A MAXIMAL INEQUALITY ASSOCIATED TO SCHRÖDINGER TYPE EQUATION

YONGGEUN CHO, SANGHYUK LEE, AND YONGSUN SHIM

Abstract. In this note, we consider a maximal operator sup_{t \in \mathbb{R}} |u(x, t)| = sup_{t \in \mathbb{R}} |e^{it\Omega(D)} f(x)|, where u is the solution to the initial value problem \( u_t = i\Omega(D)u \), \( u(0) = f \) for a \( C^2 \) function \( \Omega \) with some growth rate at infinity. We prove that the operator sup_{t \in \mathbb{R}} |u(x, t)| has a mapping property from a fractional Sobolev space \( H^{1/4} \) with additional angular regularity in which the data lives to \( L^2((1 + |x|)^{-b} \, dx) \) (\( b > 1 \)). This mapping property implies the almost everywhere convergence of \( u(x, t) \) to \( f \) as \( t \to 0 \), if the data \( f \) has an angular regularity as well as \( H^{1/4} \) regularity.

1. Introduction

We consider the following free Schrödinger type equation:
\[
\frac{\partial}{\partial t} u(x, t) = i\Omega(D)u(x, t) \quad \text{in} \quad \mathbb{R}^{n+1} (n \geq 2), \quad u(x, 0) = f(x),
\]
where \( \Omega(D) \) is a generalized differential operator defined by a \( C^2 \) function \( \Omega \) and \( D = (-\Delta)^{1/2} \). For smooth initial data \( f \), the solution \( u(x, t) = e^{it\Omega(D)} f \) can be written as
\[
u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t\Omega(\xi))} \hat{f}(\xi) \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),
\]
where \( \hat{f}(\xi) = \int e^{-ix \cdot \xi} f(x) \, dx \). In this note, we assume that the initial data \( f \) has \( H^s \) regularity for some \( s > 0 \) as well as some regularity in the angular direction. For \( \alpha, \beta \geq 0 \), we define an initial data space \( H^\alpha_r H^\beta_\omega \) by
\[
H^\alpha_r H^\beta_\omega = \left\{ f : \|f\|_{H^\alpha_r H^\beta_\omega} := \|(1 - \Delta)^{\beta/2} f\|_{L^2_r H^\alpha_\omega} < \infty \right\},
\]
where \( \|g\|_{L^2_r}^2 = \int_0^\infty |g(r)|^2 r^{n-1} \, dr \), \( \|g\|_{L^2_r H^\beta_\omega} = \|1 - \Delta_\omega\| \|g\|_{L^2_r H^\beta_\omega} \) (here, \( (r, \omega) \in \mathbb{R}_+ \times S^{n-1} \) is the spherical coordinates), and \( \Delta_\omega \) is the Laplace-Beltrami operator on \( S^{n-1} \). Since \( \Delta_\omega \) commutes with \( \Delta \), one can readily check that \( \|g\|_{H^\alpha_r H^\beta_\omega} \sim \|(1 - \Delta_\omega)^{\beta/2} g\|_{H^\alpha} \) (for instance, see [9]). Since not every function in \( H^\alpha_r H^\beta_\omega \) has radial regularity higher than \( \alpha \), there is no embedding from or into a usual Sobolev

2000 Mathematics Subject Classification. Primary 42B25; Secondary 42A45.
Keywords and phrases. Schrödinger type equation, maximal operator, angular regularity.
The first author was supported by JSPS and the second author is partially supported by the Post-doctoral Fellowship Program of Korea Science & Engineering Foundation (KOSEF).
space. In particular, it should be noted that $H^\alpha H^\beta \not\subseteq H^{\alpha+\gamma}$ ($0 < \gamma < \beta$) and $H^\alpha H^\beta \not\subseteq H^{\alpha+\gamma}$ ($\gamma \geq \beta$).

We also assume that $\Omega \in C^2(\mathbb{R}^n)$ is radially symmetric and satisfies
\[
c_1|\rho|^{a-k} \leq |\Omega^{(k)}(\rho)| \leq c_2|\rho|^{a-k} \quad (k = 0, 1, 2), \quad \text{if} \quad |\rho| \geq N
\]
for some $c_1, c_2, a > 0$ with $a \neq 1$ and a large $N > 0$. With the above assumptions, let us define a maximal function $u^*(x)$ by $u^*(x) = \sup_{t \in \mathbb{R}} |u(x, t)|$. We prove Theorem 1.1.

