)}.$$

Then the above, (48) and (43) give

$$A_{2,\frac{n+1}{n}}(R) \le (1 + Cc)A_{2,\frac{n+1}{n}}(CR/2^{C_0}) + (1 + Cc)(2^{CC_0} + 2^{Ck/N}).$$

We can iterate this as long as $R \gg 2^k$, hence we obtain

$$A_{2,\frac{n+1}{n}}(R) \le (1+Cc)^{C\log R} A_{2,\frac{n+1}{n}}(C2^k) + (2^{CC_0} + 2^{Ck/N})(1+Cc)^{C\log R}.$$

Lemma 4.3. *Let $n \ge 2$. Then, for $1 \le q, r \le 2$, $2/q \le (n+1)(1-1/r)$ and $\epsilon > 0$, $A_{q,r}(C2^k) \le C2^{(1/q-1/2)k}$.*

Proof. First we handle the case $n \ge 3$. It is needed to show the cases $(q,r) = (2,2), (1,2), (2, \frac{n+1}{n}), (1, \frac{n+1}{n-1})$. The case $(2,2)$ is contained in (42). Then, the case $(1,2)$ follows from Hölder's inequality in t.

Now we consider the case $(q,r) = (2, \frac{n+1}{n})$. Let $R = C2^k$ for some large $C > 0$ and let $\widetilde{Q}$ and Q be the ones as in (52). Repeating the previous argument on Q, we consider wave tables ϕ_k, $\widetilde{\Psi}$. From the trace lemma and (43) we see that $\|[\phi_k]_k[\widetilde{\Psi}]_{C_0}\|_{L^2_t L^1_x(X(Q))} \le C$. Then by Hölder's inequality $\|[\phi_k]_k[\widetilde{\Psi}]_{C_0}\|_{L^1_t L^1_x(X(Q))} \le C2^{k/2}$. From the above inequalities (including the estimates for $(q,r) = (2,2), (1,2)$) we see that for $1 \le q, r \le 2$,

$$\|[\phi_k]_k[\widetilde{\Psi}]_{C_0}\|_{L^q_t L^r_x(X(Q))} \le C2^{(1/q-1/2)k}.$$

Then by (52) we see $A_{2,\frac{n+1}{n}}(C2^k) \le C2^{k\frac{n-1}{2(n+1)}}$ by choosing sufficiently large N. And the case $(q,r) = (1, \frac{n+1}{n-1})$ can be similarly handed by using inequality (59) below.

When $n = 2$, it is enough to show the cases $(q,r) = (2, \frac{3}{2}), (\frac{4}{3}, 2)$. The case $(q,r) = (2, \frac{3}{2})$ is already obtained in the above reasoning. The estimate for $(q,r) = (\frac{4}{3}, 2)$ can be shown using (61) by the same argument. ☐

Using Lemma 4.3, we get

$$A_{2,\frac{n+1}{n}}(R) \le (C2^{k\frac{n-1}{2(n+1)}} + 2^{CC_0} + 2^{Ck/N})(1+Cc)^{C\log R}.$$

Since $(1+Cc)^{C\log R} \le R^{C^2 c}$, by choosing $c = \epsilon/C^2$ and $N = (C\epsilon)^{-1}$ we get (41) for $(q,r) = (2, \frac{n+1}{n})$.

4.1.2. *Proof* (41) *for* $(q,r) = (1, \frac{n+1}{n-1})$. By Hölder's inequality and (46), we can get

$$\|([\phi_j]_j - [\phi_{j+C_0}]_{j+C_0})\psi\|_{L^1_t L^2_x(X(Q))} \le C2^{C(k-j)/N} 2^{j\frac{n-1}{4}} R^{-\frac{n-3}{4}} \tag{53}$$

and from (45) and Hölder's inequality it follows that

$$\|([\phi_j]_j - [\phi_{j+C_0}]_{j+C_0})\psi\|_{L^1_{t,x}(X(Q))} \le 2^{CC_0} 2^{-j/2} R. \tag{54}$$

Interpolation between these two estimates gives

$$\|([\phi_j]_j - [\phi_{j+C_0}]_{j+C_0})\psi\|_{L^1_t L^{\frac{n+1}{n-1}}_x(X(Q))} \le C2^{C(k-j)/N} 2^{j/2}. \tag{55}$$

Similarly from (49), (50) and Hölder's inequality we have

$$\|[\phi_k]_k(|\psi| - [\widetilde{\Psi}]_{C_0})\|_{L^1_t L^2_x(X(Q))} \le C2^{k\frac{n-1}{4}} R^{-\frac{n-3}{4}}, \tag{56}$$

$$\|[\phi_k]_k(|\psi| - [\widetilde{\Psi}]_{C_0})\|_{L^1_t L^1_x(X(Q))} \le 2^{CC_0} 2^{-k/2} R. \tag{57}$$

Interpolation gives

$$\|[\phi_k]_k(|\psi| - [\widetilde{\Psi}]_{C_0})\|_{L_t^1 L_x^{\frac{n+1}{n-1}}(X(Q))} \le 2^{CC_0} 2^{k/2} \tag{58}$$

Telescoping summation of (55) and using (58), we have

$$\|\phi\psi\|_{L_t^1 L_x^{\frac{n+1}{n-1}}(Q)} \le (1+Cc)\|[\phi_k]_k[\widetilde{\Psi}]_{C_0}\|_{L_t^1 L_x^{\frac{n+1}{n-1}}(X(Q))} + 2^{CC_0}2^{k/2}. \tag{59}$$

Hence following the same argument as before, we obtain

$$A_{1,\frac{n+1}{n-1}}(R) \le (1+Cc)A_{1,\frac{n+1}{n-1}}(CR/2^{C_0}) + C2^{k/2}.$$

Using this iteratively for $R \ge C2^k$, we see

$$A_{1,\frac{n+1}{n-1}}(R) \le (1+Cc)^{C\log R}A_{1,\frac{n+1}{n-1}}(C2^k) + (1+Cc)^{C\log R}2^{CC_0}2^{k/2}$$

Finally using Lemma 4.3 and choosing suitable c, we get the required estimate (41) for $(q,r) = (1, \frac{n+1}{n-1})$.

4.1.3. *Proof* (41) *for* $(q,r) = (1,2)$. By telescoping summation of (53) and using (56) it follows that

$$\|\phi\psi\|_{L_t^1 L_x^2(X(Q))} \le C\|[\Phi]_k[\widetilde{\Psi}]_{C_0}\|_{L_t^1 L_x^2(X(Q))} + 2^{CC_0}2^{k\frac{n-1}{4}}R^{-\frac{n-3}{4}}. \tag{60}$$

It gives

$$A_{1,2}(R) \le (1+Cc)A_{1,2}(CR/2^{C_0}) + 2^{CC_0}2^{k\frac{n-1}{4}}R^{-\frac{n-3}{4}}.$$

Using this iteratively, we obtain

$$A_{1,2}(R) \le (1+Cc)^{C\log R}A_{1,2}(C2^k) + (1+Cc)^{C\log R}2^{CC_0}2^{k/2}.$$

Hence using Lemma 4.3 and choosing suitable c we get the required estimate.

