Topics in Applied Mathematics

Introduction to Game Theory

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Purpose of this course

Learn the basics of game theory and be ready for mean-field game theory

References

- Lecture notes on differential games by Alberto Bressan.
- Introduction to differential games by Caraliaguet Pierre.
- Game theory: An introduction by E. N. Barron.
- An introduction to game theory by Martin J. Osborne.
- An introductory course on mathematical game theory by Julio Gonzalez-Diaz et al.
- Differential Games by Rufus Isaacs.

Lecture 1: Normal form game and Examples

What is a game theory ?

Game theory = Multi-party Decision Theory

Mathematical theory of interactive decision situations, and these situations are characterized by the following elements:

- there is a group of agents (decision-makers, players).
- each agent has to make a decision.
- an outcome results as a function of the decisions of all agents.
- each agent has his own preferences on the set of possible outcomes.

Robert J. Aumann said

" Game theory is optimal decision making in the presence of others with different objectives" .

Situations = Games, agents = players, available decisions = strategies, action.

Classical game theory deals with question

" How rational players should behave ? "

Rational player means one who

1. knows what he wants
2. has the only objective of getting what he wants
3. is able to identify the strategies that best fit his objective.

Brief History of Game Theory

- Emile Borel (7 January 1871 – 3 February 1956)

The Forgotten Father of Game Theory ? In 1921-1927, He defined "Games of Strategy".

- John von Neumann (28 December 1903 – 8 February 1957)

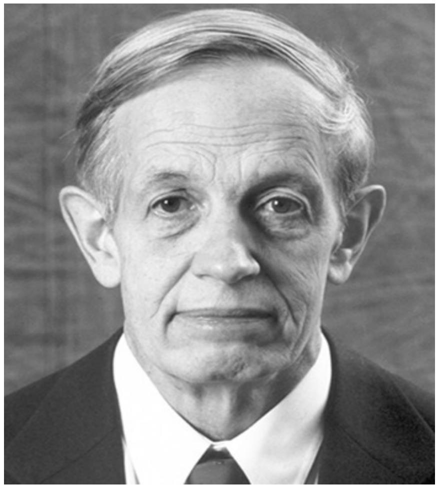
Minimax Theorem (1928)

Minimax theorem says that in zero-sum games with perfect information (players know at each time all moves that have taken place so far), there exists a pair of strategies for both players that allows each to minimize his maximum losses.

- Oskar Morgenstern (24 January 1902 – 26 July 1977)

Theory of Games and Economic Behavior (1944)

- John Nash (13 June 1928 –)

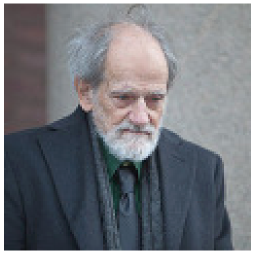


Equlibrium points in N-person games (1950)

- Robert John Aumann (8 June 1930 –)

Nobel prize in 2005 for conflict and cooperation through game theory

- Lloyd Shapley (2 June 1923 –)



Nobel prize in 2012 for the theory of stable allocations and the practice of market design

Nobel prize winners in game theory

Paul A. Samuelson (70), Kenneth J. Arrow (72), Reinhard Selten, John F. Nash, John C. Harsanyi ('94), Robert E. Lucas Jr ('95), William Vickrey ('96), Thomas C. Schelling, Robert J. Aumann ('05), Eric S. Maskin, Leonid Hurwicz, Roger B. Myerson ('07), Alvin E. Roth, Lloyd S. Shapley ('12).

Strategic games

Strategic games (simultaneous-move game, normal form game) is a static model that describes interactive situations among several players. All the players make their decision simultaneously and independently. They are characterized by players, strategies (action) and payoff functions (preferences).

Complete Information: Each player knows other players strategies and payoffs

Definition: An n-player strategic game $G = (\mathcal{N}, \mathcal{S}, \mathcal{U})$ with set of players $\mathcal{N}$ is a triple whose elements are the following:

- The set of players:

$$\mathcal{N} := \{1, 2, \cdots, n\}.$$

- Sets of strategies: For each $i \in N$, S_i is the nonempty set of strategies of player i and $\mathcal{S} = \prod_{i=1}^{n} S_i$ is the set of strategy profile of all players.

- Payoff(utility) functions: For each $i \in \mathcal{N}$, $u_i : \mathcal{S} \to R$ is the payoff function of player i and $u := \prod_{i=1}^{n} u_i$; u_i assigns to each strategy profile $S \in \mathcal{S}$, the payoff that player i gets if S is played.