**Theorem 1.1.** For any $\varepsilon > 0$ and $b > 1$, if $f \in H^{\frac{1}{4}} L^{\frac{n-1}{2} - \frac{1}{4} + \varepsilon}$, then there exists a constant $C$, depending only on $a, c_1, c_2, N, n, \varepsilon, b$, such that
\[
||u^*||_{L^2((1+|x|)^{-b}dx)} \leq C||f||_{H^{\frac{1}{4}} L^{\frac{n-1}{2} - \frac{1}{4} + \varepsilon}}.
\]

Now let us define a linear operator $T$ and a maximal operator $T^*$ for a fixed $s > 0$ by
\[
Tf(x, t) = w(|x|) \int e^{i(x \cdot \xi + \Omega(\xi))} \hat{f}(\xi) \frac{d\xi}{(1+|\xi|^2)^{\frac{s}{2}}},
\]
where $w(r) = (1 + r)^{-\frac{3}{2}}, b > 0$ and
\[
T^*f(x) = \sup_{t \in \mathbb{R}} |Tf(x, t)|.
\]
Then Theorem 1.1 follows immediately from

**Theorem 1.2.** For any $\varepsilon > 0$ and $b > 1$, if $f \in L^2 L^{\frac{n-1}{2} - s + \varepsilon}$ for some $s \in \left[\frac{1}{4}, \frac{1}{2}\right)$, there exists a constant $C$, depending only on $a, c_1, c_2, N, n, \varepsilon, b$, such that
\[
||T^*f||_{L^2} \leq C||f||_{L^2 L^{\frac{n-1}{2} - s + \varepsilon}}.
\]

The maximal function $u^*$ and operator $T^*$ have been studied extensively by many authors ([1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 18, 19, 21]). P. Sjölin [14] and L. Vega [19] showed that for some ball $B_R$ of radius $R$

\[
\|u^*\|_{L^2(B_R)} \leq C\|f\|_{H^s},
\]
only if $s \geq \frac{1}{4}$. Up to now, it is known that (1.1) is true if $n = 1$ ([5, 8]) or the initial data is radial ([4, 12]), or $s > \frac{1}{2}$ and $n \geq 2$ ([11, 19]). Recently, T. Tao [18] obtained (1.1) for $s > \frac{7}{2}$ and $n = 2$. However, the sufficiency remains open widely.

On the other hand, Theorem 1.1 shows that it is true for $s = \frac{1}{4}$ if we assume the additional angular regularity. If the initial data is a finite linear combination of
radial functions and spherical harmonics such that $f = \sum_{k \leq L} f_k Y_k$, it was proved by the first and third authors in [4] that $\|u^*\|_{L^{1 \over 2 + \varepsilon}} \leq C_L \|f\|_{H^{1 \over 2}}$, where

$$C_L \leq CL^{1 \over 2 + \varepsilon} (n + 2L)^{n+2 \over 2} \max_{1 \leq k \leq L} \|Y_k\|_{L^2}$$

(0 < \varepsilon \ll 1). The factor $(n + 2L)^{n+2 \over 2}$ follows from the asymptotic behavior of Bessel function $J_n(t) \sim b_n t^{-1 \over 2} e^{it} + b_{-1} t^{-1 \over 2} e^{-it} + O((n + 2\nu)^{n+2 \over 2}) t^{-1 \over 2}$ for $t > 1$). The tail $t^{-1 \over 2}$ seems inevitable to obtain the non-weighted global $L^{4n \over 2n-1} (4n \over 2n-1 > 2)$ estimate for which a big cost of $C_L$ is paid. In this connection, Theorem 1.1 improves significantly the dependency on the order of spherical harmonic up to $L^{1 \over 4 + \varepsilon}$ (see (2.2) below). This improvement occurs from an estimate for the tail of Bessel function $C t^{-1}$ for $t > 2\nu$, which enables us to use the $L^2$ method. The weighted $L^2$ estimate as in Theorem 1.1 is absolutely necessary for a global estimate in view of the negative result that the non-weighted global $L^2$ estimate [11] and any local estimate in $L^p(p > 2)$ [22] are impossible for the data $f \in H^{1 \over 2}$.