When $n = 2$, the estimate (41) for $(q,r) = (2, 3/2)$ is already obtained in the proof of the case $(q,r) = (2, \frac{n+1}{n})$. For the remaining $(q,r) = (4/3, 2)$ observe

$$\|\phi\psi\|_{L_t^{\frac{4}{3}} L_x^2(X(Q))} \le C\|[\Phi]_k[\widetilde{\Psi}]_{C_0}\|_{L_t^{\frac{4}{3}} L_x^2(X(Q))} + 2^{CC_0}2^{k/4} \tag{61}$$

which follows from (46) and (49) using telescoping summation and Hölder's inequality. Then the required estimates can be obtained repeating the previous argument.

Lemma 4.4 (Globalization lemma)**.** *Let S_1 and S_2 be compact surfaces with boundary $S_i = \{(\xi, \phi_i(\xi)) \ : \ \xi \in U_i\}$ and the induced Lebesgue measures $d\sigma_i(\xi) = d\xi$, $i = 1, 2$, which satisfy $\|d\sigma_i\| \le M_i$, $\sigma_i(B(z,\rho)) \le C\rho^{n-1}$ for any $z, \rho > 0$ and*

$$|\widehat{d\sigma_i}(x,t)| \le M_i(1 + |x| + |t|)^{-\sigma}$$

for some $M_i \ge 1$ and $0 < \sigma$. Suppose that for some $\frac{2+2\sigma}{\sigma} \ge q_0, r_0 \ge 1$ and some $0 < \epsilon \ll \sigma$,

$$\|\prod_{i=1}^{2}\widehat{f_i d\sigma_i}\|_{L_t^{q_0}L_x^{r_0}(Q)} \le R^\epsilon M \prod_{i=1}^{2}\|f_i\|_{L^2(d\sigma_i)} \tag{62}$$

for all cube Q with side length R. Set

$$\frac{1}{q_1}=\frac{1}{q_0}-\frac{2\epsilon}{2\epsilon+\sigma}\Big(\frac{1}{q_0}-\frac{\sigma}{2(\sigma+1)}\Big),\quad \frac{1}{r_1}=\frac{1}{r_0}-\frac{2\epsilon}{2\epsilon+\sigma}\Big(\frac{1}{r_0}-\frac{\sigma}{2(\sigma+1)}\Big).$$

Then, for all q, satisfying $q_1<q$

$$\|\prod_{i=1}^{2}\widehat{f_id\sigma_i}\|_{L^q_tL^{r_1}_x}\le CM^{1-\frac{\epsilon}{\sigma}}(\max(M_1,M_2))^{a\epsilon+(1-\frac{q_1}{q})(1-\frac{1}{r_1})}\prod_{i=1}^{2}\|f_i\|_{L^2(d\sigma_i)} \tag{63}$$

with some $a>0$ depending on σ.

This is a mixed norm generalization of Lemma 2.4 in [17]. We give a proof in the Appendix. This lemma can be further strengthened in various ways by taking account of particular situations.

Remark 4.5. *The restriction $\frac{2+2\sigma}{\sigma}\ge q_0,r_0\ge 1$ is not a serious one because we can choose any $0<\sigma'<\sigma$ to extend the range. The range of q,r_1 can be extended by interpolation with the obvious estimates $\|\prod_{i=1}^2\widehat{f_id\sigma_i}\|_{L^\infty_tL^\infty_x}\le (M_1M_2)^{1/2}\prod_{i=1}^2\|f_i\|_{L^2(d\sigma_i)}$, $\|\prod_{i=1}^2\widehat{f_id\sigma_i}\|_{L^\infty_tL^1_x}\le\prod_{i=1}^2\|f_i\|_{L^2(d\sigma_i)}$. Hence if we have (62) for all $\epsilon>0$, then for any $\delta>0$ we have*

$$\|\prod_{i=1}^{2}\widehat{f_id\sigma_i}\|_{L^q_tL^r_x}\le C\max(M,M^{1-\delta})(\max(M_1,M_2))^{\delta}\prod_{i=1}^{2}\|f_i\|_{L^2(d\sigma_i)}$$

provided $r/q<r_0/q_0$, $q(1-1/r)>q_0(1-1/r_0)$ and $(1/r,1/q)$ is close enough to $(1/r_0,1/q_0)$.

4.2. **Proof of Proposition 2.4.** Taking conjugation it is enough to consider the case $(++)$.

For f with Fourier transform supported in $\{\xi:|\xi'|\le 100\xi_n, 1/2\le\xi_n\}$, we define

$$U_lf(x,t)=\int e^{i(x'\xi'+x_n\xi_n+t\xi_n2^{2l}\theta(2^{-l}\xi'/\xi_n))}\widehat{f}(\xi)d\xi$$

where θ is given by (17). Making the change of variables $x_n\to x_n-t$, for the proof of Proposition 2.4 it is enough to show that for $1<q,r\le 2$ satisfying $1/q<\min(1,\frac{n+1}{4})$, $1/q<\frac{n+1}{2}(1-\frac{1}{r})$,

$$\|U_0fU_0g\|_{L^q_tL^r_x}\le C2^{k(\frac{1}{q}-\frac{1}{2}+\epsilon)}2^{l(\frac{2}{q}-(n-1)(1-\frac{1}{r}))}\|f\|_2\|g\|_2$$

provided $\widehat{f}$, $\widehat{g}$ are supported in

$$\Theta_l=\{(\xi_1,\xi_2,\xi''):\xi_n\sim 1,\ \xi_{n-1}\sim 2^{-l},\ |\xi''|\ll 2^{-l}\},$$
$$2^k\Theta_l'=2^k\{(\xi_1,\xi_2,\xi''):\xi_n\sim 1,\ \xi_{n-1}\sim -2^{-l},\ |\xi''|\ll 2^{-l}\},$$

respectively. Then by re-scaling $(\xi',\xi_n)\to(2^{-l}\xi',\xi_n)$, $(x,t)\to(2^lx',x_n,2^{2l}t)$, we are reduced to showing that if the Fourier transforms of f,g are supported in Θ_0, $2^k\Theta_0'$, respectively, then for $1<q,r\le 2$ satisfying $1/q<\min(1,\frac{n+1}{4})$, $1/q<\frac{n+1}{2}(1-\frac{1}{r})$,

$$\|U_lfU_lg\|_{L^q_tL^r_x}\le C2^{k(\frac{1}{q}-\frac{1}{2}+\epsilon)}\|f\|_2\|g\|_2$$

with C independent of l. Since $2^{2l}\theta(2^{-l}\eta)$ converges to $|\eta|^2/2$ as $l \to \infty$, the conic surface given as

$$(\xi', \xi_n) \to (\xi', \xi_n, \xi_n 2^{2l}\theta(2^{-l}\xi'/\xi_n))$$

is not much different from the cone (given by $(\xi', \xi_n) \to (\xi', \xi_n, |\xi'|^2/\xi_n)$). In fact, retracing the proof Proposition 4.1 in [15] one can see that it is valid for U_l uniformly in l. Then the crucial estimates (46) and (49) hold uniformly and so does the trace lemma for U_l. Hence we can repeat the argument used for the proof of Theorem 2.1 to obtain the required uniform estimates for U_l.

