A MAXIMAL INEQUALITY ASSOCIATED TO SCHRÖDINGER TYPE EQUATION

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Abstract. In this note, we consider a maximal operator $\sup_{t\in\mathbb{R}}|u(x,t)| = \sup_{t\in\mathbb{R}}|e^{it\Omega(D)}f(x)|$, where u is the solution to the initial value problem $u_t = i\Omega(D)u$, $u(0) = f$ for a C^2 function Ω with some growth rate at infinity. We prove that the operator $\sup_{t\in\mathbb{R}}|u(x,t)|$ has a mapping property from a fractional Sobolev space $H^{\frac14}$ with additional angular regularity in which the data lives to $L^2((1+|x|)^{-b}dx)$ $(b>1)$. This mapping property implies the almost everywhere convergence of $u(x,t)$ to f as $t\to 0$, if the data f has an angular regularity as well as $H^{\frac14}$ regularity.

1. Introduction

We consider the following free Schrödinger type equation:

$$\frac{\partial}{\partial t}u(x,t) = i\Omega(D)u(x,t) \quad \text{in} \quad \mathbb{R}^{n+1}(n\geq 2), \quad u(x,0) = f(x),$$

where $\Omega(D)$ is a generalized differential operator defined by a C^2 function Ω and $D = (-\Delta)^{\frac12}$. For smooth initial data f, the solution $u(x,t) = e^{it\Omega(D)}f$ can be written as

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\Omega(\xi))}\widehat{f}(\xi)\,d\xi, \quad f\in\mathcal{S}(\mathbb{R}^n),$$

where $\widehat{f}(\xi) = \int e^{-ix\cdot\xi}f(x)\,dx$. In this note, we assume that the initial data f has H^s regularity for some $s>0$ as well as some regularity in the angular direction. For $\alpha,\beta\geq 0$, we define an initial data space $H_r^\alpha H_\omega^\beta$ by

$$H_r^\alpha H_\omega^\beta = \left\{f : \|f\|_{H_r^\alpha H_\omega^\beta} := \|(1-\Delta)^{\frac{\alpha}{2}}f\|_{L_r^2 H_\omega^\beta} < \infty\right\},$$

where $||g||_{L_r^2}^2 = \int_0^\infty |g(r)|^2 r^{n-1}\,dr$, $||g||_{L_r^2H_\omega^\beta} = ||||(1-\Delta_\omega)^{\frac{\beta}{2}}f(r\omega)||_{L_\omega^2}||_{L_r^2}$ (here, $(r,\omega)\in\mathbb{R}_+\times S^{n-1}$ is the spherical coordinates), and Δ_ω is the Laplace-Beltrami operator on S^{n-1}. Since Δ_ω commutes with Δ, one can readily check that $||g||_{H_r^\alpha H_\omega^\beta} \sim ||(1-\Delta_\omega)^{\frac{\beta}{2}}g||_{H^\alpha}$ (for instance, see [9]). Since not every function in $H_r^\alpha H_\omega^\beta$ has radial regularity higher than α, there is no embedding from or into a usual Sobolev

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space. In particular, it should be noted that $H_r^\alpha H_\omega^\beta \not\subseteq H^{\alpha+\gamma}$ $(0 < \gamma < \beta)$ and $H_r^\alpha H_\omega^\beta \not\supseteq H^{\alpha+\gamma}$ $(\gamma \geq \beta)$.

We also assume that $\Omega \in C^2(\mathbb{R}^n)$ is radially symmetric and satisfies

$$c_1|\rho|^{a-k} \leq |\Omega^{(k)}(\rho)| \leq c_2|\rho|^{a-k} \ (k = 0, 1, 2), \quad \text{if} \quad |\rho| \geq N$$

for some $c_1, c_2, a > 0$ with $a \neq 1$ and a large $N > 0$. With the above assumptions, let us define a maximal function $u^*(x)$ by $u^*(x) = \sup_{t\in\mathbb{R}} |u(x,t)|$. We prove