In case that $\Omega(D) = -\Delta$, recently G. Gigante and F. Soria [6] showed a local $L^2$ estimate, independently of our work, that $\|u^*\|_{L^2(B_R)} \leq C L^{1 \over 2 + \varepsilon} \|f\|_{H^{1 \over 2}}$. They used a finer asymptotic behavior of Bessel function $J_n(t)$ for $\nu + \nu^2 \leq t \leq 2\nu$ but their method seems not to be applied directly to the general phase $\Omega$ because the power of $L$ may depend on $\Theta$ in their argument.

From the assumption on $\Omega$, we treat $\Omega$ not only of the form $|\xi|^a$ but also $\sum_{i=1}^l m_i |\xi|^{a_i}$ for any number $a_l > a_{l-1} > \cdots > a_1 > 0$, $a_l \neq 1$ and $m_i \in \mathbb{R}$. For the more general phase $\Omega$, we refer the readers to [3] in which a weighted $L^2$ estimate is discussed for the phase $\Omega$ with $\nabla \Omega$ having zeros or singularities. For another use of angular regularity, one can refer to [9] in which the endpoint Strichartz estimates of 3-d wave and Klein-Gordon equations are considered.

If not specified, throughout this paper, $C$ denotes a generic constant that depends on $\alpha, c_1, c_2, N, u, s, b, \varepsilon$. We use the notation $A \lesssim B$ and $A \sim B$ to denote $|A| \leq CB$ and $C^{-1}B \leq |A| \leq CB$ respectively.

2. Proof of Theorem 1.2

We begin with reviewing some properties of the spherical harmonic expansion. If $f(r\omega) = g(r)Y_k(\omega)$ for a radial function $g$ and a spherical harmonic $Y_k$ of order $k$, then we have

$$\hat{f}(\rho\theta) = G(\rho)Y_k(\theta)$$

$$\|g\|_{L^2} = \|G\|_{L^2}$$.
Let us define an operator $G(\rho) = c_{n,k} \int_0^\infty g(r) r^{n-1} \frac{\rho^2}{r^2} J_\nu(\rho r) \, dr$, where $|c_{n,k}| \leq C$, $\nu = \frac{2k + n - 2}{2}$.

For the representation of $G$, see e.g. [16] or [22]. Since $-\Delta_k Y_k = k(k + n - 2)Y_k$, we also have $\|f\|_{L^2_k H^2_\infty} \sim (1 + k^2)^\frac{n}{2} \|g\|_{L^2_k} \|Y_k\|_{L^2_k}$. Furthermore, if $h \in L^2_k H^3_\infty$, there exist radial functions $\{h_k^s\}$ and spherical harmonics $\{Y_k^l\}$ such that

$$h(r \omega) = \sum_{k \geq 0} \sum_{1 \leq l \leq d(k)} h_k^l(r) Y_k^l(\omega) \quad \text{in} \quad L^2_k H^3_\infty,$$

where $d(k)$ is the dimension of the space of spherical harmonics of degree $k$, and

$$\|h\|_{L^2_k H^3_\infty} \sim \sum_{k \geq 0} \sum_{1 \leq l \leq d(k)} (1 + k^2)^\beta \|h_k^l\|_{L^2_k} \|Y_k^l\|_{L^2_k}.$$

Thus for the proof of theorem, we have only to consider the case that $f(r \omega) = g(r)Y_k(\omega)$ and to show that for large $k$

$$\|T^* f\|_{L^2_k} \lesssim k^{-s} \|g\|_{L^2_k} \|Y_k\|_{L^2_k},$$

since for the function $h(r \omega) = \sum_{k \geq 0} \sum_{1 \leq l \leq d(k)} h_k^l(r) Y_k^l(\omega)$ in $L^2_k H^3_\infty$, we have from (2.2)

$$\|T^* h\|_{L^2_k} \lesssim \sum_k \sum_{1 \leq l \leq d(k)} k^{\frac{n}{2} - s} \|h_k^l\|_{L^2_k} \|Y_k^l\|_{L^2_k}$$

$$\lesssim \sum_k k^{\frac{n}{2} - s} d(k) \left( \sum_{1 \leq l \leq d(k)} \|h_k^l\|_{L^2_k} \|Y_k^l\|_{L^2_k} \right)^\frac{1}{2}$$