4.3. **Proof of Theorem 2.2.** Note that the critical line $2/q = n(1 - 1/r)$ is the border line for the bilinear restriction for the paraboloid in $\mathbb{R}^n$. So, from the expression (18) one might be tempted to apply directly the bilinear estimates for the paraboloid freezing ξ_n variables but it does not seem to work because we still have to integrate in x_n. We again make use of the induction on scale argument and the basic L^2 estimates used to prove the sharp bilinear restriction for the paraboloid ((23) in [16]).

We first prove Theorem 2.2 for the case $n \ge 4$ and later the cases $n = 2, n = 3$.

Let $Q_R \subset \mathbb{R}^n$ be the cube of side length R centered at the origin and $I_R = [-R/2, R/2]$. Let $Q'_R \subset \mathbb{R}^{n-1}$ be the cube centered at the origin with side length R so that $Q_R = Q'_R \times I_R$. As before it is enough to show that for $2/q \le n(1 - 1/r)$ and $\epsilon > 0$,

$$\|\widehat{f_1 d\sigma_1}\widehat{f_2 d\sigma_2}\|_{L^q_t(I_R)L^r_{x',x_n}(Q'_R\times\mathbb{R})} \le Ca^{1-1/r}R^\epsilon\|f_1\|_2\|f_2\|_2. \tag{64}$$

The R^ϵ can be removed by using Lemma 4.4.

By Cauchy–Schwarz's inequality and Plancherel's theorem, $\|\widehat{f_1 d\sigma_1}\widehat{f_2 d\sigma_2}\|_{L^\infty_t L^1_x} \le C\|f_1\|_2\|f_2\|_2$. Hence, to prove (64) it is enough to show it for the case $(q, r) = (1, n/(n-2))$. Then, by interpolation with the case $(\infty, 1)$ we get (64) for $2/q = n(1 - 1/r)$. Finally, since the Fourier transform of $(\widehat{f_1 d\sigma_1 f_2 d\sigma_2})(\cdot, t)$ is supported in a slab of thickness a, we get all estimate (19) for $2/q < n(1 - 1/r)$ by Bernstein's inequality and interpolation.

Let us set

$$B_{q,r}(R) = \inf\{C : \|\widehat{fd\sigma}\widehat{gd\sigma}\|_{L^q_t(I_R)L^r_{x',x_n}(Q'_R\times\mathbb{R})} \le C\|f\|_2\|g\|_2\}$$

We will show

$$B_{1,\frac{n}{n-2}}(R) \le C(R^\epsilon B_{1,\frac{n}{n-2}}(R^{1-\delta}) + a^{2/n}R^{c\delta}). \tag{65}$$

From Bernstein's inequality it is easy to see $B_{q,r}(1) \le Ca^{1-1/r}$, $1 \le q, r \le \infty$. Hence iterating estimate (65) with $\epsilon = \delta^2$, we get for any $\delta > 0$

$$B_{1,\frac{n}{n-2}}(R) \le Ca^{2/n}R^{C\delta}.$$

This proves the required estimate.

Proof of (65). For $i = 1, 2$ and $\rho \in [b, b + a]$, we denote by S_i^ρ the surface given by

$$S_i^\rho = \{(\eta, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R} : \tau = \rho\theta(\eta/\rho), (\eta, \rho) \in S_i\}.$$

Also let $d\sigma_i^\rho(\eta, \rho) = d\eta$ be the induced Lebesgue measure on S_i^ρ and let $f^\rho(\eta) = f(\eta, \rho)$. Then we may write

$$\widehat{f_i d\sigma_i}(x, t) = \int_b^{b+a} \widehat{f_i^s d\sigma_i^s}(x', t) e^{isx_n} ds. \tag{66}$$

For fixed s, we decompose $\widehat{f_i^s d\sigma_i^s}$ (in (66)) into wave packets on the R-cube $Q_R = Q'_R \times I_R$. They have one to one correspondence with a collection of tubes of dimension $(R^{1/2})^{n-1} \times R$. Hence, we can write for $(x', t) \in Q'_R$

$$\widehat{f_i^s d\sigma_i^s}(x', t) = \sum_{\tau \in \mathcal{T}_i^s} \widehat{f_{i,\tau}^s d\sigma_i^s}(x', t)$$

where $\mathcal{T}_1^s$, $\mathcal{T}_2^s$, are the collections of tubes associated to the packet decomposition for the extension operators $\widehat{f_1^s d\sigma_1^s}$, $\widehat{f_2^s d\sigma_2^s}$, respectively. Then, for each $s \in [b, b + a]$ and any subset $A \subset \mathcal{T}_i^s$ (see Lemma 4.1 in [16]),

$$(\sum_{\tau \in A} \|f_{i,\tau}^s\|_2^2)^{1/2} \le C\|f_i^s\|_2, \tag{67}$$

$$\|\sum_{\tau \in A} \widehat{f_{i,\tau}^s d\sigma_i^s}(\cdot, t)\|_2 \le C(\sum_{\tau \in A} \|f_{i,\tau}^s\|_2^2)^{1/2}. \tag{68}$$

Let $\{B\}$ be a collection of $R^{1-\delta}$ cubes which partition Q_R. Then, there are relations $\sim_1^\rho$, $\sim_2^\rho$ between B and $\tau \in \mathcal{T}_i^\rho$ such that for any s, $t \in [b, b+a]$, $0 < \delta \ll 1$,

$$\sum_B \|\sum_{\tau \sim_i^s B} f_{i,\tau}^s\|_2^2 \le CR^\epsilon \|f_i^s\|_2^2, \tag{69}$$

$$\|\sum_{\tau \not\sim_1^s B \text{ or } \tau' \not\sim_2^t B} \widehat{f_{1,\tau}^s d\sigma_1^s} \widehat{f_{2,\tau'}^t d\sigma_2^t}\|_{L^2(B)} \le CR^{-(n-2)/4 + c\delta} \|f_1^s\|_2 \|f_2^t\|_2 \tag{70}$$

with c independent of δ. This is a slight modification of the inequality (23) in [16]. It is not hard to see that the constants C in (67), (68), (69) and (70) are independent of s, t because the surfaces S_1^s, S_2^t are uniformly elliptic in $s, t \in [b, b + a]$ and the separation condition between S_1^s, S_2^t is also satisfied uniformly.