Theorem 1.1. *For any $\varepsilon > 0$ and $b > 1$, if $f \in H_r^{\frac14} H_\omega^{\frac{n-1}{2}-\frac14+\varepsilon}$, then there exists a constant C, depending only on $a, c_1, c_2, N, n, \varepsilon, b$, such that*

$$||u^*||_{L^2((1+|x|)^{-b}dx)} \leq C||f||_{H_r^{\frac14} H_\omega^{\frac{n-1}{2}-\frac14+\varepsilon}}.$$

Now let us define a linear operator T and a maximal operator T^* for a fixed $s > 0$ by

$$Tf(x,t) = w(|x|) \int e^{i(x\cdot\xi + t\Omega(\xi))} \widehat{f}(\xi) \, \frac{d\xi}{(1+|\xi|^2)^{\frac{s}{2}}},$$

where $w(r) = (1+r)^{-\frac{b}{2}}, b > 0$ and

$$T^*f(x) = \sup_{t\in\mathbb{R}} |Tf(x,t)|.$$

Then Theorem 1.1 follows immediately from

Theorem 1.2. *For any $\varepsilon > 0$ and $b > 1$, if $f \in L_r^2 H_\omega^{\frac{n-1}{2}-s+\varepsilon}$ for some $s \in [\frac14, \frac12)$, there exists a constant C, depending only on $a, c_1, c_2, N, n, s, \varepsilon, b$, such that*

$$||T^*f||_{L^2} \leq C||f||_{L_r^2 H_\omega^{\frac{n-1}{2}-s+\varepsilon}}.$$

The maximal function u^* and operator T^* have been studied extensively by many authors ([1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 18, 19, 21]). P. Sjölin [14] and L. Vega [19] showed that for some ball B_R of radius R

$$\|u^*\|_{L^2(B_R)} \leq C\|f\|_{H^s}, \tag{1.1}$$

only if $s \geq \frac14$. Up to now, it is known that (1.1) is true if $n = 1$ ([5, 8]) or the initial data is radial ([4, 12]), or $s > \frac12$ and $n \geq 2$ ([11, 19]). Recently, T. Tao [18] obtained (1.1) for $s > \frac25$ and $n = 2$. However, the sufficiency remains open widely.

On the other hand, Theorem 1.1 shows that it is true for $s = \frac14$ if we assume the additional angular regularity. If the initial data is a finite linear combination of radial functions and spherical harmonics such that $f = \sum_{k\le L} f_k Y_k$, it was proved by the first and third authors in [4] that $\|u^*\|_{L^{\frac{4n}{2n-1}}} \le C_L \|f\|_{H^{\frac14}}$, where

$$C_L \le C L^{\frac12+\varepsilon}(n+2L)^{\frac{n+2L}{2}} \max_{1\le k\le L} \frac{\|Y_k\|_{L^{\frac{4n}{2n-1}}}}{\|Y_k\|_{L^2}} \quad (0<\varepsilon\ll 1).$$

The factor $(n+2L)^{\frac{n+2L}{2}}$ follows from the asymptotic behavior of Bessel function ($J_\nu(t) \sim b_+ t^{-\frac12} e^{it} + b_- t^{-\frac12} e^{-it} + O((n+2\nu)^{\frac{n+2\nu}{2}})t^{-\frac32}$ for $t>1$). The tail $t^{-\frac32}$ seems inevitable to obtain the non-weighted global $L^{\frac{4n}{2n-1}}$ ($\frac{4n}{2n-1} > 2$) estimate for which a big cost of C_L is paid. In this connection, Theorem 1.1 improves significantly the dependency on the order of spherical harmonic up to $L^{3/4+\varepsilon}$ (see (2.2) below). This improvement occurs from an estimate for the tail of Bessel function Ct^{-1} for $t > 2\nu$, which enables us to use the L^2 method. The weighted L^2 estimate as in Theorem 1.1 is absolutely necessary for a global estimate in view of the negative result that the non-weighted global L^2 estimate [11] and any local estimate in $L^p (p>2)$ [22] are impossible for the data $f \in H^{\frac14}$.

