Introduction to Optimal Control Theory and Hamilton-Jacobi equations

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A priori message from SYHA

"The main purpose of these series of lectures is to make you understand ABC of OCT and help you prepare advanced course on stochastic optimal control theory that you might have in future. "

Lecture Schedules

- Lecture 1: ABC of Optimal Control Theory
- Lecture 2: PMP v.s. Bellman's dynamic programming
- Lecture 3: Hamilton-Jacobi equations (classical theory)
- Lecture 4: Hamilton-Jacobi equations (modern theory)

References

- An introduction to mathematical optimal control theory by Craig Evans. Available at www.math.berkeley.edu/ evans.
- Optimal Control Theory by Donald E. Kirk.
- Introduction to the mathematical theory of control by Alberto Bressan and Benedetto Piccoli.

What do you mean by control of system ?

Control of a system has a double meaning:

- (Weak sense): Checking or testing whether the system's behavior is satisfactory.

- (Strong sense): Putting things in order to guarantee that the system behaves as desired.

Maxims

• "*Since the building of the universe is perfect and is created by the wisdom creator, nothing arises in the universe in which one cannot see the sense of some maximum or minimum.*" by Leonhard Euler

• "The words control theory are, of course, of recent origin, but the subject itself is much older, since it contains the classical calculus of variations as a special case, and the first calculus of variations problems go back to classical Greece." by Hector J. Sussmann.

Lecture 1: ABC of Optimal Control Theory

Goal of OCT

The objective of OCT is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion.

Minimize cost and maximize payoff, utility

"A problem well put is a problem half solved"

Key words of Lecture 1

- Controlled system, state, control, Performance measure
- Controllability, reachable set, linear time-invariant system
- Bang-bang control (principle), Kalman's rank theorem, observability,

Three giants of modern control theory

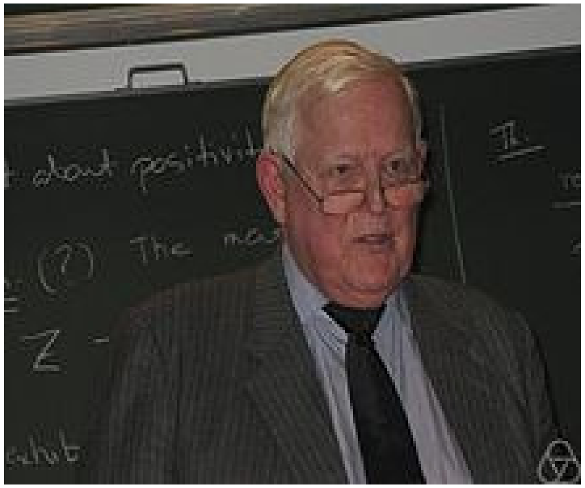
- Richard Bellman (August 26, 1920 - March 19, 1984)

Bellman's dynamic programming (1953)

• Lev Pontryagin (3 September 1908 – 3 May 1988)

Pontryagin's Maximum principle (PMP) (1956)

- Rudolf E. Kalman (19 May 1930)



Kalman filter, Kalman's rank theorem (1960)

Controlled dynamical systems

- (Dynamical system)

$$\dot{x} = f(x,t), \quad x = (x^1, \cdots, x^n), \quad x = x(t) \in R^n.$$

x: state (variable)

- (Controlled dynamical system)

$$\dot{x} = f(x,u,t), \quad u = (u^1, \cdots, u^m), \quad u = u(t) \in R^m.$$

u: control (parameter)

• Example 1: Merton's optimal investment and consumption

◇ Step A: (Modeling of a problem)

Consider an individual whose wealth today is W^0, and who will live exactly T years. His task is to plan the rate of consumption of wealth $C(s)$ for $0 < s < T$. All wealth not yet consumed earns interest at a fixed rate r. We have, for simplicity, assigned no utility to final-time wealth (a bequest).

Let $W(t)$ to be the amount of wealth at time t.

$$\begin{cases} \dot{W} = rW - C, & 0 < t \leq T, \\ W(0) = W^0, & \end{cases}$$

$\diamond$ Step B: (Identification of physical constraints)

$$W(t) \geq 0, \quad W(T) = 0, \quad C(t) \geq 0.$$

$\diamond$ Step C: (Performance measure)

$$P[C] = \int_0^T e^{-\rho t} U(C(s))ds, \qquad \max P[C].$$

where ρ is a discounting rate, and U is the utility function of consumption.

