SHARP NULL FORM ESTIMATES FOR THE WAVE EQUATION IN \mathbb{R}^{3+1}

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ABSTRACT. We prove an equivalence between certain null form estimates for the wave equation in \mathbb{R}^{n+1} and the mixed norm bilinear restriction estimates for the paraboloid in \mathbb{R}^n . By constructing a counterexample and improving the positive results, we also fill the gap between the necessary and sufficient conditions when n = 3.

1. INTRODUCTION

Let ϕ , ψ be solutions of the homogeneous wave equation in \mathbb{R}^{n+1} ;

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$$\Box \phi = 0, \quad \Box \psi = 0; \qquad \Box = \Delta_x - \partial_t^2, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}.$$

We will mainly be concerned with null form estimates, which take the form

(1.1)
$$\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi \psi)\|_{L^q_t L^r_x} \le C(\|\phi(0)\|_{\dot{H}^{\alpha_1}} + \|\partial_t \phi(0)\|_{\dot{H}^{\alpha_1-1}}) \times (\|\psi(0)\|_{\dot{H}^{\alpha_2}} + \|\partial_t \psi(0)\|_{\dot{H}^{\alpha_2-1}}).$$

Here \dot{H}^{α} is the homogeneous L^2 -Sobolev space with α derivatives, D_0, D_+, D_- denote the Fourier multiplier operators defined by

$$\begin{split} \widehat{D_0 f}(\xi,\tau) &= |\xi| \widehat{f}(\xi,\tau), \\ \widehat{D_+ f}(\xi,\tau) &= (|\xi| + |\tau|) \widehat{f}(\xi,\tau), \\ \widehat{D_- f}(\xi,\tau) &= ||\xi| - |\tau|| \widehat{f}(\xi,\tau), \end{split}$$

and ξ, τ are the Fourier variables corresponding to x, t respectively.

This can be thought of as a bilinear generalization of the well known Strichartz's estimates (see [4]). The additional multiplier weights $D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}$ compensate the interaction between two waves making it possible to get further estimates which are not allowed in the linear setting.

These estimates were first considered by Beals [1], and by Klainerman and Machedon [5, 6, 7], who used them in their study of nonlinear wave equations. This work was furthered by Klainerman and Selberg [8], Klainerman and Tataru [9], and Foschi and

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Klainerman [3] who determined the whole range of $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ for which (1.1) holds when q = r = 2.

By considering the various interactions between two waves it can be shown that for (1.1) to hold, the following conditions are necessary:

• Scaling invariance:

(1.2)
$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 + \frac{1}{q} - n(1 - \frac{1}{r}).$$

• Geometry of the cones:

(1.3)
$$\frac{1}{q} \le \frac{n+1}{2}(1-\frac{1}{r}), \quad \frac{1}{q} \le \frac{n+1}{4}.$$

• Concentration near null directions:

(1.4)
$$\beta_{-} \ge \frac{1}{q} - \frac{n-1}{2}(1-\frac{1}{r}).$$

• Low frequency interactions (++):

(1.5)
$$\beta_0 \ge \frac{1}{q} - n(1 - \frac{1}{r}),$$

(1.6)
$$\beta_0 \ge \frac{2}{q} - (n+1)(1-\frac{1}{r}),$$

(1.7)
$$\beta_0 \ge \frac{2}{q} - n(1 - \frac{1}{r}) - \frac{1}{2}$$

• Low frequency interactions (+-):

(1.8)
$$\alpha_1 + \alpha_2 \ge \frac{1}{q},$$

(1.9)
$$\alpha_1 + \alpha_2 \ge \frac{3}{q} - n(1 - \frac{1}{r}).$$

• Interaction between high and low frequency:

(1.10)
$$\alpha_i \le \beta_- + \frac{n}{2},$$

(1.11)
$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r}),$$

(1.12)
$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}),$$

(1.13)
$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}) + (\frac{1}{2} - \frac{1}{q}),$$

(1.14)
$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + \frac{1}{2}$$

(1.15)
$$\alpha_i \le \beta_- + \frac{n}{2} - \frac{1}{q} + n(\frac{1}{2} - \frac{1}{r}) + \frac{1}{r} - \frac{1}{q}$$

These were proven in [3], except for (1.7), (1.14) and (1.15) that were subsequently proven in [11].

The first estimates for q = r < 2 are due to Bourgain [2], who considered bilinear estimates with separated frequency supports, but without the multiplier weights. Tao, Vargas and Vega [16, 17] improved these results, and the sharp L^p -estimates were obtained by Wolff [19] and Tao [12] (also see [10] for some generalizations).

Estimates for q = r < 2 that included the multiplier weights were proven by Tao and Vargas [16], and some sharp L^p -estimates were subsequently obtained by Tao [12]. This was extended to hyperbolic equations with rough coefficients by Tataru [18].

Recently, two of the authors [11] obtained the sharp null form estimates up to endpoints when $n \ge 4$. In lower dimensions, gaps remain between the necessary and sufficient conditions; in particular when

$$\frac{4}{n+1} \le q \le \frac{4}{n} \quad \text{and} \quad 2 < r \le \infty.$$

This is related to the *low frequency interactions* (+-); where ϕ and ψ are frequency supported in $\{(\xi, +|\xi|) : |\xi| \sim 1\}$ and $\{(\xi, -|\xi|) : |\xi| \sim 1\}$, respectively.