In case that $\Omega(D) = -\Delta$, recently G. Gigante and F. Soria [6] showed a local L^2 estimate, independently of our work, that $\|u^*\|_{L^2(B_R)} \le C L^{\frac12+\varepsilon}\|f\|_{H^{\frac14}}$. They used a finer asymptotic behavior of Bessel function $J_\nu(t)$ for $\nu + \nu^{\frac13} \le t \le 2\nu$ but their method seems not to be applied directly to the general phase Ω because the power of L may depend on Ω in their argument.

From the assumption on Ω, we treat Ω not only of the form $|\xi|^a$ but also $\sum_{i=1}^l m_i|\xi|^{a_i}$ for any number $a_l > a_{l-1} > \cdots > a_1 > 0$, $a_l \ne 1$ and $m_i \in \mathbb{R}$. For the more general phase Ω, we refer the readers to [3] in which a weighted L^2 estimate is discussed for the phase Ω with $\nabla\Omega$ having zeros or singularities. For another use of angular regularity, one can refer to [9] in which the endpoint Strichartz estimates of 3-d wave and Klein-Gordon equations are considered.

If not specified, throughout this paper, C denotes a generic constant that depends on $a, c_1, c_2, N, n, s, b, \varepsilon$. We use the notation $A \lesssim B$ and $A \sim B$ to denote $|A| \le CB$ and $C^{-1}B \le |A| \le CB$ respectively.

2. Proof of Theorem 1.2

We begin with reviewing some properties of the spherical harmonic expansion. If $f(r\omega) = g(r)Y_k(\omega)$ for a radial function g and a spherical harmonic Y_k of order k, then we have

$$\widehat{f}(\rho\theta) = G(\rho)Y_k(\theta), \quad ||g||_{L_r^2} = ||G||_{L_r^2},$$

where

$$G(\rho) = c_{n,k}\int_0^\infty g(r)r^{n-1}(r\rho)^{-\frac{n-2}{2}}J_\nu(r\rho)\,dr,\ |c_{n,k}| \le C,\ \nu = \frac{2k+n-2}{2}.$$

For the representation of G, see e.g. [16] or [22]. Since $-\Delta_\omega Y_k = k(k+n-2)Y_k$, we also have $||f||_{L_r^2 H_\omega^\beta} \sim (1+k^2)^{\frac{\beta}{2}}||g||_{L_r^2}||Y_k||_{L_\omega^2}$. Furthermore, if $h \in L_r^2 H_\omega^\beta$, then there exist radial functions $\{h_k^l\}$ and spherical harmonics $\{Y_k^l\}$ such that

$$h(r\omega) = \sum_{k\ge 0}\sum_{1\le l\le d(k)} h_k^l(r)Y_k^l(\omega) \quad \text{in} \quad L_r^2 H_\omega^\beta,$$

where $d(k)$ is the dimension of the space of spherical harmonics of degree k, and

$$||h||^2_{L_r^2 H_\omega^\beta} \sim \sum_{k\ge 0}\sum_{1\le l\le d(k)} (1+k^2)^\beta ||h_k^l||^2_{L_r^2}||Y_k^l||^2_{L_\omega^2}. \tag{2.1}$$

Thus for the proof of theorem, we have only to consider the case that $f(r\omega) = g(r)Y_k(\omega)$ and to show that for large k

$$||T^*f||_{L^2} \lesssim k^{\frac12 - s}||g||_{L_r^2}||Y_k||_{L_\omega^2}, \tag{2.2}$$

since for the function $h(r\omega) = \sum_{k\ge 0}\sum_{1\le l\le d(k)} h_k^l(r)Y_k^l(\omega)$ in $L_r^2 H_\omega^\beta$, we have from (2.2)

$$\begin{aligned}
\|T^*h\|_{L^2} &\lesssim \sum_k \sum_{1\le l\le d(k)} k^{\frac12 - s}\|h_k^l\|_{L_r^2}\|Y_k^l\|_{L_\omega^2} \\
&\lesssim \sum_k k^{\frac12 - s} d(k)^{\frac12}\left(\sum_{1\le l\le d(k)} \|h_k^l\|^2_{L_r^2}\|Y_k^l\|^2_{L_\omega^2}\right)^{\frac12} \\
&\lesssim \sum_k k^{\frac{n-1}{2} - s}\left(\sum_{1\le l\le d(k)} \|h_k^l\|^2_{L_r^2}\|Y_k^l\|^2_{L_\omega^2}\right)^{\frac12} \\
&\lesssim \left(\sum_k \sum_{1\le l\le d(k)} k^{n-1-2s+\varepsilon}\|h_k^l\|^2_{L_r^2}\|Y_k^l\|^2_{L_\omega^2}\right)^{\frac12},
\end{aligned}$$