Examples:

Players = Firms, Strategies = prices, preferences = firm's profits.

Players = Candidates for political office, Strategies = campaign expenditures, preferences = a reflection of the candidate's probabilities of winning.

Examples of strategic games

1. Prisoner's dilemma (Conceived by Merrill Flood and Melvin Dresher, and coined by Albert W. Tucker)

Alice and Bob in a major crime are held in separate rooms. There is enough evidence to convict each of them of minor, but not enough evidence to convict either of their offense of major crime, unless one of them acts as an informer against the other. Both of them are given the chance to confess. If both confess the crime, each of them will spend 10 years in jail. If only one confesses, he will act as a witness against the other, who will spend 15 years in jail, and will receive no punishment. Finally if no one confesses, they will be in jail for 1 years.

	C	RS
C	$(-10, -10)$	$(0, -15)$
RS	$(-15, 0)$	$(-1, -1)$

2. Working on a joint project

You are working with a friend on a joint project. Each of you can either work hard or goof off. If your friend works hard, then you prefer to goof off (the outcome of the project would be better if you worked hard too, but the increment in its value to you his not worth the extra effort). You prefer the outcome of your both working hard to the outcome of your both goofing off, and the worst outcome for you is that you work hard and your friend goofs off (you hate to be "exploited").

	Work hard	Goof off
Work hard	$(2,2)$	$(0,3)$
Goof off	$(3,0)$	$(1,1)$

3. Duopoly

Two firms produce the same good, for which each firm charges either a low price or a high price. Each firm wants to achieve the highest possible profit. If both firms choose high, then each earns a profit of 1000 dollars. If one firm choose high and the other chooses low, then the firm choosing high obtains no customers and makes a los of 200 dollars, whereas the firm choosing low earns a profit of 1200 dollars. If both firms choose low, then each earns a profit of 600 dollars. Each firm cares only about its profit, so we can represent its preferences by the profit it obtains.

	High	Low
High	$(1000, 1000)$	$(-200, 1200)$
Low	$(1200, -200)$	$(600, 600)$

4. The arms race

An arms race can be modeled as the Prisoner's Dilemma. 1950s, the United States and the Soviet Union were involved in a nuclear arms race. Assume that each country can build an arsenal of nuclear bombs, or can refrain from doing so. Assume also that each country's favorite outcome is that it has bombs and the other counter does not; the next best outcome is that neither country has any bombs; the next best outcome is that both countries have bombs; and the worst outcome is that only the other country has bombs.

5. Battle of Sexes

Alice and Bob want to date, and they can either go to the football game (F) or the opera (O). Bob prefers seeing football and Alice prefers the opera, but both would rather meet than not.

	F	O
F	$(2, 1)$	$(0, 0)$
O	$(0, 0)$	$(1, 2)$

6. Matching Pennies

Alice and Bob choose simultaneously whether to show the head or tail of a coin. If they show the same side, then Bob pays Alice a dollar, whereas if they show different sides, Alice pays Bob a dollar. In this game, the player's interests are diametrically opposed.

	H	T
H	$(1,-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

7. Game of Chicken

The game of chicken models two drivers, both headed for a single lane bridge from opposite directions. The first to swerve away yields the bridge to the other. If neither player swerves, the result is a costly deadlock in the middle of the bridge, or a potentially fatal head-on collision. It is presumed that the best thing for each driver is to stay straight while the other swerves (since the other is the "chicken" while a crash is avoided). Additionally, a crash is presumed to be the worst outcome for both players. This yields a situation where each player, in attempting to secure his best outcome, risks the worst.

	Swerve	Straight
Swerve	$(0, 10)$	$(-1, +1)$
Straight	$(+1, -1)$	$(-10, -10)$

8. Game of Hawk-Dove

Hawk and dove are finghting for food. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome in which its opponent is passive to that in which its opponent is aggressive.

7. The Driving Game

Alice and Bob can each choose whether to drive on the left or right side of the road

	L	R
L	$(0, 0)$	$(-10, -10)$
R	$(-10, -10)$	$(0, 0)$

Pure Strategy

Definition: If a player has a strategy set $S = (s_1, s_2, \cdots, s_n)$, a pure strategy is an element of the set S. That is $s_1, \cdots, s_n$ are all pure strategies.

cf. feasible strategy, Mixed strategy

Example: C and RS are pure strategies in Prisoner's Dilemma.