We will prove the following extra necessary conditions:

• Low frequency interactions (+-):

(1.16)
$$\alpha_1 + \alpha_2 \ge \frac{3}{q} + \frac{1}{r} - 2, \qquad n = 3$$

(1.17)
$$\alpha_1 + \alpha_2 \ge \frac{3}{q} + \frac{1}{2r} - \frac{5}{4}, \quad n = 2.$$

We will also fill the gap when n = 3, yielding estimates which are sharp up to endpoints.

Theorem 1.1. Let n = 3. If (1.2) holds and (1.3)-(1.16) are satisfied with strict inequalities, then (1.1) holds for all $1 < r, q \leq \infty$.

Mixed-norm bilinear restriction estimates for the paraboloid in \mathbb{R}^n (which corresponds to the Schrödinger equation in $\mathbb{R}^{n-1} \times \mathbb{R}$) were also obtained in [11] that are sharp up to endpoints when $n \geq 4$. We will make reductions to show that when $q, r \geq r'$, these estimates are essentially equivalent to certain null form estimates. Thus, we will see that the mixed-norm estimates for the paraboloid in \mathbb{R}^3 are also sharp up to endpoints. We note that in the remaining unsettled cases, when n = 2, the condition $q, r \geq r'$ is satisfied.

2. The necessary conditions (1.16) and (1.17)

Let c be a small positive constant to be chosen at the end. In order to prove the necessity of (1.16), we take $\widehat{\phi(0)}$ to be a nonzero smooth function adapted to the set

$$\left\{ \xi \in \mathbb{R}^3 : 1 \le \xi_3 \le 1 + c\epsilon, \left| \frac{\xi_2}{\xi_3} - \epsilon \right| \le \epsilon^2, \left| \frac{\xi_1}{\xi_3} \right| \le \epsilon^2 \right\},\$$

so that $\|\widehat{\phi(0)}\|_2^2 \leq c\epsilon^5$. One can also calculate that

$$|\phi(x,t)| = \frac{1}{(2\pi)^n} \left| \int e^{i(x\xi+t|\xi|)} \widehat{\phi(0)}(\xi) \, d\xi \right| \ge c\epsilon^5$$

in the parallelepiped

$$R = \left\{ (x,t) : |x_1| \le \epsilon^{-2}, |x_2 + \epsilon t| \le \epsilon^{-2}, |x_3 + \left(1 - \frac{\epsilon^2}{2}\right)t \right| \le \epsilon^{-1}, |t| \le \epsilon^{-4} \right\}.$$

To see this, we note that by Taylor's expansion,

$$\begin{aligned} |\xi| &= \xi_3 \sqrt{1 + \left(\frac{\xi_1}{\xi_3}\right)^2 + \left(\frac{\xi_2}{\xi_3} - \epsilon + \epsilon\right)^2} \\ &= \xi_3 \sqrt{1 + 2\epsilon \left(\frac{\xi_2}{\xi_3} - \epsilon\right) + \left(\frac{\xi_1}{\xi_3}\right)^2 + \left(\frac{\xi_2}{\xi_3} - \epsilon\right)^2 + \epsilon^2} \\ &= \xi_3 \left(1 - \frac{\epsilon^2}{2}\right) + \epsilon \xi_2 + 0(\epsilon^2). \end{aligned}$$

On the other hand, if we take $\widehat{\psi_0(0)}$ to be a nonzero smooth function adapted to the set

$$\left\{ \xi \in \mathbb{R}^3 : 1 - (1+c)\epsilon \le -\xi_3 \le 1 - \epsilon, \ \left| \frac{\xi_2}{\xi_3} \right| \le \epsilon^{3/2}, \ \left| \frac{\xi_1}{\xi_3} \right| \le \epsilon^{3/2} \right\}$$

then $\|\psi_0(0)\|_2^2 \leq c\epsilon^4$, and by a similar calculation we have

$$|\psi_0(x,t)| = \frac{1}{(2\pi)^n} \left| \int e^{i(x\xi - t|\xi|)} \widehat{\psi(0)}(\xi) \, d\xi \right| \ge c\epsilon^4$$

in the parallelepiped

$$R_0 = \left\{ (x,t) : |x_1| \le \epsilon^{-3/2}, |x_2| \le \epsilon^{-3/2}, \left| x_3 + \left(1 - \frac{\epsilon^2}{2} \right) t \right| \le \epsilon^{-1}, |t| \le \epsilon^{-3} \right\}.$$

Note that R_0 is contained in R, and the angle between the two is approximately ϵ .