For each B, we break

$$\widehat{f_i d\sigma_i}(x, t) = \int_b^{b+a} \sum_{\tau \sim_i^s B} \widehat{f_{i,\tau}^s d\sigma_s^i}(x', t) e^{isx_n} ds + \int_b^{b+a} \sum_{\tau \not\sim_i^s B} \widehat{f_{i,\tau}^s d\sigma_s^i}(x', t) e^{isx_n} ds$$

For simplicity let us set $F_{i,\tau}^s(x', t) = \widehat{f_{i,\tau}^s d\sigma_s^i}(x', t)$ and break

$$\widehat{f_1 d\sigma_1} \widehat{f_2 d\sigma_2}(x, t) = \iint (\sum_{\tau \sim_1^s B \text{ and } \tau' \sim_2^{s'} B} + \sum_{\tau \not\sim_1^s B \text{ or } \tau' \not\sim_2^{s'} B}) [F_{1,\tau}^s F_{2,\tau'}^{s'}](x', t) e^{i(s+s')x_n} ds ds'.$$

By triangle inequality

$$\|\widehat{f_1 d\sigma_1} \widehat{f_2 d\sigma_2}\|_{L_t^q(I_R) L_{x', x_n}^r(Q'_R \times \mathbb{R})} \le \sum_{B = B' \times I} \|\widehat{f_1 d\sigma_1} \widehat{f_2 d\sigma_2}\|_{L_t^q(I) L_{x', x_n}^r(B' \times \mathbb{R})}.$$

Using the above decomposition

$$\|\widehat{f_1 d\sigma_1 f_2 d\sigma_2}\|_{L^q_t(I_R)L^r_{x',x_n}(Q'_R\times\mathbb{R})} \le I + II$$

where

$$I = \sum_{B=B'\times I} \Big\| \iint \sum_{\tau\sim^s_1 B \text{ and } \tau'\sim^{s'}_2 B} [F^s_{1,\tau}F^{s'}_{2,\tau'}](x',t)e^{i(s+s')x_n}dsds'\Big\|_{L^q_t(I)L^r_{x',x_n}(B'\times\mathbb{R})},$$

$$II = \sum_{B=B'\times I} \Big\| \iint \sum_{\tau\not\sim^s_1 B \text{ or } \tau'\not\sim^{s'}_2 B} [F^s_{1,\tau}F^{s'}_{2,\tau'}](x',t)e^{i(s+s')x_n}dsds'\Big\|_{L^q_t(I)L^r_{x',x_n}(B'\times\mathbb{R})}.$$

For I, it is easy to see that

$$\begin{aligned}\Big\| \iint \sum_{\tau\sim^s_1 B \text{ and } \tau'\sim^{s'}_2 B} (\cdot)\Big\|_{L^q_t(I)L^r_{x',x_n}(B'\times\mathbb{R})}& \\ \le B_{q,r}(R^{1-\delta})\Big\|\sum_{\tau\sim^s_1 B} f^s_{1,\tau}\Big\|_{L^2_s(L^2)}&\Big\|\sum_{\tau\sim^s_2 B} f^s_{2,\tau}\Big\|_{L^2_s(L^2)}\end{aligned}$$

Then, using Cauchy–Schwarz's inequality and (69) we see that

$$I \le CR^{\epsilon}B_{q,r}(R^{1-\delta})\|f_1\|_2\|f_2\|_2.$$

Since the number of B is $\lesssim R^{c\delta}$, for (65) it is enough to show that for $B = B'\times I$

$$\Big\| \iint \sum_{\tau\not\sim^s_1 B \text{ or } \tau'\not\sim^{s'}_2 B} (\cdot)\Big\|_{L^1_t(I)L^{\frac{n}{n-2}}_{x',x_n}(B'\times\mathbb{R})} \le Ca^{2/n}R^{c\delta}\|f\|_2\|g\|_2 \tag{71}$$

Using (66) we apply Plancherel's theorem and Minkowski's inequality to see

$$\begin{aligned}&\Big\| \int_b^{b+a}\int_b^{b+a} \sum_{\tau\not\sim^s_1 B \text{ or } \tau'\not\sim^{s'}_2 B} [F^s_{1,\tau}F^{s'}_{2,\tau'}]e^{i(s+s')x_n}dsds'\Big\|_{L^2_{x',t}(B)L^2_{x_n}} \\ &\le C\int_b^{b+a} \Big\|\chi_{[b,b+a]}(s'-s)\Big(\sum_{\tau\not\sim^s_1 B \text{ or } \tau'\not\sim^{s'-s}_2 B} F^s_{1,\tau}F^{s'-s}_{2,\tau}\Big)\Big\|_{L^2_{x',t}(B)L^2_{s'}}ds \\ &\le CR^{-(n-2)/4+C\delta}\int_b^{b+a}\|f^s_1\|_2\Big(\int_b^{b+a}\|f^{s-s'}_2\|_2^2ds'\Big)^{1/2}ds \\ &\le CR^{-(n-2)/4+C\delta}a^{1/2}\|f\|_2\|g\|_2.\end{aligned} \tag{72}$$

For the second inequality we used (70) taking x',t-integration first. From (72) and Hölder's inequality in t, we get

$$\Big\| \iint \sum_{\tau\not\sim^s_1 B \text{ or } \tau'\not\sim^{s'}_2 B} (\cdot)\Big\|_{L^1_t(I)L^2_{x',x_n}(B'\times\mathbb{R})} \le Ca^{1/2}R^{-(n-4)/4+c\delta}\|f\|_2\|g\|_2.$$

From Plancherel, (67) and (68) it is easy to see that

$$\Big\| \iint \sum_{\tau\not\sim^s_1 B \text{ or } \tau'\not\sim^{s'}_2 B} (\cdot)\Big\|_{L^1_t(I)L^1_{x',x_n}(B'\times\mathbb{R})} \le CR^{1-\delta}\|f_1\|_2\|f_2\|_2.$$

Interpolation between these two estimates gives (71). This completes the proof for the case $n \geq 4$. $\square$

Finally, we prove the cases $n = 2, 3$. When $n = 2$, as before it is enough to show $\|\widehat{f_1 d\sigma_1 f_2 d\sigma_2}\|_2 \leq C a^{1/2} \|f_1\|_2 \|f_2\|_2$. But this is easy to show because $\|d\sigma_1 * d\sigma_2\|_\infty \leq Ca$.