$$\lesssim \sum_k k^{\frac{n}{2} - s - \epsilon} \left( \sum_{1 \leq l \leq d(k)} \|h_k^l\|_{L^2_k} \|Y_k^l\|_{L^2_k} \right)^\frac{1}{2}$$

$$\lesssim \left( \sum_k \sum_{1 \leq l \leq d(k)} k^{n - 2s + \epsilon} \|h_k^l\|_{L^2_k} \|Y_k^l\|_{L^2_k} \right)^\frac{1}{2},$$

where we used the estimate $d(k) = \frac{n + 2k - 2}{k} \left( \frac{n + k - 3}{k - 1} \right) \lesssim k^{n/2}$ for the third inequality (see [16]).

Now if $\tilde{f}(\rho \omega) = G(\rho) Y_k(\omega)$, from the definition of $T$, it follows that

$$Tf(r \omega, t) = w(r) \int_{S_{n-1}} e^{i(r \omega \cdot \rho \theta + t \Omega(\rho))} G(\rho) Y_k(\theta) \rho^{n-1} \frac{d \rho}{(1 + \rho^2)^{\frac{n}{2}}} d \theta$$

$$= c_{n,k} w(r) \int_0^\infty e^{it \Omega(\rho)} (\rho)^{-\frac{n-2}{2}} J_\nu(\rho r) \rho^{n-1} G(\rho) \frac{d \rho}{(1 + \rho^2)^{\frac{n}{2}}} Y_k(-\omega).$$

Let us define an operator $S$ by

$$SG(r, t) = c_{n,k} w(r) \int_0^\infty e^{it \Omega(\rho)} (\rho)^{-\frac{n-2}{2}} J_\nu(\rho r) \rho^{n-1} G(\rho) \frac{d \rho}{(1 + \rho^2)^{\frac{n}{2}}}.$$
Let us denote by \( \|F\|_{L^p L^q} \) the mixed norm \( \|(\|F(r,t)\|_{L^p(dt)})\|_{L^q(dr)} \). Here we use the notation \( \|F\|_{L^p L^q}^p \) for \( \int |F(r)|^p dr \) to avoid the confusion with \( \|F\|_{L^p} \). To prove (2.2) it suffices to show that

\[
(2.3) \quad \|S\tilde{G}\|_{L^q L^q} \lesssim k^{1/2} \|\tilde{G}\|_{L^q L^q},
\]

where \( \tilde{G}(\rho) = \rho^{n+1} G(\rho) \). Now we define the dual operator \( S^d \) of \( S \) by

\[
S^d F(\rho) = \frac{c_{n,k}}{(1 + \rho^2)^{1/2}} \int_{\mathbb{R}} \int_0^\infty e^{-it\Omega(\rho)} (\rho r)^{1/2} J_{i/2}(\rho r) w(r) F(r, t) dr dt
\]

for \( F \in C_0^\infty (\mathbb{R}_+ \times \mathbb{R}) \). Then, by duality (2.3) follows from

\[
(2.4) \quad \|S^d F\|_{L^q L^q} \leq C k^{1/2} \|F\|_{L^q L^q}.
\]

Choose smooth cut-off functions \( \phi_0, \phi_1 \) and \( \phi_3 \) so that \( \phi_0 = 1 \) on \( \{|s| < \frac{1}{4}\} \), \( \phi_0 = 0 \) on \( \{|s| \geq \frac{1}{4}\} \), \( \phi_1 = 1 \) on \( \{|s| \sim 1\} \), \( \phi_1 = 0 \) otherwise, \( \phi_2 = 0 \) on \( \{|s| < 2\} \), \( \phi_2 = 1 \) on \( \{|s| > 2\} \), and \( \phi_0 + \phi_1 + \phi_2 = 1 \). Then we decompose \( S^d \) as

\[
S^d F(\rho) = S_0 F + S_1 F + S_2 F,
\]

where for \( i = 0, 1, 2 \),

\[
S_i F(\rho) = \frac{c_{n,k}}{(1 + \rho^2)^{1/2}} \int_{\mathbb{R}} \int_0^\infty e^{-it\Omega(\rho)} (\rho r)^{1/2} J_{i/2}(\rho r) \phi_i (\frac{r \rho}{\nu}) w(r) F(r, t) dr dt.
\]