Best Response

Definition: For a given strategic game $(\mathcal{N}, \mathcal{S}, \mathcal{U})$,

Player i's best response to other player's strategies $BR_i(s_{-i})$ is the set of strategies out of S_i that maximize i's payoff, when the other players play s_{-i}. Formally,

$$BR_i(s_{-i}) := \{s_i \in S_i : \; u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \quad \forall \;\; s_i' \in S_i\}.$$

Example 1: Prisoner's Dilemma

	C	RS
C	$(-10, -10)$	$(0, -15)$
RS	$(-15, 0)$	$(-1, -1)$

Find BR_1(C), BR_1(RS), BR_2(C), BR_2(RS).

Example 2: Battle of Sexes

	F	O
F	$(2, 1)$	$(0, 0)$
O	$(0, 0)$	$(1, 2)$

Find BR_1(F), BR_1(O), BR_2(F), BR_2(O).

Example 3: In the Driving game, Alice's best response function is

$$BR_A(s_B) = \begin{cases} L, & s_B = L; \\ R, & s_B = R \end{cases}$$

Remark: There can be more than one strategy that satisfies the definition of best response, so in general $BR_i(s_{-i})$ is a set.

Definition: A set of strategies $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ is a pure strategy Nash equilibrium if for every $i = 1, \cdots, n$,

$$s_i^* \in BR_i(s_{-i}^*).$$

- In a Nash Equilibrium, every player's strategy is optimal, given every other players strategy in the equilibrium.

- In a Nash Equilibrium, no player has an incentive to deviate.

Pure strategy Nash equilibrium

A Nash equilibrium is a strategy(action) profile s^* with the property that no player i can do better by choosing an action different from s_i^* given that every other player j adheres to s_j^*.

Let $s = (s_1, \cdots, s_n)$ be an action(strategy) profile, in which the action of player i is s_i. Let s_i' be any action of player i. Then (s_i', s_{-i}) denote the action profile in which every player j except i chooses action s_j as specified by a, whereas player i chooses s_i'.

Definition: Let $G = (\mathcal{N}, \mathcal{S}, \mathcal{U})$ be a strategic game. (Pure strategy) A Nash equilibrium of G is a strategy profile $s^* \in A$ such that, for each $i \in \mathcal{N}$ and each $s_i' \in \mathcal{S}_i$,

$$u_i(s^*) \geq u_i(s_i', s_{-i}^*).$$

Remark. In strategic games, some games have a single Nash equilibrium, some possess no Nash equilibrium, and others have many Nash equilibria.

How to find Nash Equilibrium ?

- Step 1: Find each players best response $\text{BR}_i(s_{-i})$.

- Step 2: Impose that for every i:

$$s_i^* = \text{BR}_i(s_{-i}).$$

Examples for pure strategy Nash Equilibrium

1. Single person decision

- Bob is arrested and charged with a crime.
- He is thinking about what to do when he is interrogated
- If he confesses to the crime, he goes to jail for 15 years
- If he remains silent, theres only enough evidence to send him to jail for 1 year.

Question: What should Bob do?

2. Prisoner's Dilemma

- Both Alice and Bob are accused of a major crime and could go to jail for 10 years. They are being interrogated independently and cant communicate.

- Both A and B confess ? 10 year off sentences

- Both remain silent ? just 1 year for minor crimes

- Only 1 confesses ? confessor gets immunity, other gets 15 years

Question: Which actions do Alice and Bob should take ?

	C	RS
C	$(-10, -10)$	$(0, -15)$
RS	$(-15, 0)$	$(-1, -1)$

For Alice side;

If Bob confess, $u_A(C) = -10, \quad u_A(RS) = -15.$
If Bob does not confess, $u_A(C) = 0, \quad u_A(RS) = -1.$

Thus, Alice always should confess

For Bob side;

If Alice confess, $u_B(C) = -10, \quad u_B(RS) = -15.$
If Alice does not confess, $u_B(C) = 0, \quad u_B(RS) = -1.$

Thus, Bob always should confess.

Therefore, the pure strategy Nash equilibrium is

(Alice, Bob) = (Confess, Confess).

In Bible (Mathew 7: 12): Golden rule.

"So whatever you wish that others would do to you, do also to them, for this is the Law and the Prophets."