Now, for all $k = 1, \ldots, \lfloor \epsilon^{-1} \rfloor$ we define ψ_k and ψ by

$$\psi_k(x,t) = \psi_0\left(x_1, x_2 - k\epsilon^{-2}, x_3 - k\epsilon^{-3}\left(1 - \frac{\epsilon^2}{2}\right), t + k\epsilon^{-3}\right),$$
$$\psi(x,t) = \sum_{k=0}^{\lfloor \epsilon^{-1} \rfloor} \omega_k \psi_k(x,t),$$

where ω_k are random variables taking values in $\{-1, 1\}$ with equal probability. Independent of the choice of ω_k , we have that $|\psi_k(x,t)| \ge c\epsilon^4$ on the set R_k given by

$$\left\{ (x,t) : |x_1| \le \epsilon^{-3/2}, |x_2 - k\epsilon^{-2}| \le \epsilon^{-3/2}, |x_3 + \left(1 - \frac{\epsilon^2}{2}\right)t \right| \le \epsilon^{-1}, |t + k\epsilon^{-3}| \le \epsilon^{-3} \right\}.$$

Integrating by parts, one can calculate that

$$|\psi_k(x,t)| \le C_N \epsilon^4 (1 + \epsilon^{3/2} |x_2 - k\epsilon^{-2}|)^{-N},$$

so that $\sum_{k \neq k'} |\psi_k(x,t)| \leq C \epsilon^N$ on $R_{k'}$. Thus, $|\psi(x,t)| \geq c \epsilon^4/2$ in the union of the parallelepipeds R_k which are contained in the parallelepiped R.

In order to calculate D_0 , D_+ , and D_- , we consider ξ , ξ' in the frequency supports of ϕ and ψ respectively. We see that

$$\xi_1 + \xi'_1 = O(\epsilon^{3/2}), \quad \xi_2 + \xi'_2 = \epsilon + O(\epsilon^{3/2}), \quad \xi_3 + \xi'_3 = \epsilon + O(c\epsilon).$$

From here it is easy to see that $D_0, D_+ \sim \epsilon$. To calculate D_- , we note that $|\xi + \xi'| = 2^{1/2}\epsilon + O(c\epsilon)$, and that

$$\begin{aligned} |\tau + \tau'| &= \left| |\xi| - |\xi'| \right| \\ &= \left| |\xi_3| - |\xi'_3| \right| + O(c\epsilon) \le \epsilon + O(c\epsilon) \end{aligned}$$

Thus, provided c is small enough, $D_{-} \sim \epsilon$.

Now, one can calculate that

$$\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\psi\phi)\|_{L^q L^r(\bigcup R_j)} \ge C\epsilon^9 \epsilon^{-4/r} \epsilon^{-4/q} \epsilon^{\beta_0+\beta_++\beta_-},$$

so that by Khintchin's inequality and (1.1), we have

$$\epsilon^9 \epsilon^{-4/r} \epsilon^{-4/q} \epsilon^{\beta_0 + \beta_+ + \beta_-} < C \epsilon^4$$

for all $\epsilon > 0$. This yields

$$\beta_0 + \beta_+ + \beta_- \ge 4/q + 4/r - 5$$

which combined with the scaling relation (1.2), gives the condition (1.16).

To prove the condition (1.17), we simply kill the x_1 variable and repeat the calculation.

3. Null form estimates imply bilinear restriction for the paraboloid

By an affine change of variables, we will consider the phase

$$|\xi| - \xi_n = \xi_n(\sqrt{1 + |\xi'/\xi_n|^2} - 1), \qquad \xi' = (\xi_1, \xi_2, \dots, \xi_{n-1}).$$

As was pointed out in [11, 12], the null form estimates are closely related to the bilinear adjoint restriction estimates to the paraboloid. Indeed, after scaling we will consider

$$\theta_a(\xi') = a^{-2}(\sqrt{1+|a\xi'|^2}-1),$$

where $0 < a \ll 1$, so that by Taylor's expansion

(3.1)
$$\theta_a(\xi') = \frac{|\xi'|^2}{2} + O(a^2).$$

Define the extension operator $\widehat{fd\sigma}$ by

$$\widehat{fd\sigma}(x,t) = \int_{1}^{1+a} \int_{[-1,1]^{n-1}} e^{i(x'\cdot\xi'+x_n\xi_n+t\xi_n\theta_a(\xi'/\xi_n))} f(\xi',\xi_n) d\xi' d\xi_n.$$

and the angular support of f by

$$\Theta(f) = \left\{ \xi' / \xi_n : (\xi', \xi_n) \in \operatorname{supp} f \cap \left([-1, 1]^{n-1} \times [1, 1+a] \right) \right\}.$$

In the case of *low frequency interaction* (+-), the estimate (1.1) for given $q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ is essentially equivalent to

(3.2)
$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma}\|_{L_{t}^{q}L_{x}^{r}} \leq Ca^{1-\frac{1}{r}+\frac{1}{q}-\alpha_{1}-\alpha_{2}}\prod_{j=1}^{2}\|f_{j}\|_{2}$$

whenever dist $(\Theta(f_1), \Theta(f_2)) \sim 1$. One half of this equivalence is presented in the following proposition. Later we will see that the converse is also essentially true.

Proposition 3.1. If (1.1) holds for some $q, r, \alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$, then (3.2) holds for q, r, α_1, α_2 whenever $dist(\Theta(f_1), \Theta(f_2)) \sim 1$.