where we used the estimate $d(k) = \frac{n+2k-2}{k}\begin{pmatrix} n+k-3 \\ k-1 \end{pmatrix} \lesssim k^{n-2}$ for the third inequality (see [16]).

Now if $\widehat{f}(\rho\omega) = G(\rho)Y_k(\omega)$, from the definition of T, it follows that

$$\begin{aligned}
Tf(r\omega, t) &= w(r)\int_{S^{n-1}}\int_0^\infty e^{i(r\omega\cdot\rho\theta + t\Omega(\rho))}G(\rho)Y_k(\theta)\rho^{n-1}\frac{d\rho}{(1+\rho^2)^{\frac{s}{2}}}d\theta \\
&= c_{n,k}w(r)\int_0^\infty e^{it\Omega(\rho)}(r\rho)^{-\frac{n-2}{2}}J_\nu(r\rho)\rho^{n-1}G(\rho)\,\frac{d\rho}{(1+\rho^2)^{\frac{s}{2}}}Y_k(-\omega).
\end{aligned}$$

Let us define an operator S by

$$SG(r,t) = c_{n,k}r^{\frac{n-1}{2}}w(r)\int_0^\infty e^{it\Omega(\rho)}(r\rho)^{-\frac{n-2}{2}}J_\nu(r\rho)\rho^{\frac{n-1}{2}}G(\rho)\,\frac{d\rho}{(1+\rho^2)^{\frac{s}{2}}}.$$

Let us denote by $||F||_{L^pL^q}$ the mixed norm $\|(\|F(r,t)\|_{L^q(dt)})\|_{L^p(dr)}$. Here we use the notation $||F||^p_{L^p(dr)}$ for $\int |F(r)|^p\,dr$ to avoid the confusion with $||F||_{L^p_r}$. To prove (2.2) it suffices to show that

$$\|S\tilde{G}\|_{L^2L^\infty} \lesssim k^{\frac{1}{2}-s}||\tilde{G}||_{L^2(dr)}, \tag{2.3}$$

where $\tilde{G}(\rho) = \rho^{\frac{n-1}{2}}G(\rho)$. Now we define the dual operator S^d of S by

$$S^dF(\rho) = \frac{c_{n,k}}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{\frac{1}{2}}J_\nu(r\rho)w(r)F(r,t)\,drdt$$

for $F \in C_0^\infty(\mathbb{R}_+\times\mathbb{R})$. Then, by duality (2.3) follows from

$$||S^dF||_{L^2(dr)} \le Ck^{\frac{1}{2}-s}||F||_{L^2L^1}. \tag{2.4}$$

Choose smooth cut-off functions ϕ_0, ϕ_1 and ϕ_3 so that $\phi_0 = 1$ on $\{|s| < \frac{1}{4}\}$, $\phi_0 = 0$ on $\{|s| > \frac{1}{2}\}$, $\phi_1 = 1$ on $\{|s| \sim 1\}$, $\phi_1 = 0$ otherwise, $\phi_2 = 0$ on $\{|s| < 2\}$, $\phi_2 = 1$ on $\{|s| > 3\}$, and $\phi_0 + \phi_1 + \phi_2 = 1$. Then we decompose S^d as

$$S^dF(\rho) = S_0F + S_1F + S_2F,$$

where for $i = 0, 1, 2$,

$$S_iF(\rho) = \frac{c_{n,k}}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{\frac{1}{2}}J_\nu(r\rho)\phi_i\left(\frac{r\rho}{\nu}\right)w(r)F(r,t)\,drdt.$$