3. Matching Pennies (Two player-zero sum game)

	H	T
H	$(1,-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

For Alice side;

If Bob shows a head, then

$$u_A(H) = 1, \quad u_A(T) = -1.$$

If Bob shows a tail, then

$$u_A(H) = -1, \quad u_A(T) = 1.$$

For Bob, the situation is similar. Thus there is no pure strategy Nash equilibrium.

Example 4.

	C	D
A	$(6,4)$	$(1,1)$
B	$(1,1)$	$(2,2)$

Example 5. Driving game

	L	R
L	$(0,0)$	$(-10,-10)$
R	$(-10,-10)$	$(0,0)$

Example 6. Battle of Sexes

	F	O
F	$(1,3)$	$(0,0)$
O	$(0,0)$	$(3,1)$

Nash equilibria: (A, C) and (B, D).
Nash equilibria: (L, L) and (R, R).
Nash equilibria: (F, F) and (O, O).

Example 7:

	A	B	C
A	$(1,0)$	$(2,2)$	$(1,2)$
B	$(2,3)$	$(1,0)$	$(1,1)$
C	$(0,1)$	$(0,2)$	$(3,0)$

Example 8:

	A	B	C
A	$(0,8)$	$(1,1)$	$(5,6)$
B	$(3,4)$	$(2,8)$	$(1,3)$
C	$(2,5)$	$(3,5)$	$(8,9)$

Dominated strategy

In any game, a player's strategy (action) "strictly dominates" another strategy if it is superior, no matter what the other players do.

Definition (Strict domination): In a strategic game $(\mathcal{N}, \mathcal{S}, \mathcal{U})$, player i's strategy s_i'' strictly dominates the strategy s_i' if

$$u_i(s_i'', s_{-i}) > u_i(s_i', s_{-i}),$$

for every list s_{-i} of the other players' strategy.

We call the strategy s_i' is strictly dominated. Thus, strictly dominated action is not a best response to any actions of the other players.

Example 1: Prisoner's Dilemma

	C	RS
C	$(-10, -10)$	$(0, -15)$
RS	$(-15, 0)$	$(-1, -1)$

Which strategy is a strictly dominating one ?

Example 2: Battle of sexes

	F	O
F	$(2,1)$	$(0,0)$
O	$(0,0)$	$(1,2)$

Which strategy is a strictly dominating one ?

Definition (Weak domination): In a strategic game, player i's strategy s_i'' weakly dominates the strategy s_i' if

1. $$u_i(s_i'', s_{-i}) \geq u_i(s_i', s_{-i}), \qquad \forall\ s_{-i}.$$

2. $$u_i(s_i'', s_{-i}) > u_i(s_i', s_{-i}), \qquad \text{for some } s_{-i}.$$

Example

	B	C
B	$(1, 1)$	$(2, 0)$
C	$(0, 2)$	$(2, 2)$

Cournot's model of oligopoly (1838)

Oligopoly = competition between a small number of sellers.

Consider two firms who produce identical products and sell to the same market. The market has an inverse demand curve given by:

$$P(q_1, q_2) = a - b \cdot (q_1 + q_2).$$

Each firm has constant marginal cost c. Firm is profit function is given by

$$\pi_i(q_1, q_2) = P(q_1, q_2)q_i - cq_i$$

Each firm takes the others quantity output as given.

Step 1: Find the best response function(maximizing profits).

Taking Firm 2s action q2 as given, firm 1 solves:

$$\max_{q_1} \pi_1(q_1, q_2) = \max_{q_1}(a - b \cdot (q_1 + q_2)) \cdot q_1 - c \cdot q_1$$

Firm 1 chooses q1 as a best response to a given q_2: $q_1^* = BR_1(q_2)$
Here

$$q_1^* = BR_1(q_2) = \frac{a - c - bq_2}{2b}.$$

And Similarly,

$$q_2^* = BR_2(q_1) = \frac{a - c - bq_1}{2b}.$$

Step 2 is applying the fact that in the Nash Equilibrium, every players strategy is a best response to the opponents

- Note that the Nash Equilibrium here is given by (q_1^*, q_2^*) such that
$$q_1^* = BR_1(q_2^*), \quad q_2^* = BR_2(q_1^*).$$

- Solving these two equations, we find
$$q_1^* = \frac{a-c}{3b} = \frac{q_0}{3}, \quad q_2^* = \frac{a-c}{3b}.$$

- And the equilibrium payoff for both firms is
$$\pi = P(q_1^*, q_2^*)q_1 - cq_1 = \frac{(a-c)^2}{9b} =: \frac{1}{9}\pi_0.$$