Proof. If we set $\widehat{F}_1(\xi',\xi_n) = f_1(\xi'/a,\xi_n)$ and $\widehat{F}_2(\xi',\xi_n) = f_2(-\xi'/a,-\xi_n)$, then by dilation and rotation we can suppose that supp \widehat{F}_1 and supp \widehat{F}_2 are respectively contained in

$$\{(\xi',\xi_n) \in \mathbb{R}^n : 1 \le \xi_n \le (1+a), \ |\xi'/\xi_n - ae'_{n-1}| \le a/2\}, \\ \{(\xi',\xi_n) \in \mathbb{R}^n : -(1+a) \le \xi_n \le -1, \ |\xi'/\xi_n + ae'_{n-1}| \le a/2\};$$

here $e'_{n-1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n-1}$. Let ϕ, ψ be frequency supported in the forward, backward light cones with initial data F_1, F_2 , respectively. The frequency support of $\phi \psi$ is contained in

$$\{(\xi',\xi_n,\tau)\in\mathbb{R}^{n+1}: |\xi_n|\leq a, |\xi'-2ae'_{n-1}|\leq 3a/2, |\tau|\leq 4a\},\$$

so it is easy to see that the multiplier weight $D_0^{\beta_0}D_+^{\beta_+-\beta_-} \sim a^{\beta_0+\beta_+-\beta_-}$. One can also calculate that $|\Box|^{\beta_-} \equiv D_+^{\beta_-}D_-^{\beta_-} \sim a^{2\beta_-}$ and

$$a^{\beta_0+\beta_+-\beta_-}a^{2\beta_-}\|\phi\psi\|_{L^q_xL^r_t} \le C\|D_0^{\beta_0}D_+^{\beta_+}D_-^{\beta_-}(\phi\psi)\|_{L^q_xL^r_t}.$$

(This is made rigorous by rescaling and Fourier series expansion; see [11, Lemma 2.6]).

Now, assuming (1.1), by Plancherel's theorem, we have

$$\|\phi\psi\|_{L^q_xL^r_t} \le Ca^{-\beta_0-\beta_+-\beta_-+n-1} \|f_1\|_2 \|f_2\|_2.$$

By the change of variables $x_n \to x_n - t$, $x' \to a^{-1}x'$ and $t \to a^{-2}t$ and $\xi' \to a\xi'$, we see that

$$\|\phi\psi\|_{L^q_x L^r_t} = a^{-2/q + (n-1)(2-1/r)} \|\prod_{j=1}^2 \widehat{f_j d\sigma}\|_{L^q_x L^r_t}$$

Combining this with the previous estimate, and using the condition (1.2) which is necessary for (1.1) to hold, we obtain (3.2).

We define the extension operator E by

$$Eg(x',t) = \int_{[-1,1]^{n-1}} e^{i(x'\xi'+t|\xi'|^2/2))} g(\xi')d\xi',$$

and note that when $\operatorname{supp} \widehat{g} \subset [-1, 1]^{n-1}$,

$$e^{-i\frac{t}{2}\Delta}g(x') = (2\pi)^{-n}E\,\widehat{g}(x',t),$$

where $e^{-it\Delta}$ is the Schrödinger operator. This is also the adjoint restriction operator to the paraboloid, which is why we often refer to the following estimates as restriction estimates.

Proposition 3.2. Suppose that

$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma}\|_{L^{q}_{t}L^{r}_{x}} \leq Ca^{1-\frac{1}{r}}\prod_{j=1}^{2}\|f_{j}\|_{2}$$

holds whenever $\operatorname{dist}\left(\Theta(f_1),\Theta(f_2)\right)\sim 1$. Then

$$\|\prod_{j=1}^{2} Eg_{j}\|_{L_{t}^{q}L_{x'}^{r}} \leq C\prod_{j=1}^{2} \|g_{j}\|_{2}$$

holds whenever $dist(supp g_1, supp g_2) \sim 1$.

Proof. By Fatou's lemma and (3.1),

$$\|\prod_{j=1}^{2} Eg_{j}\|_{L_{t}^{q}L_{x'}^{r}} \leq \liminf_{a \to 0} a^{-2} \|\prod_{j=1}^{2} \widehat{f_{j}d\sigma}(\cdot, 0, \cdot)\|_{L_{t}^{q}L_{x'}^{r}}$$

where $f_j(\xi', \xi_n) = \chi_{[1,1+a]}(\xi_n)g_j(\xi')$. Note that there is no dependence on x_n . By integrating,

(3.3)
$$a^{-\frac{1}{r}} \| \prod_{j=1}^{2} Eg_j \|_{L^q_t L^r_{x'}} \le Ca^{-2} \| \prod_{j=1}^{2} \widehat{f_j d\sigma}(\cdot, 0, \cdot) \|_{L^q_t(\mathbb{R}, L^r_x(\mathbb{R}^{n-1} \times [-a^{-1}, a^{-1}])}.$$

Now, as a function of ξ_n , the phase is almost stationary when $x_n \in [-a^{-1}, a^{-1}]$, so that, essentially

$$|\widehat{f_j d\sigma}(\cdot, 0, \cdot)| \le C |\widehat{f_j d\sigma}(\cdot, x_n, \cdot)|.$$