When $n = 3$, we need to show

$$B_{4/3,2}(R) \leq C(R^\epsilon B_{4/3,2}(R^{1-\delta}) + a^{1/2} R^{c\delta}).$$

This can be obtained by using the estimate

$$\| \iint \sum_{\tau \not\sim_1^s B \text{ or } \tau' \not\sim_2^{s'} B} (\cdot) \|_{L_t^{4/3}(I) L^2_{x', x_n}(B' \times \mathbb{R})} \leq C a^{1/2} R^{c\delta} \|f\|_2 \|g\|_2,$$

which follows from (72) by Hölder's inequality.

4.4. **Proof of Proposition 2.5.** For $l \gg 1$, we set

$$U_l f(x,t) = \int e^{i(x'\xi' + x_n \xi_n + t\xi_n 2^{2l} \theta(2^{-l}\xi'/\xi_n))} \widehat{f}(\xi) d\xi.$$

By conjugation, change of variable $x_n \to x_n - t$ and Plancherel's theorem, it is enough to show that for q, r as in Proposition 2.5,

$$\|U_0 f_1 U_0 f_2\|_{L_t^q L_x^r} \leq C 2^{l(\frac{2}{q} - (n-1)(1-\frac{1}{r}))} a^{1-1/r-\epsilon} \|f_1\|_2 \|f_2\|_2$$

if f_1, f_2 are supported in the sets S_1^l, S_2^l, respectively, where

$$S_i^l = \{\xi : b \leq \xi_n \leq b + a, |2^l \xi'/\xi_n + (-1)^i e_n'| \leq 1/2\}, \ i = 1, 2.$$

Then by rescaling $(\xi', \xi_n) \to (2^{-l}\xi', \xi_n)$, $\quad (x,t) \to (2^l x', x_n, 2^{2l} t)$, we are reduced to showing that

$$\|U_l f_1 U_l f_2\|_{L_t^q L_x^r} \leq C a^{1-1/r-\epsilon} \|f_1\|_2 \|f_2\|_2$$

if f_1, f_2 are supported in the sets S_1, S_2, respectively. However $2^{2l}\theta(2^{-l}\cdot)$ is uniformly elliptic in l. So, it is not hard to see the estimates (67), (68), (69) and (70) are valid with C, independent of l. Then retracing the proof of Theorem 2.2, one can obtain the required estimate.

5. The additional necessary conditions (13)-(15)

Here we derive the additional necessary conditions. Unlike [3] where the necessary conditions obtained considering pairs of waves supported in various sets, the additional conditions (13) and (15) are obtained by considering collections of waves, which make it possible to capture additional concentration.

We also discuss briefly the necessary conditions for (16) with the standard null forms $Q = Q_{0,j}$, or $Q_{i,j}$. Taking account of the additional multiplier weights, the necessity of the conditions (3)–(12) for (16) can be seen easily from the examples given in [3] with minor modifications if needed. We make remarks only about the new necessary conditions (13), (14) and (15).

In this section, the symbol $\widehat{\ }$ denotes the spatial Fourier transform. Let ϕ^+ stand for a wave which is Fourier supported in the forward light cone.

5.1. **Necessity of** (13)**.** Let $\psi(=\psi^+)$ be the wave defined from $\widehat{\psi(0)}(\xi)$ which is a smooth function supported in a small ball centered at $e_1 \in \mathbb{R}^n$. To construct ϕ_k, consider $\widehat{\phi}(\xi)$ a smooth bump function, supported on a small ball centered at e_2. Write

$$F(x,t) = \int \widehat{\phi}(\eta) e^{i(x\eta + t|\eta|)} d\eta.$$

Let $R \gg 1$ and define a wave ϕ_k by

$$\phi_k(x,t) = 2^{kn} \sum_{m=1}^{2^k/10} \omega_n F(2^k x, 2^k(t - R2^{-k}m))$$

where $\omega_n = \pm 1$. Since $|F(x,t)| \ge c > 0$ for $|(x,t)| < 1/10$ and $|F(x,t)| \le Ct^{-2}$ for $|x| < 1/10$ and $|t| > 10$, we see that if $|x| \le 2^{-k}/10$ and $t \in \cup_m (2^{-k}Rm, 2^{-k}Rm + 2^{-k}/10)$, then $|\phi_k(x,t)| \sim 2^{kn}$ provided R is sufficiently large. Therefore,

$$\|\phi_k \psi\|_{L^q_t L^r_x} \sim 2^{kn - k\frac{n}{r}}. \tag{73}$$

Note that $\widehat{\phi_k(0)}(\xi) = \sum_{m=1}^{2^k/10} \omega_n \widehat{\phi}(2^{-k}\xi) e^{-i2^{-k}Rm|\xi|}$. Since $D_0, D_+ \sim 2^k$ and $D_- \sim 1$ on the Fourier support of $\phi_k \psi$,

$$2^{k(\beta_0+\beta_+)} \|\phi_k\psi\|_{L^q_t L^r_x} \le C \|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-} (\phi_k \psi)\|_{L^q_t L^r_x}.$$

This can be shown by the same argument in proof of Lemma 2.6. Hence if (1) is true, using (2) we have

$$\|\phi_k \psi\|_{L^q_t L^r_x} \le C 2^{k(\beta_- - \alpha_2 - \frac{1}{q} + n(1-\frac{1}{r}))} \|\phi_k(0)\|_2 \|\psi(0)\|_2.$$

Note that (73) and the above does not depend on a particular choice of ω_n. Thus, by Khintchin's inequality

$$\begin{aligned} 2^{kn-k\frac{n}{r}} &\le C 2^{k(\beta_- - \alpha_2 - \frac{1}{q} + n(1-\frac{1}{r}))} \big\| \big(\sum |\widehat{\phi}(2^{-k}\cdot) e^{-i2^{-k}10m|\cdot|}|^2 \big)^{1/2} \big\|_2 \\ &\le C 2^{k(\beta_- - \alpha_2 - \frac{1}{q} + n(1-\frac{1}{r}))} 2^{\frac{k(n+1)}{2}}. \end{aligned}$$

This gives (13).