- Reformulation as calculus of variation problem

$$\max_{W(\cdot)} \int_0^T e^{-\rho t} h(rW - \dot{W})ds, \quad \text{subject to} \quad W(0) = W^0.$$

• Example 2. (automobile problem): Minimum-time optimal problem

◇ Step A: (Modeling of a problem)

The car is to be driven in a strainght line away from the point 0 to the point e. The distance of the car from 0 at time t is given by $d(t)$. For simplicity, we assume that the car is denoted by the unit point mass that can be accelerate by using the throttle or decelerated by using the brake.

We set

$$\ddot{d}(t) = \alpha(t) + \beta(t),$$

where α and β stand for throttle accelerate and braking deceleration respectively.

Again we set

$$x_1 := d, \quad x_2 := \dot{d}, \quad u_1 := \alpha, \quad u_2 := \beta.$$

Then our controlled dynamical system is given by

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} u(t),$$

and two point boundary conditions:

$$x(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad x(t_f) = \begin{pmatrix} e \\ 0 \end{pmatrix}.$$

$\diamond$ Step B: (Identification of physical constraints)

State constraints

$$0 \le x_1 \le e, \quad 0 \le x_2.$$

Control constraints

$$0 \le u_1 \le M_1, \quad -M_2 \le u_2 \le 0.$$

Assume that the amount of gasoline at time $t = t_0$ is G gallons, and the rate of gas consumption is proportional to both accelerate and speed, thus the amount of gasoline used

$$\int_{t_0}^{t_f} \Big(k_1 u_1(t) + k_2 x_2(t)\Big) dt \le G.$$

$\diamond$ Step C: (Performance measure)

Minimum-time control

$$\text{minimize } J := t_f - t_0.$$

In general, the performance measure takes the form of

$$J = g(x(t_f), t_f) + \int_{t_0}^{t_f} r(x(t), u(t), t)dt,$$

$g \equiv 0$; Lagrange problem, $r \equiv 0$: Mayer problem

Admissible controls: controls satisfying physical constraints, admissible trajectory

Optimal control problem

Optimal control problem is to find an admissible control u^* which causes the system $\dot{x} = f(x, u, t)$ to follow an admissible trajectory x^* that maximize the performance measure (payoff)

$$P = \underbrace{g(x(t_f), t_f)}_{\text{terminal payoff}} + \int_{t_0}^{t_f} \underbrace{r(x(t), u(t), t)}_{\text{running payoff}} \, dt.$$

In this lecture, we assume that $\mathcal{U}_{ad}$ denotes the set of all admissible controls:

$$\begin{aligned} \mathcal{U}_{ad} \quad & := \{u : R_+ \ \to \ U : \ u = u(\cdot) \text{ is measurable} \\ & \text{and satisfies constraints}\}, \end{aligned}$$

and $U = [-1, 1]^n$: symmetric and convex.

- Basic problem

To find a control $u^* = u^*(t) \in \mathcal{U}_{ad}$ which maximize th payoff

$$P[u^*] \geq P[u], \quad u \in \mathcal{U}_{ad}.$$

◇ Main questions

1. Does an optimal control exist ? (Existence of optimal control)

2. How can we characterize an optimal control mathematically ? (characterization of optimal control)

3. How can we construct an optimal control ? (realization of optimal control)

Two examples

1. Control of production and consumption

$$x(t) := \text{amount of output produced at time } t \geq 0$$

$\diamond$ Assumptions:

- We consume some fraction of output at each time

- We reinvest the remaining fraction

Let $u = u(t)$ be the fraction of output reinvested at time $t \geq 0$, $0 \leq u \leq 1$. In this case

$$\begin{cases} \dot{x} = ux, & 0 < t \leq T, \\ x(0) = x^0, \end{cases}$$

where

$$x \geq 0, \quad 0 \leq u \leq 1, \qquad U = [0, 1].$$

and

$$P[u] = \int_0^T (1 - u(t))x(t)dt.$$

2. A pedulum problem

$$\begin{cases} \ddot{\theta} + \lambda\dot{\theta} + \omega^2 \sin\theta = u, \quad 0 < t \le T, \\ \theta(0) = \theta_0, \quad \dot{\theta}(0) = \omega_0. \end{cases}$$

We use the approximation

$$\sin\theta \approx \theta, \quad |\theta| \ll 1$$

to get the linear approximate equation

$$\begin{cases} \ddot{\theta} + \lambda\dot{\theta} + \omega^2 \theta = u, \quad 0 < t \le T, \\ \theta(0) = \theta_0, \quad \dot{\theta}(0) = \omega_0. \end{cases}$$

So the main question is to determine the control u so that $(\theta, \dot{\theta}$ approaches $(0, 0)$ as soon as possible. (minimum-time control problem)

We set

$$x_1 = \theta, \quad x_2 = \dot{\theta},$$

then, we have

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

We now set

$$\tau = \tau[\alpha(\cdot)] \;\; : \;\; \text{first time such that} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We also define

$$P[\alpha] = -\int_0^\tau 1 dt = -\tau.$$

Controllability

In order to be able to do whatever we want with the given dynamical system under control input, the system must be controllable.