To make this inequality rigorous, write

$$\widehat{f_j d\sigma}(x',0,t) = e^{ix_n} \int_1^{1+a} e^{-ix_n(\xi_n-1)} \int_{[-1,1]^{n-1}} e^{i(x'\cdot\xi'+x_n\xi_n+t\xi_n\theta_a(\xi'/\xi_n))} f_j(\xi',\xi_n) \, d\xi' d\xi,$$

and using Taylor's expansion of the exponential this is equal to

$$e^{ix_n} \sum_{k=0}^{\infty} \frac{i^k}{k!} \int_1^{1+a} (x_n(\xi_n-1))^k \int_{[-1,1]^{n-1}} e^{i(x'\cdot\xi'+x_n\xi_n+t\xi_n\theta_a(\xi'/\xi_n))} f_j(\xi',\xi_n) \, d\xi' d\xi_n.$$

Defining $f_j^k(\xi',\xi_n) = (\xi_n - 1)^k f_j(\xi',\xi_n)$, we may write

$$\widehat{f_j d\sigma}(x', 0, t) = e^{ix_n} \sum_{k=0}^{\infty} \frac{i^k}{k!} x_n^k \widehat{f_j^k d\sigma}(x', x_n, t)$$

for all x_n . Substituting into (3.3), we see that

$$\begin{split} \|\prod_{j=1}^{2} Eg_{j}\|_{L_{t}^{q}L_{x'}^{r}} &\leq Ca^{-2+\frac{1}{r}} \sum_{k_{1}=0}^{\infty} \sum_{k_{2}=0}^{\infty} \frac{1}{k_{1}!} \frac{1}{k_{2}!} \|\prod_{j=1}^{2} x_{n}^{k_{j}} \widehat{f_{j}^{k_{j}}} d\sigma \|_{L_{t}^{q}(\mathbb{R}, \ L_{x}^{r}(\mathbb{R}^{n-1} \times [-a^{-1}, a^{-1}]))} \\ &\leq Ca^{-2+\frac{1}{r}} \sum_{k_{1}=0}^{\infty} \sum_{k_{2}=0}^{\infty} \frac{1}{k_{1}!} \frac{1}{k_{2}!} a^{-k_{1}-k_{2}} \|\prod_{j=1}^{2} \widehat{f_{j}^{k_{j}}} d\sigma \|_{L_{t}^{q}(\mathbb{R}, \ L_{x}^{r}(\mathbb{R}^{n-1} \times [-a^{-1}, a^{-1}]))} \\ \end{split}$$

and using the hypothesis, we obtain

$$\|\prod_{j=1}^{2} Eg_{j}\|_{L_{t}^{q}L_{x'}^{r}} \leq Ca^{-1} \sum_{k_{1}=0}^{\infty} \sum_{k_{2}=0}^{\infty} \frac{1}{k_{1}!} \frac{1}{k_{2}!} a^{-k_{1}-k_{2}} \prod_{j=1}^{2} \|f_{j}^{k_{j}}\|_{L^{2}(\mathbb{R}^{n})}.$$

Finally, since $\|f_j^k\|_{L^2(\mathbb{R}^n)} \leq a^k \|f_j\|_{L^2(\mathbb{R}^n)}$, we conclude that

$$\begin{split} \|\prod_{j=1}^{2} Eg_{j}\|_{L_{t}^{q}L_{x'}^{r}} &\leq Ca^{-1}\sum_{k_{1}=0}^{\infty}\sum_{k_{2}=0}^{\infty}\frac{1}{k_{1}!}\frac{1}{k_{2}!}\prod_{j=1}^{2}\|f_{j}\|_{L^{2}(\mathbb{R}^{n})} \\ &\leq Ca^{-1}\prod_{j=1}^{2}\|g_{j}\chi_{[1,1+a]}\|_{L^{2}(\mathbb{R}^{n})} \leq C\prod_{j=1}^{2}\|g_{j}\|_{L^{2}(\mathbb{R}^{n-1})}, \end{split}$$

as desired.

Remark 3.3. Combining Propositions 3.1 and 3.2, we see that null form estimates with $\alpha_1 + \alpha_2 = 1/q$ imply mixed norm bilinear restriction estimates for the paraboloid. For q = r = (n+2)/n, such an implication had already been observed in [12].