Now we need to show each \( S_i \) satisfies (2.4) in the place of \( S^d \). Each estimate is to be shown using the following asymptotic behavior of Bessel functions:

\[
(2.5) \quad |J_\nu(t)| \leq C \exp(-Ct), \quad \text{if} \quad t \leq \frac{\nu}{2},
\]

\[
(2.6) \quad \frac{1}{r} \int_0^r |J_\nu(t)|^2 t dt \leq C \quad \text{for all} \quad r > 0,
\]

\[
(2.7) \quad J_\nu(t) \phi_2 (\frac{t}{\nu}) = t^{-\frac{1}{2}} (b_+ e^{it} + b_- e^{-it}) \phi_2 (\frac{t}{\nu}) + \Phi_\nu(t) \phi_2 (\frac{t}{\nu}),
\]

where \( |\Phi_\nu(t)| \leq \frac{C}{\tau}, \quad |b_\pm| \leq C \) and the constant \( C \) is independent of \( \nu \). For the proof of (2.5), see [17]. The mean value estimate (2.6) can be found in Section 4.10 of [20]. Invoking the Schl"afli’s integral representation (see p.176 in [23]):

\[
J_\nu(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(t \sin \theta - \nu \theta)} d\theta - \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu t - t \sinh \tau} d\tau,
\]

the last two asymptotic behavior (2.7) follow from the easy estimate

\[
\left| \frac{\sin(\nu \pi)}{\pi} \int_0^\infty e^{-\nu t - t \sinh \tau} d\tau \right| \leq \frac{C}{\nu + t}
\]

and the method of stationary phase such that

\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i(t \sin \theta - \nu \theta)} d\theta = (b_+ e^{it} + b_- e^{-it}) t^{-1/2} + O(t^{-3/2}) \quad \text{for} \quad t > 2\nu.
\]
Using (2.5), we now see
\[ |S_0 F(\rho)| \lesssim \nu^{1/2} e^{-C\nu}(1 + \rho^2)^{-\frac{1}{2}} \int_0^\infty w(r) (|F(r, \cdot)|_{L^1})^2 dr \]

\[ = \nu^{1/2} e^{-C\nu}(1 + \rho^2)^{-\frac{1}{2}} \left( \int_0^{\min(\rho, 2)} (|F(r, \cdot)|_{L^1})^2 dr + \int_0^\infty \chi_{[2, \infty]}(r) w(r) (|F(r, \cdot)|_{L^1})^2 dr \right) \]

\[ \lesssim \nu^{1/2} e^{-C\nu}(1 + \rho^2)^{-\frac{1}{2}} \left( (\min(\nu, 2))^{\frac{1}{2}} + \chi_{[0, \frac{1}{2}]}(\rho) \right) ||F||_{L^2 L^1}. \]

Thus we have
\[ ||S_0 F||_{L^2(\rho dr)} \lesssim \nu^{1/2} e^{-C\nu} \left( \int_0^\infty (1 + \rho^2)^{-s} \left( \min(\nu, 2) + \chi_{[0, \frac{1}{2}]}(\rho) \right) d\rho \right)^{\frac{1}{2}} ||F||_{L^2 L^1} \]

\[ \lesssim \nu^{1-s} e^{-C\nu} ||F||_{L^2 L^1}. \]  

For \( S_1 \), we have
\[ |S_1 F(\rho)| \lesssim (1 + \rho^2)^{-\frac{1}{2}} \left( \int_0^\infty J^2_{\nu}(r\rho) r\rho \phi_2^2 \left( \frac{r\rho}{\nu} \right) w(r)^2 dr \right)^{\frac{1}{2}} ||F||_{L^2 L^1} \]

\[ \lesssim (1 + \rho^2)^{-\frac{1}{2}} \left( \int_0^2 + \int_2^\infty \right)^{\frac{1}{2}} ||F||_{L^2 L^1}. \]

Using the change of variable \( r \mapsto r/\nu \), the first part in the middle parenthesis is bounded by \( \chi_{[\frac{\nu}{2}, \infty]}(\rho) \frac{1}{\nu} \int_0^{2\nu} J^2_{\nu}(r\rho) r\rho \phi_2^2 (r/\nu) dr \). By (2.6), it follows that
\[ \int_0^2 \lesssim \nu^{-1} \chi_{[\frac{\nu}{2}, \infty]}(\rho). \]