The necessity of (13) *for* (16). To obtain the condition (13), by symmetry it is enough to consider $Q_{0,1}$, $Q_{1,2}$. Using the same ϕ_k and ψ as above, it is enough to observe that

$$\begin{aligned} \|D_0^{\beta_0} D_+^{\beta_+-1/2} D_-^{\beta_- -1/2} Q_{0,1}(\phi_k,\psi)\|_{L^q_t L^r_x} &\gtrsim 2^{k(\beta_0+\beta_+-1/2)} 2^k \|\phi_k\psi\|_{L^q_t L^r_x}, \\ \|D_0^{\beta_0-1} D_+^{\beta_++1/2} D_-^{\beta_- -1/2} Q_{1,2}(\phi_k,\psi)\|_{L^q_t L^r_x} &\gtrsim 2^{k(\beta_0+\beta_+-1/2)} 2^k \|\phi_k\psi\|_{L^q_t L^r_x}. \end{aligned}$$

This is easy to see because $D_0, D_+ \sim 2^k$, $D_- \sim 1$, $Q_{0,1} \sim 2^k$ and $Q_{1,2} \sim 2^k$ on the Fourier supports of ϕ_k, ψ. Then the remaining details are straightforward.

5.2. **Necessity of** (14)**.** Let ψ $(= \psi^+)$ be the wave defined by $\widehat{\psi(0)}(\xi)$ being the characteristic function of the set

$$F = \{\xi : 1/2 \le \xi_n \le 1, ||\xi| + \xi_n - 2| \le 2^{-k}\} \cap \{\xi : \xi_1 \ge \frac{1}{10\sqrt{n}}\}.$$

This set has measure $\sim 2^{-k}$ and is obtained by scaling the set called F in Example 14.14 in [3]. Then

$$\|\psi(0)\|_2 \sim 2^{-k/2}$$

and $|\psi(x,t)| \ge 2^{-k}$ if $|x_n - t| \lesssim 1$, $|x'| \lesssim 1$ and $|t| \lesssim 2^k$. Here $A \lesssim B$ means $A \le CB$ for some constant $C > 0$.

Let $\phi_k (= \phi_k^+)$ be a wave defined by $\widehat{\phi_k(0)}$ being the characteristic function of the set $\{\xi : \xi_n \sim -2^k,\ |\xi'| \lesssim 1\}$. Then $\|\phi_k(0)\|_2 \sim 2^{k/2}$ and $|\phi_k(x,t)| \gtrsim 2^k$ if $|x_n - t| \lesssim 2^{-k}$, $|x'| \lesssim 1$ and $|t| \lesssim 2^k$ because $||\xi| + \xi_n| \lesssim 2^{-k}$ on the support of $\widehat{\phi_k(0)}$.

Note that $\phi_k\psi$ is Fourier supported in the set $\{(\xi,\tau) : |\xi| - \tau \sim 1, |\xi'| \lesssim 1, \xi_n \sim 2^k\}$. Hence, by a suitable affine transformation it is not hard to see

$$2^{k(\beta_0+\beta_+)}\|\phi_k\psi\|_{L_t^qL_x^r} \le C\|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}(\phi_k\psi)\|_{L_t^qL_x^r}.$$

Hence if (1) holds, then by (2)

$$\|\phi_k\psi\|_{L_t^qL_x^r} \le C2^{k(\beta_- - \alpha_2 - \frac{1}{q} + n(1-\frac{1}{r}))}\|\phi_k(0)\|_2\|\psi(0)\|_2. \tag{74}$$

On the other hand $\|\phi_k\psi\|_{L_t^qL_x^r} \ge C2^{k(\frac{1}{q}-\frac{1}{r})}$ from our choice of ψ, ϕ_k. Hence we get (14).

The necessity of (14) *for* (16)*.* As before it is enough to consider $Q_{0,1}$, $Q_{1,n}$. Using the same ϕ_k and ψ, it suffices to observe that

$$\|D_0^{\beta_0}D_+^{\beta_+-1/2}D_-^{\beta_--1/2}Q_{0,1}(\phi_k,\psi)\|_{L_t^qL_x^r} \gtrsim 2^{k(\beta_0+\beta_+-1/2)}2^k\|\phi_k\psi\|_{L_t^qL_x^r},$$
$$\|D_0^{\beta_0-1}D_+^{\beta_++1/2}D_-^{\beta_--1/2}Q_{1,n}(\phi_k,\psi)\|_{L_t^qL_x^r} \gtrsim 2^{k(\beta_0+\beta_+-1/2)}2^k\|\phi_k\psi\|_{L_t^qL_x^r}.$$

This is easy to see because $D_0, D_+ \sim 2^k$, $D_- \sim 1$, $|Q_{0,1}| \sim 2^k$ and $|Q_{1,n}| \sim 2^k$ on the Fourier supports of ϕ_k, ψ.

5.3. **Necessity of** (15)**.** For $m \ge 0$, let B, B' be balls in $\mathbb{R}^n$ given by

$$B = B(e_n, 2^{-m}), \quad B' = B(-e_n + 2^{3-m}e_{n-1}, 2^{-m})$$

and let $\phi_B (= \phi_B^+)$ and $\psi_{B'} (= \psi_{B'}^+)$ be waves with $\widehat{\phi_B(0)}$, $\widehat{\psi_{B'}(0)}$ supported in B, B', respectively. Since $D_-, D_+ \sim 1$ and $D_0 \sim 2^{-m}$ on the Fourier support of $\phi_B\psi_{B'}$,

$$2^{-\beta_0 m}\|(\phi_B\psi_{B'})\|_{L_t^qL_x^r} \le C\|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}(\phi_B\psi_{B'})\|_{L_t^qL_x^r}.$$

Hence, from (15) it is enough to show that there are ϕ_B and $\psi_{B'}$ such that

$$\|\phi_B\psi_{B'}\|_{L_t^qL_x^r} \ge C2^{m(2/q - n(1-/r) - 1/2)}\|\phi_B(0)\|_2\|\psi_{B'}(0)\|_2.$$

Let η be a wave so that $\widehat{\eta}(0)$ is a smooth function supported on B. Then $|\eta(x,t)| \sim 2^{-mn}$ if $|x'|, |x_n + t| \lesssim 2^m$ and $|x_n - t| \lesssim 2^{2m}$. For $R \gg 1$ we set

$$\phi_B = \sum_{|k| \le 2^m} \eta(x', x_n - Rk2^m, t).$$

And let $\psi_{B'}$ be the wave with $\widehat{\psi_{B'}}(0) = \chi_{B'}$. Then $|\psi_{B'}(x,t)| \sim 2^{-mn}$ if $|x'|, |x_n - t| \lesssim 2^m$ and $|x_n + t| \lesssim 2^{2m}$.

By routine computation it is easy to see that

$$\|\phi_B \psi_{B'}\|_{L^q_t L^r_x} \gtrsim 2^{m(n/r+2/q-2n)}, \ \|\phi_B(0)\|_2 \lesssim 2^{m(1/2-n/2)}$$

if R is large enough. Obviously $\|\psi_{B'}(0)\|_2 \sim 2^{-nm/2}$. It gives the required lower bound.