Controllability

Consider a controlled dynamics

$$\begin{cases} \dot{x} = f(x,u), & t_0 < t < \infty, \\ x(t_0) = x^0, \end{cases}$$

We set the solution of the above controlled system as $x(t; t_0, x^0)$:

•Natural a prior question before optimal control

For a fixed desired state $x^0, x_f \in R^n$, can we find a control $u \in \mathcal{U}_{ad}$ and $t_f < \infty$ such that

$$x(t_f; t_0, x^0) = x_f.$$

- Controllability question

Given the initial point $x^0 \in R^n$ and a set $S \subset R^n$,

does there exist a control u steering the system to the set S in finite time ?

For the case $S = \{x_f\}$, controllability question asks us

$$\exists\ T < \infty,\ u \in \mathcal{U}_{ad} \quad \text{such that}$$
$$\begin{cases} \dot{x} = f(x,u), & 0 < t < \infty, \\ x(0) = x^0, & x(T) = x_f. \end{cases}$$

Controllability for a linear system

From now on, consider a linear-control system and $S = \{0\}$, i.e.,

$$\begin{cases} \dot{x} = Mx + Nu, \quad M \in M^{n \times n}, \quad N \in M^{n \times m}, \\ x(0) = x^0. \end{cases} \tag{1}$$

Definition:

1. A linear-control system (**??**) is completely controllable $\iff$ For any $x^0, x_f \in R^n$, there exists a control $u : [0, t_f] \to R^m$ such that $x(t_f) = x_f$.

2. Reachable set

$$\begin{aligned} \mathcal{C}(t) &:= \{x_0 : \exists\ u \in \mathcal{A} \quad \text{such that} \quad x(t,u,x^0) = 0\}, \\ \mathcal{C} &:= \cup_{t \geq 0} \mathcal{C}(t) : \text{reachable set} \end{aligned}$$

- Simple observation

$$0 \in \mathcal{C}(t) \quad \text{and}$$

$$x^0 \in \mathcal{C}(t), \quad \hat{t} > t \quad \Longrightarrow \quad x^0 \in \mathcal{C}(\hat{t}).$$

Looking for sufficient and necessary condition of controllability

Snapshot of ODE theory

Consider a homogeneous linear ODE:

$$\begin{cases} \dot{x} = Mx, \ t > 0, & M \in M^{n\times n} : \text{constant matrix} \\ x(0) = x^0. \end{cases}$$

Then we have

$$x(t) = \Phi(t)x^0,$$

where Φ is a fundamental matrix define by

$$\Phi(t) = e^{tM}\Big(:= \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}\Big).$$

Consider an inhomogeneous linear system:

$$\begin{cases} \dot{x} = Mx + f(t), \ t > 0, \\ x(0) = x^0. \end{cases}$$

Then the solution x can be given by the variation of parameters formula (Duhamel's formula)

$$x(t) = \Phi(t)x^0 + \int_0^t \Phi(t-s)f(s)ds,$$

where $\Phi(t-s) = \Phi(t)\Phi^{-1}(s)$.

We now return to

$$\begin{cases} \dot{x} = Mx + Nu, \ \ t > 0, \\ x(0) = x^0. \end{cases}$$

Then by Duhamel's formula, we have

$$x(t) = \Phi(t)x^0 + \Phi(t)\int_0^t \Phi^{-1}(s)Nu(s)ds.$$

Note that

$$x^0 \in \mathcal{C}(t) \quad \Longleftrightarrow \quad x(t) = 0$$

$$\Longleftrightarrow \quad x^0 = -\int_0^t \Phi^{-1}(s)Nu(s)ds, \quad \text{for some } u \in \mathcal{U}_{ad}.$$

- **Theorem** (Geometry of reachable set)

Rechable set $\mathcal{C}$ is symmetric and convex, i.e.,

$$(i)\ x^0 \in \mathcal{C} \quad \Longrightarrow \quad -x^0 \in \mathcal{C}.$$
$$(ii)\ x^0, \hat{x}^0 \in \mathcal{C}, \lambda \in [0,1] \quad \Longrightarrow \quad \lambda x^0 + (1-\lambda)\hat{x}^0 \in \mathcal{C}.$$

Kalman's rank theorem (1960)

Consider

$$\dot{x} = Mx + Nu, \quad M \in M^{n\times n}, \quad N \in M^{n\times m}. \qquad (2)$$

We define a controllability matrix

$$G(M,N) := [N|MN|\cdots|M^{n-1}N].$$

- **Definition**

The linear control system (**??**) is controllable $\iff$ $\mathcal{C} = R^n$.