For the second part, we also use the change of variable \( r \mapsto r/\nu \) and then by (2.6) have that
\[ \int_2^\infty \lesssim \rho^{-1} \int_{\max(2\nu, \frac{1}{\nu})}^{3\nu} J^2_{\nu}(r) r^{1-b} dr \lesssim \nu \rho^{b-1} (\max(2\nu, \frac{1}{\nu}))^{-b}. \]

We thus obtain
\[ ||S_1 F||_{L^2(\rho dr)} \lesssim \nu^{\frac{1}{2}-s} ||F||_{L^2 L^1}. \]

(2.9) \[ \lesssim \left( \int_0^\infty (1 + \rho^2)^{-s} (\nu \rho^{-1} \chi_{[\nu, \infty]}(\rho) + \nu \rho^{b-1} (\max(2\nu, \frac{1}{\nu}))^{-b}) d\rho \right)^{\frac{1}{2}} ||F||_{L^2 L^1} \]

\[ \lesssim \nu^{\frac{1}{2}-s} ||F||_{L^2 L^1}. \]

Now we estimate \( S_2 F \). Let us set \( S_2 F = S_+ F + S_- F + S_3 F \), where
\[ S_{\pm} F(\rho) = \frac{c_n b^2}{(1 + \rho^2)^{\frac{1}{2}}} \int_0^\infty e^{i\pm \rho t} \Omega(\rho) \Phi_2(\frac{\rho^2}{\nu}) w(r) F(r, t) dr dt \]

\[ S_3 F(\rho) = \frac{c_n}{(1 + \rho^2)^{\frac{1}{2}}} \int_0^\infty e^{-i \Omega(\rho) t} r^2 \Phi_2(\frac{\rho^2}{\nu}) w(r) F(r, t) dr dt. \]
For the estimate $S_\pm F$, it suffices to consider $S_+ F$. We decompose it into two parts as follows:

$$S_+ F(\rho) = I + II$$

where

$$I = \frac{c_n k_+}{(1 + \rho^2)^2} \int_{\mathbb{R}} \int_0^\infty e^{i(\rho \Omega(\rho))} w(r) F(r, t) \, dr \, dt,$$

$$II = \frac{c_n k_+}{(1 + \rho^2)^2} \int_{\mathbb{R}} \int_0^\infty e^{i(\rho \Omega(\rho))} (\phi_2(\frac{\rho}{\nu}) - 1) w(r) F(r, t) \, dr \, dt.$$

For $II$, we have

$$|II(\rho)| \lesssim (1 + \rho^2)^{-\frac{1}{2}} \int_0^{\frac{\rho^2}{2}} w(r) ||F(r, \cdot)||_{L^1} \, dr$$

$$\lesssim (1 + \rho^2)^{-\frac{1}{2}} \left( \int_0^{\frac{\rho^2}{2}} w(r)^2 \, dr \right)^{\frac{1}{2}} ||F||_{L^2 L^1},$$

and hence by the similar estimate to (2.8) for $S_0 F$

(2.10)

$$||II||_{L^2 L^1} \lesssim \nu^{-1-s} ||F||_{L^2 L^1}.$$

Now we estimate $I$. Since $F$ is in $C_0^\infty(\mathbb{R} \times \mathbb{R})$, obviously we may assume

$$I = \frac{c_n k_+}{(1 + \rho^2)^2} \int_{\mathbb{R}^2} e^{i(\rho \Omega(\rho))} w(|r|) F(r, t) \, dr \, dt.$$

Squaring and integrating $I$ over $\{|\rho| \leq N\}$, where $N$ is the number in the condition of $\Omega$, we have

(2.11)

$$\int_{|\rho| \leq N} |I|^2 \, d\rho \leq C||F||_{L^2 L^1}^2.$$

Now it is easy to see

$$\int_{|\rho| > N} |I|^2 \, d\rho$$

$$\leq C \int_{|\rho| > N} \int_{|r|} |K(r - r', t - t') w(|r|) F(r, t) w(|r'|) F(r', t')| \, dr \, dr' \, dt \, dt',$$

where

$$K(r, t) = \int_{|\rho| > N} e^{i(\rho \Omega(\rho))} \frac{d\rho}{|\rho|^{2s}}.$$

For the kernel estimate, we introduce a lemma which shows uniform bound of kernel $K$ on $t$.