4. BILINEAR RESTRICTION FOR THE PARABOLOID IMPLIES NULL FORM ESTIMATES

In this section, a, b, h and ρ will be variables satisfying $0 < h < a \ll 1$ and $|b|, |\rho| \sim 1$. For the converse, we need to consider small perturbations of the previous operators, and we will require the estimates to be uniform in these perturbations. Define $\widehat{fd\sigma_b^h}$ by

$$\widehat{fd\sigma_b^h}(x,t) = \int_b^{b+a} \int_{[-1,1]^{n-1}} e^{i(x'\cdot\xi'+x_n\xi_n+t\xi_n\theta_h(\xi'/\xi_n))} f(\xi',\xi_n) d\xi' d\xi_n$$

and E^h_{ρ} by

$$E^{h}_{\rho}g(x',t) = \int_{[-1,1]^{n-1}} e^{i(x'\xi'+t\rho\theta_{h}(\xi'/\rho))}g(\xi')d\xi'.$$

Proposition 4.1. Let $q, r \ge r'$, and suppose that

$$\|\prod_{j=1}^{2} E_{\rho_{j}}^{h} g_{j}\|_{L_{t}^{q} L_{x'}^{r}} \leq C \prod_{j=1}^{2} \|g_{j}\|_{2}$$

holds uniformly in ρ_1 , ρ_2 , a, h whenever dist (supp g_1 , supp g_2) ~ 1. Then

$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma_{b_{j}}^{h}}\|_{L_{t}^{q}L_{x}^{r}} \leq Ca^{1-\frac{1}{r}}\prod_{j=1}^{2}\|f_{j}\|_{2}$$

holds uniformly in b_1 , b_2 , a, h whenever $dist(\Theta(f_1), \Theta(f_2)) \sim 1$.

Proof. By the Hausdorff-Young inequality with respect to x_n , we get

$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma_{b_{j}}^{h}}\|_{L_{t}^{q}L_{x}^{r}} \leq \left\|\int E_{\rho-\rho'}^{h}g_{1}(\cdot,\rho-\rho')E_{\rho'}^{h}g_{2}(\cdot,\rho')d\rho'\right\|_{L_{t}^{q}L_{x'}^{r}L_{\rho'}^{r'}},$$

where $g_j(\cdot, \rho) = \chi_{[b_j, b_j+a]}(\rho) f_j(\cdot, \rho)$. Note that we can suppose that $|\rho| \sim 1$ and $|\rho - \rho'| \sim 1$, as $g_j \equiv 0$ otherwise. Now as $q, r \geq r'$, by Minkowski's inequality, the left hand side is

bounded by $\|\cdot\|_{L^{r'}_{\rho}L^q_tL^r_{\pi'}}$. Thus, by hypothesis, we have

$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma_{b_{j}}}\|_{L_{t}^{q}L_{x}^{r}} \leq C \left\|\int \|g_{1}(\cdot,\rho-\rho')\|_{2}\|g_{2}(\cdot,\rho')\|_{2}d\rho'\right\|_{L_{\rho}^{r'}}$$

Finally, by Young's inequality the left side is bounded by $||g_1||_{L^p_\rho L^2_{\xi'}} ||g_2||_{L^p_\rho L^2_{\xi'}}$ where 1/p = 1 - 1/2r, and the desired bound follows by Hölder's inequality.

Proposition 4.2. Suppose that (1.2) is satisfied, (1.4) is satisfied with strict inequality, and for some $\epsilon > 0$,

(4.1)
$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma_{b_{j}}^{h}}\|_{L_{t}^{q}L_{x}^{r}} \leq Ca^{1-\frac{1}{r}+\frac{1}{q}-\alpha_{1}-\alpha_{2}+\epsilon}\prod_{j=1}^{2}\|f_{j}\|_{2}$$

holds uniformly in b_1, b_2, a, h whenever $dist(\Theta(f_1), \Theta(f_2)) \sim 1$. Then (1.1) holds for low frequency interactions (+-).

Proof. We follow the argument of Section 3 in [11]. By spatial rotation and dilation it is enough to show that for $0 \le m \le l$ and some $\epsilon_0 > 0$,

(4.2)
$$\|D_0^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\|_{L^q_x L^r_t} \le C 2^{-\epsilon_0 l} 2^{-\epsilon_0 m} \|\phi(0)\|_2 \|\psi(0)\|_2$$

whenever $\widehat{\phi}(0)$ and $\widehat{\psi}(0)$ are supported in

$$\{\xi \in \mathbb{R}^{n} : b \leq \xi_{n} \leq (b+2^{m-l}), |\xi'/\xi_{n}-2^{-l}e'_{n-1}| \leq 2^{-l}/2 \}, \\ \{\xi \in \mathbb{R}^{n} : -(b+3\cdot 2^{m-l}) \leq \xi_{n} \leq -(b+2\cdot 2^{m-l}), |\xi'/\xi_{n}+2^{-l}e'_{n-1}| \leq 2^{-l}/2 \},$$

respectively, where $b \sim 1$.

As before, one can calculate that $D_0^{\beta_0} D_+^{\beta_+ - \beta_-} \sim 2^{(m-l)(\beta_0 + \beta_+ - \beta_-)}$, and $|\Box|^{\beta_-} \sim 2^{-2l\beta_-}$ on the frequency support of $\phi\psi$. Hence, by the rescaling argument of Lemma 2.6 in [11],

where the $\widehat{f_j d\sigma_{b_j}^h}$ are defined with $a = 2^{m-l}$, $b_1 = b$, $b_2 = -(b+3a)$, $h = 2^{-l}$, and the functions f_1 , f_2 are defined by

$$f_1(\xi',\xi_n) = 2^{-\frac{(n-1)l}{2}}\widehat{\phi(0)}(2^{-l}\xi',\xi_n)$$
 and $f_2(\xi',\xi_n) = 2^{-\frac{(n-1)l}{2}}\widehat{\psi(0)}(2^{-l}\xi',\xi_n).$