• **Theorem**

$$\text{rank}G(M, N) = n \quad \Longleftrightarrow \quad 0 \in \mathcal{C}^0.$$

Proof. • ($\Longleftarrow$)

Suppose that $0 \in \mathcal{C}^0$. Note that

$$\text{rank}G(M, N) \leq n.$$

If $\text{rank}G(M, N) < n$, then there exists $b \neq 0$ such that

$$b^t G(M, N) = 0.$$

This yields

$$b^t N = b^t M N = \cdots = b^t M^{n-1} N = O.$$

By Cayley-Hamilton's theorem, we also have

$$b^t M^k N = O, \quad k \geq 0, \quad b^t \Phi^{-1}(t) N = O.$$

We now claim

$$b \text{ is perpendicular to } \mathcal{C}(t), \quad \text{i.e.,} \quad \mathcal{C}^0 = \emptyset.$$

If $x^0 \in \mathcal{C}(t)$, then

$$x^0 = -\int_0^t \Phi^{-1}(s) N u(s) ds, \quad u \in \mathcal{U}_{ad}.$$

Therefore,

$$b^t x^0 = -\int_0^t b^t \Phi^{-1}(s) N u(s) ds = 0.$$

• ($\Longrightarrow$) Suppose that $0 \notin \mathcal{C}^0$, i.e.,

$$0 \in (\mathcal{C}^0)^c \subset \cap_{t \geq 0} (\mathcal{C}^0(t))^c.$$

Then $0 \notin \mathcal{C}^0(t), \quad \forall\ t \geq 0$. Therefore

$$0 \in \partial \mathcal{C}(t).$$

Since $\mathcal{C}(t)$ is convex, there exists $b \neq 0$ such that

$$b^t x^0 \leq 0, \quad x^0 \in \mathcal{C}(t).$$

For $x^0 \in \mathcal{C}(t)$,

$$x^0 = -\int_0^t \Phi^{-1}(s) N u(s) ds, \quad u \in \mathcal{U}_{ad}.$$

Thus

$$b^t x^0 = -\int_0^t b^t \Phi^{-1}(s) N u(s) ds \leq 0.$$

This yields

$$b^t\Phi^{-1}(s)N = 0.$$

By differentiating the above relation, we have

$$b^tN = b^tMN = \cdots = b^tM^{n-1}N = O, i.e., b^tG(M,N) = 0.$$

Hence rank$G(M,N) < n$.

- **Theorem** Let λ be the eigenvalue of M.

rank$G(M,N) = n$ and Re$(\lambda) \leq 0 \quad \Longrightarrow \quad$ The system (**??**) is controllable.

- **Theorem** Kalman (1960)

The system (**??**) is controllable $\iff$ $rank(G) = n.$

This rank indicates how many components of the system are sensitive to the action of the control

- **Examples**

1.

$$n = 2, \quad m = 1, \quad A = [-1, 1]$$
$$\begin{cases} \dot{x}_1 = 0, \ t > 0, \\ \dot{x}_2 = u(t). \end{cases}$$

2.

$$\begin{cases} \dot{x}_1 = x_2, \ t > 0, \\ \dot{x}_2 = u(t). \end{cases}$$

3.

$$\begin{cases} \dot{x}_1 = u(t), \ t > 0, \\ \dot{x}_2 = u(t). \end{cases}$$

Observability

In order to see what is going on inside the system under observation, the system must be observable

Observability

- Consider uncontrolled system:

$$\begin{cases} \dot{x} = Mx, \; t > 0, \quad M \in M^{n \times n}, \quad x \in R^n, \\ x(0) = x^0 \; : \; \text{unknown}. \end{cases} \tag{3}$$

Note that

$$x(t) = e^{tM} x^0. \tag{4}$$

Once we know x^0, then we know everything !!

Suppose that we have observed data y:

$$y(t) = Nx(t), \quad N \in M^{m \times n}, \quad m \ll n.$$

• Observability question

"Given the observation $y = y(\cdot)$ which is low-dimensional, can we in principle construct high-dimensional real dynamics $x = x(\cdot)$?

Thus problem is how to recover $x^0 \in R^n$ from the observed data y.