Now we need to show each S_i satisfies (2.4) in the place of S^d. Each estimate is to be shown using the following asymptotic behavior of Bessel functions:

$$|J_\nu(t)| \le C\exp(-C\nu), \quad \text{if} \quad t \le \frac{\nu}{2}, \tag{2.5}$$

$$\frac{1}{r}\int_0^r |J_\nu(t)|^2 t\,dt \le C \quad \text{for all } r > 0, \tag{2.6}$$

$$J_\nu(t)\phi_2(\frac{t}{\nu}) = t^{-\frac{1}{2}}(b_+e^{it} + b_-e^{-it})\phi_2(\frac{t}{\nu}) + \Phi_\nu(t)\phi_2(\frac{t}{\nu}), \tag{2.7}$$

where $|\Phi_\nu(t)| \le \frac{C}{t}$, $|b_\pm| \le C$ and the constant C is independent of ν. For the proof of (2.5), see [17]. The mean value estimate (2.6) can be found in Section 4.10 of [20]. Invoking the Schläfli's integral representation (see p.176 in [23]):

$$J_\nu(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(t\sin\theta - \nu\theta)}\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau - t\sinh\tau}\,d\tau,$$

the last two asymptotic behavior (2.7) follow from the easy estimate

$$\left|\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau-t\sinh\tau}\,d\tau\right| \le \frac{C}{\nu+t}$$

and the method of stationary phase such that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(t\sin\theta-\nu\theta)}\,d\theta = (b_+e^{it} + b_-e^{-it})t^{-\frac{1}{2}} + O(t^{-\frac{3}{2}}) \quad \text{for} \quad t > 2\nu.$$

Using (2.5), we now see

$$
\begin{aligned}
|S_0F(\rho)| &\lesssim \nu^{\frac12}e^{-C\nu}(1+\rho^2)^{-\frac{s}{2}}\int_0^{\frac{\nu}{\rho}} w(r)||F(r,\cdot)||_{L^1}\,dr\\
&= \nu^{\frac12}e^{-C\nu}(1+\rho^2)^{-\frac{s}{2}}\left(\int_0^{\min(\frac{\nu}{\rho},2)}||F(r,\cdot)||_{L^1}\,dr\right.\\
&\qquad\qquad\left.+\int_0^{\frac{\nu}{\rho}}\chi_{[2,\infty)}(r)w(r)||F(r,\cdot)||_{L^1}\,dr\right)\\
&\lesssim \nu^{\frac12}e^{-C\nu}(1+\rho^2)^{-\frac{s}{2}}\left((\min(\frac{\nu}{\rho},\,2))^{\frac12}+\chi_{[0,\frac{\nu}{2}](\rho)}\right)||F||_{L^2L^1}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&||S_0F||_{L^2(dr)}\\
&\lesssim \nu^{\frac12}e^{-C\nu}\left(\int_0^\infty(1+\rho^2)^{-s}(\min(\frac{\nu}{\rho},\,2)+\chi_{[0,\frac{\nu}{2}]}(\rho))\,d\rho\right)^{\frac12}||F||_{L^2L^1} \qquad (2.8)\\
&\lesssim \nu^{1-s}e^{-C\nu}||F||_{L^2L^1}.
\end{aligned}
$$

For S_1, we have

$$
\begin{aligned}
|S_1F(\rho)| &\lesssim (1+\rho^2)^{-\frac{s}{2}}\left(\int_0^\infty J_\nu^2(r\rho)r\rho\phi_1^2\left(\frac{r\rho}{\nu}\right)w(r)^2\,dr\right)^{\frac12}||F||_{L^2L^1}\\
&\lesssim (1+\rho^2)^{-\frac{s}{2}}\left(\int_0^2+\int_2^\infty\right)^{\frac12}\|F\|_{L^2L^1}.
\end{aligned}
$$