It is easy to see that $||f_1||_2 = ||\phi(0)||_2$, $||f_2|| = ||\psi(0)||_2$ and dist $(\Theta(f_1), \Theta(f_2)) \sim 1$. Using the assumption (4.1) with $a = 2^{m-l}$ and the condition (1.2) we get

$$\begin{split} \|D_0^{\beta_0} D_+^{\beta_+ - \beta_-} \|\Box\|^{\beta_-} (\phi \psi)\|_{L^q_t L^r_x} \\ &\leq C 2^{-\epsilon l} 2^{m(2/q - (n-1)(1 - 1/r) - 2\beta_- + \epsilon)} \|\phi(0)\|_2 \|\psi(0)\|_2. \end{split}$$

Finally, as (1.4) holds with strict inequality, we can choose ϵ sufficiently small to obtain (4.2), and we are done.

Remark 4.3. Combining Propositions 4.1 and 4.2, we see that mixed norm bilinear restriction estimates for small perturbations of the paraboloid, where $q, r \ge r'$, imply null form estimates for low frequency interactions (+-), whenever (1.4) and (1.8) are satisfied strictly.

5. Results for the paraboloid

In [11, Theorem 2.3] it was shown that the mixed norm bilinear restriction estimate

(5.1)
$$\|\prod_{j=1}^{2} Eg_{j}\|_{L_{t}^{q}(\mathbb{R}, L_{x'}^{r}(\mathbb{R}^{n-1}))} \leq C \prod_{j=1}^{2} \|g_{j}\|_{L^{2}(\mathbb{R}^{n-1})}$$

holds whenever dist (supp g_1 , supp g_2) ~ 1 for $q > \max(1, 4/n)$ and 2/q < n(1 - 1/r). These estimates are stable under small perturbations. Since the condition

(5.2)
$$\frac{2}{q} \le n\left(1 - \frac{1}{r}\right)$$

is necessary, this gives the optimal result up to endpoints when $n \ge 4$. By a similar construction to that used to prove condition (1.16), or indeed via the equivalence, one can calculate that

(5.3)
$$\frac{1}{r} \le 2\left(1 - \frac{1}{q}\right)$$

is also necessary for (5.1) to hold when n = 3. We will also see that when a strict version of condition (5.3) holds, (5.1) follows trivially, so we obtain the following result which is sharp up to endpoints.

Theorem 5.1. Let n = 3. If (5.2) and (5.3) are satisfied with strict inequalities, then (5.1) holds whenever dist(supp g_1 , supp g_2) ~ 1. Conversely, if (5.1) holds whenever dist(supp g_1 , supp g_2) ~ 1, then (5.2) and (5.3) are satisfied.

Proof. When q > 1 and $r = \infty$, the estimate (5.1) follows from the linear Strichartz estimates for the paraboloid by applying Cauchy-Schwarz. (In [14] it is shown that the endpoint estimate $L^2 \times L^2 \rightarrow L_t^1 L_x^\infty$ fails.) In [11, Theorem 2.3], the estimates when q > 4/3 and 2/q < 3(1 - 1/r) were obtained; in particular, for q > 4/3 and r = 2. Interpolation yields the remaining estimates when $q \le 4/3$ and 1/r < 2(1 - 1/q).

For clarity, we provide a direct proof of the necessary condition (5.3). Take η_0 and g_1 to be nonzero smooth functions adapted to the sets

$$\{\xi \in \mathbb{R}^2 : |\xi| \le \epsilon\},\$$
$$\{\xi \in \mathbb{R}^2 : |\xi - e_1| \le \epsilon^2\},\$$

respectively.



Figure 1. T and T_k

One can calculate that

$$|E\eta_0(x,t)| \ge c\epsilon^2$$
 and $|Eg_1(x,t)| \ge c\epsilon^4$

on the tubes T_0 and T defined by

$$T_0 = \{ (x,t) \in \mathbb{R}^{2+1} : |x| \le \epsilon^{-1}, \quad 0 \le t \le \epsilon^{-2} \},$$
$$T = \{ (x,t) \in \mathbb{R}^{2+1} : |x+te_1| \le \epsilon^{-2}, \quad 0 \le t \le \epsilon^{-4} \}$$

Now, for all $k = 1, \ldots, \lfloor \epsilon^{-2} \rfloor$ define η_k and g_2 by

$$\eta_k(\xi) = e^{ik\epsilon^{-2}(\xi_1 - |\xi|^2/2)} \eta_0(\xi),$$
$$g_2 = \sum_{k=0}^{\lfloor \epsilon^{-1} \rfloor} \omega_k \eta_k,$$

where ω_k are random variables taking values in $\{-1, 1\}$ with equal probability.

As in the second section, regardless of the choice of ω_k , we have $|Eg_2(x,t)| \ge c\epsilon^2$ on the union of the tubes T_k given by

$$T_k = \{ (x,t) \in \mathbb{R}^{2+1} : |x + k\epsilon^{-2}e_1| \le \epsilon^{-1}, \quad k\epsilon^{-2} \le t \le (k+1)\epsilon^{-2} \},\$$

which are contained in the tube T. (See Figure 1.)