Note that

$$\begin{aligned} y(0) &= Nx(0) = Nx^0, \\ \dot{y}(0) &= N\dot{x}(0) = NMx^0, \\ \cdots &= \cdots, \\ y^{(n-1)}(0) &= Nx^{(n-1)}(0) = NM^{n-1}x^0. \end{aligned}$$

This yields

$$\begin{pmatrix} y(0) \\ y'(0) \\ . \\ y^{(n-1)}(0) \end{pmatrix} = \begin{pmatrix} N \\ NM \\ . \\ NM^{n-1} \end{pmatrix} x^0.$$

Thus, we have

$$\text{(\textbf{??}) and (\textbf{??}) is observable} \quad \Longleftrightarrow \quad \text{rank} \begin{pmatrix} N \\ NM \\ . \\ NM^{n-1} \end{pmatrix} = n.$$

- **Definition**

$$\left\{\begin{array}{l}\dot{x} = Mx,\ \ t > 0,\\ x(0) = x^0, \quad y = Nx\end{array}\right. \text{ is observable}$$

$$\Longleftrightarrow \qquad \text{For two solutions } x^1 \text{ and } x^2 \text{ such that}$$

$$Nx^1 = Nx^2 \text{ on } [0,t] \quad x^1(0) = x^2(0).$$

Example: $N = I, N = 0$.

- **Theorem** (Observability is a dual concept of controllability)

$$\left\{\begin{array}{l}\dot{x} = Mx,\ \ t > 0,\\ y = Nx,\end{array}\right. \qquad \text{is observable}$$

$$\Longleftrightarrow \qquad \dot{z} = M^t z + N^t u \ \text{ is controllable.}$$

La Salle's Bang-bang control

$$\mathcal{U}_{ad} := \{u \ : [0,\infty) \to U = [-1,1]^m \ : \ u \text{ is measurable}\}.$$

- **Definition** Let $u \in \mathcal{U}_{ad}$.

$$u = (u^1, \cdots, u^m) \ \text{ is a bang-bang control}$$
$$\iff \quad |u^i(t)| = 1, \quad \forall\ t > 0,\ i = 1, \cdots, m.$$

- **Theorem** (Bang-bang principle)

$$\begin{cases} \dot{x} = Mx + Nu, \\ x(0) = x^0, \end{cases} \qquad x^0 \in \mathcal{C}(t)$$
$$\Longrightarrow \quad \exists\ u^* = u^*(\cdot) \ : \ \text{bang-bang control such that}$$
$$x^0 = -\int_0^t \Phi^{-1}(s) N u^*(s) ds.$$

Preliminaries for bang-bang principle

$$L^\infty = L^\infty(0,t;R^m) = \{u : [0,t] \to R^m : \sup_{0\le s\le t} |u(s)| < \infty\},$$
$$||u||_{L^\infty} := \sup_{0\le s\le t} |u(s)|, \quad L^1 \subset (L^\infty)^*.$$

- **Definition** Let $u_n, u \in L^\infty$.

$$\begin{aligned} & u_n \to u \text{ in weak-star topology} \\ \iff & \int_0^t u_n(s)\varphi(s)ds \to \int_0^t u(s)\varphi(s)ds, \\ \iff & u_n \text{ converges to } u \text{ in weak-star topology,} \end{aligned}$$

where φ is a L^1-test function with $\int_0^t |\varphi(s)|ds < \infty$.

- **Theorem** (Banach-Alaoglu)

Any bounded set in L^∞ is weak-star compact.

- **Corollary**

  If $u_n \in \mathcal{U}_{ad} := \{u \;:\; [0,t] \to [-1.1]^m \;:\; u$ is measurable$\}$,
  $\exists \; \{u_{n_k}\}$: subsequence of u_n such that
  $u_{n_k} \to u$ weak-star topology.

- **Definition** Let $K \subset R^n$ and $z \in K$.

1. K is convex $\iff$ $\forall\, x, y \in K,\ 0 \leq \lambda \leq 1, \quad \lambda x + (1-\lambda)y \in K$.

2. $z \in K$ is an extreme point $\iff$
there does not exist $x, \widehat{x} \in K$ and $\lambda \in (0,1)$ such that $z = \lambda x + (1-\lambda)\widehat{x}$.

- **Theorem** (Krein-Milman)

$$K \neq \emptyset \ : \text{convex and weak-star compact}$$
$$\Longrightarrow \quad K \text{ has at least one extreme pint.}$$

The proof of bang-bang's principle

Let $x^0 \in \mathcal{C}(t)$. Then, we set

$$K := \{u \in \mathcal{U}_{ad} \ : \ u \text{ steers } x^0 \text{ to } 0 \text{ at time } t\}.$$

- **Lemma**

$$K \neq \emptyset \ : \ \text{convex and weak-star compact.}$$