The necessity of (15) *for* (16). By symmetry it is enough to consider $Q_{0,n}$, $Q_{n-1,n}$. Using ϕ_B, $\psi_{B'}$ in the above, we need only to observe that

$$\|D_0^{\beta_0} D_+^{\beta_+ - 1/2} D_-^{\beta_- - 1/2} Q_{0,n}(\phi_B \psi_{B'})\|_{L^q_t L^r_x} \gtrsim 2^{-m\beta_0} \|\phi_B \psi_{B'}\|_{L^q_t L^r_x},$$
$$\|D_0^{\beta_0 - 1} D_+^{\beta_+ + 1/2} D_-^{\beta_- - 1/2} Q_{n-1,n}(\phi_B \psi_{B'})\|_{L^q_t L^r_x} \gtrsim 2^{-m\beta_0} \|\phi_B \psi_{B'}\|_{L^q_t L^r_x}.$$

It is easy to see since $D_-, D_+ \sim 1$, $D_0 \sim 2^{-m}$, $Q_{0,n} \sim 1$ and $Q_{n-1,n} \sim 2^{-m}$ on the Fourier supports of ϕ_B, $\psi_{B'}$.

Appendix: Proof of Globalization lemma

Set F to be any subset of

$$E = \left\{ (x,t) : \left| \Re \prod_{i=1}^{2} \widehat{f_i d\sigma_i}(x,t) \right| > \lambda \right\}.$$

Since $\sigma_i(B(z,\rho)) \le C\rho^{n-1}$ for any $z, \rho > 0$, following the argument [17] (see the proof of Lemma 2.4 in that paper), we have the estimate

$$\|\chi_F \prod_{i=1}^{2} \widehat{f_i d\sigma_i}\|_{L^1} \le C \left[(\max(M_1, M_2))^c \sqrt{R^{-\sigma} |F|^{\frac{\sigma+2}{\sigma+1}}} + C_2 \right] \prod_{i=1}^{2} \|f_i\|_{L^2(d\sigma_i)} \tag{75}$$

for some C, $c > 0$, where C_2 is any constant satisfying

$$\|\chi_F \widehat{g_1} \widehat{g_2}\|_{L^1} \le R^{-1} C_2 \|g_1\|_{L^2} \|g_2\|_{L^2}$$

for all g_1, g_2 supported in $O(R^{-1})$-neighborhoods of S_1, S_2, respectively. Here R is assumed to be bigger than or equal to 1.

First we try to estimate C_2 making use of (62). Let ϕ be a smooth function with its fourier transform supported in $B(0,1)$ satisfying $\sum_{k \in \mathbb{Z}^{n+1}} \phi^2(\cdot - k) = 1$. Then we set $\phi_k = \phi(\frac{x}{R} - k)$ for $k \in \mathbb{Z}^{n+1}$. Since ϕ_k is essentially supported in a ball of radius R, by a simple argument it is easy to see that (62) implies

$$\|\phi_k^2 \widehat{g_1} \widehat{g_2}\|_{L^{q_0} L^{r_0}} \le CMR^{\epsilon - 1} \|\phi_k \widehat{g_1}\|_2 \|\phi_k \widehat{g_2}\|_2$$

provided g_1, g_2 are supported in $O(R^{-1})$-neighborhoods of S_1, S_2, respectively. Hence by Schwartz inequality and Plancherel's theorem

$$\|\widehat{g_1} \widehat{g_2}\|_{L^{q_0} L^{r_0}} \le \sum_k \|\phi_k^2 \widehat{g_1} \widehat{g_2}\|_{L^{q_0} L^{r_0}} \le CMR^{\epsilon - 1} \|g_1\|_2 \|g_2\|_2.$$

Meanwhile from Hölder's inequality $\|\chi_F \widehat{g_1}\widehat{g_2}\|_{L^1} \le \|\chi_F\|_{L^{q_0'}L^{r_0'}}\|g_1g_2\|_{L^{q_0}L^{r_0}}$. Hence,

$$C_2 \le CMR^{\epsilon}\|\chi_F\|_{L^{q_0'}L^{r_0'}}. \tag{76}$$

For each t, let us set $E_t = \{x : (x,t) \in E\}$ and for a fixed $B > 0$,

$$E(B) = \bigcup_{t: B\le |E_t| < 2B} \{(x,t) : x \in E_t\}, \quad T(B) = \{t : B \le |E_t| < 2B\}.$$

We claim that for some $c > 0$,

$$\|\chi_{E(B)}\|_{L^{q_1}L^{r_1}} \le C\lambda^{-1}(\max(M_1,M_2))^{c\epsilon}M^{1-\frac{2\epsilon}{2\epsilon+\sigma}}\prod_{i=1}^{2}\|f_i\|_{L^2(d\sigma_i)}. \tag{77}$$

Proof of (77). We may assume $\|f_1\|_{L^2(d\sigma_1)} = \|f_2\|_{L^2(d\sigma_2)} = 1$. Let us set $A = |T(B)|$. Then obviously

$$AB \sim |E(B)|, \quad \|\chi_{E(B)}\|_{L^qL^r} \sim B^{1/r}A^{1/q}$$

for $1 \le p, r \le \infty$. From (75) and (76)

$$\|\chi_{E(B)}\prod_{i=1}^{2}\widehat{f_id\sigma_i}\|_{L^1} \le C\Big[(\max(M_1,M_2))^cR^{-\frac{\sigma}{2}}(AB)^{\frac{\sigma+2}{2\sigma+2}} + MR^{\epsilon}B^{1/r_0'}A^{1/q_0'}\Big]. \tag{78}$$

If $(\max(M_1,M_2))^c(AB)^{\frac{\sigma+2}{2\sigma+2}} \ge MB^{1/r_0'}A^{1/q_0'}$, we get

$$\|\chi_{E(B)}\prod_{i=1}^{2}\widehat{f_id\sigma_i}\|_{L^1} \le (\max(M_1,M_2))^{\frac{2c\epsilon}{2\epsilon+\sigma}}M^{1-\frac{2\epsilon}{2\epsilon+\sigma}}A^{1/q_1{}'}B^{1/r_1{}'}$$

by choosing $R \ge 1$ satisfying $(\max(M_1,M_2))^cR^{-\frac{\sigma}{2}}(AB)^{\frac{\sigma+2}{2\sigma+2}} = MR^{\epsilon}B^{1/r_0'}A^{1/q_0'}$. Note that $\lambda AB \sim \lambda|E(B)| \le \|\chi_{E(B)}\widehat{f_1d\sigma_1 f_2d\sigma_2}\|_{L^1}$. So (77) follows from the above because $\|\chi_{E(B)}\|_{L^{q_1}L^{r_1}} \sim A^{1/q_1}B^{1/r_1}$.