Using the change of variable $r\mapsto r/\rho$, the first part in the middle parenthesis is bounded by $\chi_{[\frac{\nu}{4},\infty)}(\rho)\frac{1}{\rho}\int_0^{2\rho}J_\nu^2(r)r\phi_1^2(r/\nu)\,dr$. By (2.6), it follows that

$$
\int_0^2 \lesssim \nu\rho^{-1}\chi_{[\frac{\nu}{4},\infty)}(\rho).
$$

For the second part, we also use the change of variable $r\mapsto r/\rho$ and then by (2.6) have that

$$
\int_2^\infty \lesssim \rho^{b-1}\int_{\max(2\rho,\frac{\nu}{2})}^{3\nu}J_\nu^2(r)r^{1-b}\,dr \lesssim \nu\rho^{b-1}(\max(2\rho,\frac{\nu}{2}))^{-b}.
$$

We thus obtain

$$
\begin{aligned}
&||S_1F||_{L^2(dr)}\\
&\lesssim \left(\int_0^\infty(1+\rho^2)^{-s}(\nu\rho^{-1}\chi_{[\frac{\nu}{4},\infty)}(\rho)+\nu\rho^{b-1}(\max(2\rho,\frac{\nu}{2}))^{-b})\,d\rho\right)^{\frac12}||F||_{L^2L^1} \qquad (2.9)\\
&\lesssim \nu^{\frac12-s}||F||_{L^2L^1}.
\end{aligned}
$$

Now we estimate S_2F. Let us set $S_2F=S_+F+S_-F+S_3F$, where

$$
\begin{aligned}
S_\pm F(\rho) &= \frac{c_{n,k}b_\pm}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{i(\pm r\rho-t\Omega(\rho))}\phi_2(\frac{r\rho}{\nu})w(r)F(r,t)\,drdt\\
S_3F(\rho) &= \frac{c_{n,k}}{(1+\rho^2)^{\frac{s}{2}}}\int_{\mathbb{R}}\int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{\frac12}\Phi_\nu(r\rho)\phi_2\left(\frac{r\rho}{\nu}\right)w(r)F(r,t)\,drdt.
\end{aligned}
$$

For the estimate $S_\pm F$, it suffices to consider S_+F. We decompose it into two parts as follows:

$$S_+F(\rho) = I + II$$

where

$$I = \frac{c_{n,k}b_+}{(1+\rho^2)^{\frac{s}{2}}} \int_{\mathbb{R}} \int_0^\infty e^{i(r\rho - t\Omega(\rho))} w(r)F(r,t)\, drdt,$$

$$II = \frac{c_{n,k}b_+}{(1+\rho^2)^{\frac{s}{2}}} \int_{\mathbb{R}} \int_0^\infty e^{i(r\rho - t\Omega(\rho))} (\phi_2(\frac{r\rho}{\nu}) - 1) w(r)F(r,t)\, drdt.$$

For II, we have

$$\begin{aligned} |II(\rho)| &\lesssim (1+\rho^2)^{-\frac{s}{2}} \int_0^{\frac{3\nu}{\rho}} w(r) ||F(r,\cdot)||_{L^1}\, dr \\ &\lesssim (1+\rho^2)^{-\frac{s}{2}} \left(\int_0^{\frac{3\nu}{\rho}} w(r)^2\, dr \right)^{\frac{1}{2}} ||F||_{L^2L^1} \end{aligned}$$

and hence by the similar estimate to (2.8) for S_0F

$$||II||_{L^2(dr)} \lesssim \nu^{\frac{1}{2}-s} ||F||_{L^2L^1}. \tag{2.10}$$

Now we estimate I. Since F is in $C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, obviously we may assume

$$I = \frac{c_{n,k}b_+}{(1+\rho^2)^{\frac{s}{2}}} \int_{\mathbb{R}^2} e^{i(r\rho - t\Omega(\rho))} w(|r|)F(r,t)\, drdt.$$

Squaring and integrating I over $\{|\rho| \le N\}$, where N is the number in the condition of Ω, we have

$$\int_{|\rho|<N} |I|^2\, d\rho \le C||F||^2_{L^2L^1}. \tag{2.11}$$

Now it is easy to see

$$\begin{aligned} &\int_{|\rho|>N} |I|^2 d\rho \\ &\le C \iiiint |K(r-r', t-t') w(|r|)|F(r,t)| w(|r'|)|F(r',t')|\, drdr'dtdt', \end{aligned}$$

where

$$K(r,t) = \int_{|\rho|>N} e^{i(r\rho - t\Omega(\rho))} \frac{d\rho}{|\rho|^{2s}}.$$

For the kernel estimate, we introduce a lemma which shows uniform bound of kernel K on t.