**Lemma 2.1** (see Lemma 2.3 in [4]). For any real number $A, B (A \neq 0)$ and $s \in [\frac{1}{2}, 1)$, there exists a constant $C$ independent of $A$ and $B$ such that

$$\int_{|\rho| > N} e^{i(A \Omega(\rho) + B \rho)} \frac{d\rho}{|\rho|^{2s}} \leq C|B|^{-\frac{1}{2-s}}.$$
Applying Lemma 2.1 with \(2s \left( \frac{1}{4} \leq s < \frac{1}{2} \right) \) and \(B = r - r'\), from fractional integration and Hölder inequality it follows

\[
\int_{|\rho| > N} |I|^2 d\rho \\
\lesssim \iint |r - r'|^{-\left(1 - 2s\right)} w(|r|) |F(r, \cdot)||F(r', \cdot)||w(|r'|)| dr dr' \\
\lesssim \|\mathcal{I}_{2s}(w||F||_{L^1})\|_{L^p} \|w||F||_{L^1}||_{L^{p'}} \left( \frac{1}{p'} = \frac{1}{p} - 2s \right) \\
\lesssim \|wF\|_{L^{\frac{3}{2}}(\mathbb{R}^3 \times \mathbb{R}^3)} \lesssim \|w\|_{L^2}^2 \|F\|_{L^2}^2 \\
\lesssim \|F\|^2_{L^2 L^1},
\]

(2.12)

where \(\mathcal{I}_{2s}\) is the Riesz potential of order \(2s\).

Finally, we estimate \(S_3 F\). From the uniform bound of \(\Phi_\nu\) on \(\nu\), for small \(\varepsilon > 0\), we have

\[
|S_3 F(\rho)| \\
\lesssim \frac{1}{(1 + \rho^2)^{\varepsilon}} \int (r\rho)^{-\frac{1}{2}} \Phi_2 \left( \frac{r\rho}{\nu} \right) w(r) |F(r, \cdot)||_{L^1} dr \\
\lesssim \rho^{-s-\frac{1}{2}} \chi_{[\nu, \infty)}(\rho) \int \frac{1}{\rho} r^{-\frac{3}{2}} |F(r, \cdot)||_{L^1} dr + \rho^{-s-\frac{1}{2}} \int_{\max(2, \frac{2\nu}{\rho})}^\infty r^{-\frac{1}{2} - \frac{1}{2}} |F(r, \cdot)||_{L^1} dr \\
\lesssim \nu^{-\delta} \rho^{-s-\frac{1}{2} + \delta} \chi_{[\nu, \infty)}(\rho) \int \frac{1}{\rho} r^{-\frac{3}{2} + \delta} |F(r, \cdot)||_{L^1} dr + \rho^{-s-\frac{1}{2}} \left( \max\left(2, \frac{2\nu}{\rho}\right) \right)^{-\frac{1}{2}} \|F\|_{L^2 L^1} \\
\lesssim \left( \nu^{-\delta} \rho^{-s-\frac{1}{2} + \delta} \chi_{[\nu, \infty)}(\rho) + \rho^{-s-\frac{1}{2}} \left( \max\left(2, \frac{2\nu}{\rho}\right) \right)^{-\frac{1}{2}} \right) \|F\|_{L^2 L^1}.
\]

Choosing \(\delta = \frac{1}{2}\), we obtain

\[
\|S_3 F\|_{L^2(\rho dr)} \lesssim \nu^{-s} \|F\|_{L^2 L^1}.
\]

(2.13)

Combining all the estimates (2.8) to (2.13) and recalling \(\nu = \frac{2k+n-2}{2}\), we get (2.4) and hence Theorem 1.2.

**Acknowledgement.** The authors thank the referees so much for their kind and valuable comments, which improve the presentation of the paper.

**References**


Yonggeun Cho: Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: ygcho@math.sci.hokudai.ac.jp

Sanghyuk Lee: Department of Mathematics, University of Wisconsin-Madison, WI 53706-1388, USA
E-mail address: slee@math.wisc.edu

Yongsun Shim: Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Korea
E-mail address: shim@postech.ac.kr