If $(\max(M_1,M_2))^c(AB)^{\frac{\sigma+2}{2\sigma+2}} < MB^{1/r_0'}A^{1/q_0'}$, then we take $R = 1$ in (78) to get

$$A^{1/q_0}B^{1/r_0} \le CM\lambda^{-1}.$$

On the other hand, we have Stein–Tomas estimate $\|\widehat{f_i\,d\sigma_i}\|_{L^{\frac{2\sigma+2}{\sigma}}} \le CM_i^{\frac{1}{2+2\sigma}}\|f_i\|_{L^2(d\sigma_i)}$ for $i = 1, 2$ (see [13], chapter VIII, section 4 and also [12]). Hence, by Chebychev's and Cauchy–Schwarz's inequalities,

$$A^{\frac{\sigma}{\sigma+1}}B^{\frac{\sigma}{\sigma+1}} \le C\lambda^{-1}(\max(M_1,M_2))^{\frac{2}{2+2\sigma}}.$$

Therefore,

$$A^{1/q_1}B^{1/r_1} = (A^{1/q_0}B^{1/r_0})^{1-\frac{2\epsilon}{2\epsilon+\sigma}}(A^{\frac{\sigma}{2\sigma+2}}B^{\frac{\sigma}{2\sigma+2}})^{\frac{2\epsilon}{2\epsilon+\sigma}} \le C\lambda^{-1}(\max(M_1,M_2))^{c\epsilon}M^{1-\frac{2\epsilon}{2\epsilon+\sigma}}.$$

(Here we used the fact that $r_0, q_0 \le 2 + 2/\sigma$.) This proves (77). □

Assuming $\|f_l\|_{L^2(\sigma_l)} = 1$ for $l = 1, 2$, we prove for $q > q_1$ the weak type inequality

$$\|\chi_E\|_{L^qL^{r_1}} \le C\lambda^{-1}(\max(M_1,M_2))^{a\epsilon+(1-q_1/q)(1-1/r_1)}M^{1-\frac{\epsilon}{2\epsilon+\sigma}} \tag{79}$$

for some $a > 0$. Since $f_1,\ f_2 \in L^2$, $\prod_{i=1}^{2}|\widehat{f_id\sigma_i}| \le (M_1M_2)^{1/2}$, and we may assume $\lambda \le (M_1M_2)^{1/2}$.

For $B = 2^k$, define $E(2^k)$ as before and decompose $E = \bigcup_k E(2^k)$. Set $A_k = |T(2^k)|$. For each fixed t, using the conservation of energy we have

$$|\{x : \Re \prod_{i=1}^{2} \widehat{f_i d\sigma_i}(x,t) > \lambda\}| \le C\lambda^{-1} \| \prod_{i=1}^{2} \widehat{f_i d\sigma_i}(\cdot, t)\|_{L^1(dx)} \le C\lambda^{-1}. \tag{80}$$

Therefore, we only need to consider the case $2^k \le C/\lambda$. Then,

$$\|\chi_E\|^q_{L^q_t L^{r_1}_x} = \int |E_t|^{q/r_1}\, dt = \sum_{k:\ 2^k \le C/\lambda} 2^{kq/r_1} A_k \le \sum_{k:\ 2^k \le C/\lambda} 2^{k(q-q_1)/r_1} \sup_k A_k 2^{kq_1/r_1}.$$

We use (77) to obtain

$$\|\chi_E\|^q_{L^q_t L^{r_1}_x} \le C\lambda^{-q} \lambda^{-(q-q_1)(1/r_1 - 1)} (\max(M_1, M_2))^{\frac{2cq_1\epsilon}{2\epsilon+\sigma}} M^{(1-\frac{\epsilon}{2\epsilon+\sigma})q_1}.$$

Since $q > q_1$, $r_1 \ge 1$ and $\lambda \le (M_1 M_2)^{1/2} \le \max(M_1, M_2)$, we get (79).

Assuming $\|f_l\|_{L^2(\sigma_l)} = \|f_2\|_{L^2(\sigma_2)} = 1$, we now obtain the strong type estimate. Since $\prod_{i=1}^2 |\widehat{f_i d\sigma_i}| \le (M_1 M_2)^{1/2} \le \max(M_1, M_2)$, we can write

$$\prod_{i=1}^{2} |\widehat{f_i d\sigma_i}| \le \sum_{k: 2^{-k} \le (M_1 M_2)^{1/2}} 2^{-k} \chi_{F^{(k)}}$$

where $F^{(k)} = \{(x,t) : \prod_{i=1}^2 |\widehat{f_i d\sigma_i}(x,t)| \sim 2^{-k}\}$. Fix $p > q_1$ and choose q satisfying $p > q > q_1$. Then,

$$\| \prod_{i=1}^{2} \widehat{f_i d\sigma_i}\|^p_{L^p L^{r_1}} = \int \Big[\sum_{k: 2^{-k} \le (M_1 M_2)^{1/2}} 2^{-kr_1} |(F^{(k)})_t| \Big]^{p/r_1} dt.$$

For $\beta > 0$, we bound this by

$$\| \prod_{i=1}^{2} \widehat{f_i d\sigma_i}\|^p_{L^p L^{r_1}} \le C \max(M_1, M_2)^\beta \sum_{k: 2^{-k} \le (M_1 M_2)^{1/2}} \int 2^{k\beta} 2^{-kp} |(F^{(k)})_t|^{p/r_1}\, dt.$$

(Actually, if $p \le r_1$, we can take $\beta = 0$.) Since $|(F^{(k)})_t| \le C 2^k$ for any t by (80),

$$\| \prod_{i=1}^{2} \widehat{f_i d\sigma_i}\|^p_{L^p L^{r_1}} \le C \max(M_1, M_2)^\beta \sum_{k: 2^{-k} \le (M_1 M_2)^{1/2}} 2^{k\beta} 2^{-kp} 2^{k(p/r_1 - q/r_1)} \|\chi_{F^{(k)}}\|^q_{L^q L^r_1}.$$

By (79), the right hand side of the above is bounded by

$$C \max(M_1, M_2)^\beta \sum_{k: 2^{-k} \le (M_1 M_2)^{1/2}} 2^{k(\beta - (p-q)(1-1/r_1))} \times (\max(M_1, M_2))^{a\epsilon + (q-q_1)(1-1/r_1)} M^{(1-\frac{\epsilon}{2\epsilon+\sigma})q}.$$

Thus, choosing $\beta < (p-q)(1-1/r_1)$, we get

$$\| \prod_{i=1}^{2} \widehat{f_i d\sigma_i}\|^p_{L^p L^{r_1}} \le C(\max(M_1, M_2))^{a\epsilon + (p-q_1)(1-1/r_1)} M^{(1-\frac{\epsilon}{2\epsilon+\sigma})q}.$$

This proves (63) for all $q > q_1$.

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