Lemma 2.1 (see Lemma 2.3 in [4]). *For any real number $A, B (A \neq 0)$ and $s \in [\frac{1}{2}, 1)$, there exists a constant C independent of A and B such that*

$$\left| \int_{|\rho|>N} e^{i(A\Omega(\rho) + B\rho)} \frac{d\rho}{|\rho|^s} \right| \le C|B|^{-(1-s)}.$$

Applying Lemma 2.1 with $2s$ ($\frac{1}{4} \le s < \frac{1}{2}$) and $B = r - r'$, from fractional integration and Hölder inequality it follows

$$
\begin{aligned}
&\int_{|\rho|>N} |I|^2 d\rho \\
&\qquad \lesssim \iint |r-r'|^{-(1-2s)} w(|r|)||F(r,\cdot)||_{L^1} w(|r'|)||F(r',\cdot)||\, dr dr' \\
&\qquad \lesssim ||\mathcal{I}_{2s}(w||F||_{L^1})||_{L^p} ||w||F||_{L^1}||_{L^{p'}} \qquad \left(\frac{1}{p} = \frac{1}{p'} - 2s\right) \\
&\qquad \lesssim ||wF||^2_{L^{\frac{2}{1+2s}} L^1} \lesssim \|w\|^2_{L^{\frac{1}{s}}} \|F\|^2_{L^2 L^1} \qquad \left(\frac{b}{2} \cdot \frac{1}{s} > 1\right) \\
&\qquad \lesssim \|F\|^2_{L^2 L^1},
\end{aligned}
\tag{2.12}
$$

where $\mathcal{I}_{2s}$ is the Riesz potential of order $2s$.

Finally, we estimate $S_3 F$. From the uniform bound of Φ_ν on ν, for small $\varepsilon > 0$, we have

$$
\begin{aligned}
&|S_3 F(\rho)| \\
&\lesssim \frac{1}{(1+\rho^2)^{\frac{s}{2}}} \int (r\rho)^{-\frac{1}{2}} \phi_2\left(\frac{r\rho}{\nu}\right) w(r) ||F(r,\cdot)||_{L^1}\, dr \\
&\lesssim \rho^{-s-\frac{1}{2}} \chi_{[\nu,\infty)}(\rho) \int_{\frac{2\nu}{\rho}}^{2} r^{-\frac{1}{2}} ||F(r,\cdot)||_{L^1}\, dr + \rho^{-s-\frac{1}{2}} \int_{\max(2,\frac{2\nu}{\rho})} r^{-\frac{1}{2}-\frac{b}{2}} \|F(r,\cdot)\|_{L^1}\, dr \\
&\lesssim \nu^{-\delta} \rho^{-s-\frac{1}{2}+\delta} \chi_{[\nu,\infty)}(\rho) \int_{\frac{2\nu}{\rho}}^{2} r^{-\frac{1}{2}+\delta} ||F(r,\cdot)||_{L^1}\, dr + \rho^{-s-\frac{1}{2}} \left(\max(2, \frac{2\nu}{\rho})\right)^{-\frac{b}{2}} \|F\|_{L^2 L^1} \\
&\lesssim \left(\nu^{-\delta} \rho^{-s-\frac{1}{2}+\delta} \chi_{[\nu,\infty)}(\rho) + \rho^{-s-\frac{1}{2}} \left(\max(2, \frac{2\nu}{\rho})\right)^{-\frac{b}{2}}\right) ||F||_{L^2 L^1}.
\end{aligned}
$$

Choosing δ as $\frac{1}{8}$, we obtain

$$
||S_3 F||_{L^2(dr)} \lesssim \nu^{-s} ||F||_{L^2 L^1}. \tag{2.13}
$$

Combining all the estimates (2.8) to (2.13) and recalling $\nu = \frac{2k+n-2}{2}$, we get (2.4) and hence Theorem 1.2.

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