Now, assuming (5.1), by Khintchin's inequality, one can calculate that

$$\epsilon^{-4/q} \epsilon^{-2/r} \epsilon^6 \le C \epsilon^2$$

for all $\epsilon > 0$, and the condition follows.

Finally, when n = 2, by killing the x_2 variable in the previous construction, one can calculate that the condition

(5.4)
$$\frac{2}{q} + \frac{1}{2r} \le \frac{5}{4}$$

is necessary in order for (5.1) to hold. This is stronger than $1/q \leq (1 - 1/r)$ when r > 2.



Figure 2. The regions obtained by comparing the conditions (1.8), (1.9) and (1.16) when n = 3. The marked condition is strongest in each divided region.

6. Results for null forms

Only a small modification of the proof of the higher dimensional result in [11] is required to prove Theorem 1.1. The estimate is obtained by combining the bilinear estimates for the paraboloid of the previous section with Propositions 4.1 and 4.2.

Proof of Theorem 1.1. The estimate (1.1) is shown by considering the various interactions between waves of different frequencies and decomposing the waves accordingly. In [11] the sharp estimates were obtained for all the cases apart from the low frequency interactions (+-); where ϕ and ψ are frequency supported in $\{(\xi, +|\xi|) : |\xi| \sim 1\}$ and $\{(\xi, -|\xi|) : |\xi| \sim$ 1}, respectively. By Proposition 4.2 it will suffice to prove (4.1) assuming strict versions of (1.8), (1.9) and (1.16). Comparing the three conditions (see Figure 2) it is enough to show that whenever dist $(\Theta(f_1), \Theta(f_2)) \sim 1$, we have

$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma_{b_{j}}^{h}}\|_{L^{q}_{t}L^{r}_{x}} \leq Ca^{\gamma}\prod_{j=1}^{2}\|f_{j}\|_{2}$$

when $(1/r, 1/q, \gamma) = (0, 0, 1)$, (1, 0, 0), $(0, 1 - \epsilon, 1)$, $(1/2, 3/4 - \epsilon, 1/2)$, $(1/2, 1 - \epsilon, 0)$ for all $0 < \epsilon \ll 1$. The other estimates follow by complex interpolation.

By Cauchy-Schwarz followed by Plancherel, it is easy to see that the estimate for $(1/r, 1/q, \gamma) = (1, 0, 0)$ holds. As the Fourier transform of $\widehat{f_j d\sigma_{b_j}^h}$ is contained in a slab of thickness a, by an application of Bernstein's inequality, (0, 0, 1) follows. By Proposition 4.1 and Cauchy-Schwarz, $(0, 1 - \epsilon, 1)$ is a consequence of the linear Strichartz estimates for the paraboloid in \mathbb{R}^3 . Similarly, $(1/2, 3/4 - \epsilon, 1/2)$ follows from Proposition 4.1 combined with Theorem 5.1. Finally, the estimate for $(1/2, 1 - \epsilon, 0)$ can be found in Proposition 2.4 in [11] (with scale $h = 2^{-l}$ and k = 0). It is a mixed norm generalization of the bilinear restriction estimate for the cone.

We also prove the following partial result in two spatial dimensions.

Theorem 6.1. Let n = 2. If the conditions (1.2)-(1.15) are satisfied with strict inequalities and additionally $\alpha_1 + \alpha_2 > 3/q - 1$, then (1.1) holds for all $1 \le r, q \le \infty$.

Proof. By (1.3) we have that q > 4/3, and as before we need only consider the low frequency interaction (+-). We may also assume that $q \le 2 \le r$ because the other estimates were already obtained in [11, Theorem 1.1]. By Proposition 4.2 it will suffice to show that for $4/3 < q \le 2 \le r$

$$\|\prod_{j=1}^{2}\widehat{f_{j}d\sigma_{b_{j}}^{h}}\|_{L_{t}^{q}L_{x}^{r}} \leq Ca^{2-\frac{1}{r}-\frac{2}{q}}\prod_{j=1}^{2}\|f_{j}\|_{2}$$

whenever dist $(\Theta(f_1), \Theta(f_2)) \sim 1$. Again, this follows from interpolating the estimates corresponding to (1/r, 1/q) = (1/2, 1/2), (0, 1/2), $(1/2, 3/4 - \epsilon)$ and $(0, 3/4 - \epsilon)$ for all $\epsilon > 0$, with bounds $Ca^{\frac{1}{2}}, Ca^1, Ca^0$, and $Ca^{\frac{1}{2}}$, respectively. The estimate when (1/q, 1/r) =(1/2, 1/2) is a consequence of the bilinear restriction estimates for the parabola combined with Proposition 4.1. The estimate for $(1/r, 1/q) = (1/2, 3/4 - \epsilon)$ can be found in [11, Proposition 2.4]. The remaining estimates follow by Bernstein's inequality because the Fourier transform of $\widehat{f_j d\sigma_{b_j}^h}$ is contained in a slab of thickness a.

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