Lecture 4: Ito’s Stochastic Calculus and SDE

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• What is Calculus?

Integral, Differentiation.

• Differentiation
• Integral

Riemann integral, Lebesgue integral.
• Ordinary differential equations (ODE):

\[ \dot{X}(t) = f(X(t)), \quad t > 0, \quad \dot{X} := \frac{dX}{dt}, \]
\[ X(0) = X_0, \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a Lipschitz continuous function.

or equivalently the IVP can be rewritten as an integral equation:

\[ X(t) = X_0 + \int_0^t f(X(s))ds, \quad t \geq 0. \]
We add (white) noise which is responsible for random fluctuations

\[ \dot{X}(t) = f(X(t)) + \sigma \xi(t), \quad t > 0, \quad \dot{X} := \frac{dX}{dt}, \]
\[ X(0) = X_0, \]

where \( \xi = \xi(t) \) is a white noise satisfying

\[ \langle \xi(t) \rangle = 0, \quad \langle \xi(t), \xi(s) \rangle = \delta(t - s). \]
We formally set

\[ \xi(t) = \frac{dW(t)}{dt}. \]

Then the stochastically perturbed ODE becomes

\[
\begin{aligned}
\dot{X}(t) &= f(X(t)) + \sigma \xi(t) \\
\iff \frac{dX(t)}{dt} &= f(X(t)) + \sigma \frac{dW(t)}{dt} \\
\iff dX(t) &= f(X(t))dt + \sigma dW(t).
\end{aligned}
\]
• Example 1:

\[ \dot{x} = ax, \quad x(0) = x_0. \]

• Example 2:

\[ \dot{x} = x^2, \quad x(0) = x_0. \]
Heat equation (or diffusion equation)

The fundamental solution $K = K(x, t)$ is defined to be the solution of the following IVP:

$$u_t = \sigma^2 u_{xx}, \quad x \in \mathbb{R}, \ t > 0,$$

$$u(x, 0) = \delta(x).$$

$$K(x, t) = \frac{1}{\sqrt{4\pi\sigma^2 t}} \exp \left(-\frac{x^2}{4\sigma^2 t}\right).$$
The solution to the IVP for heat equation:

\[ u_t = \sigma u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \]
\[ u(x, 0) = u_0(x) \]

is given by

\[ u(x, t) = K(x, t) * u_0. \]
• Stochastic Differential Equations (SDE)

\[ dX = b(X, t)dt + B(X, t)dW, \]
\[ X(0) = X_0. \]

which means

\[ X(t) = X_0 + \int_0^t b(X(s), s)ds + \int_0^t B(X(s), s)dW(s), \quad t \geq 0. \]
Need to define Ito's integral (1949):

\[ \int_0^T G \, dW, \quad \text{or} \quad \int_0^T G(t, \omega) \, dW(t, \omega), \quad G: \text{adapted process.} \]

If \( W \) is differentiable, (which is not true), we can define

\[ \int_0^T G \, dW = \int_0^T GW' \, dt. \]

Of course, B.M \( W \) is not differentiable in probability 1.
Construction of Ito’s integral

- General guideline:

Step 1: Construction of Ito’s Integral for simple adapted process.

Step 2: Construction of Ito’s Integral for general $L^2$-adapted process.
• **Definition** (Simple adopted process)

\( \Delta = \Delta(t) \) is a simple process if and only if for some partition \( \mathcal{P} = \{t_0 = 0 < t_1 < \cdots < t_n\} \) of \([0, T]\), \( \Delta(t) \) is constant in \( t \) on each subinterval \([t_j, t_{j+1})\).

**Question:** How to define \( I(t) := \int_0^t \Delta(s) dW(s) \)?
Hueristic interpretation

- \( W(t) \): the price per share of an asset at time \( t \).
- \( t_0, t_1, \cdots, t_n \) : trading dates in the asset.
- \( \Delta(t_0), \Delta(t_1), \cdots, \Delta(t_n) \) : the number of shares take in the asset at each trading date and held to the next trading date.

Then the gain \( I(t) \) from trading at each time \( t \) is given by

\[
I(t) = \Delta(0)[W(t) - W(t_0)] = \Delta(0)W(t), \quad 0 \leq t \leq t_1,
\]

\[
I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t) - W(t_1)], \quad t_1 \leq t \leq t_2,
\]

\[
I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t_2) - W(t_1)] + \Delta(t_2)[W(t) - W(t_2)], \quad t_2 \leq t \leq t_3,
\]

\[
I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)], \quad t_k \leq t \leq t_{k+1}.
\]

(2)
Theorem

Ito's integral is a martingale.

Proof.
• **Theorem** (Ito’s isometry)

\[ E[I^2(t)] = E\int_0^t \Delta^2(s)ds. \]

Proof.
Theorem (Quadratic Variation)

\[ [I, I](t) := \int_0^t \Delta^2(s)ds = t. \]

Proof.
• Construction of $I(t)$ for $L^2$-process $\Delta(t)$:

$$E \int_0^T \Delta^2(t) dt < \infty.$$ 

Step 1: Choose a sequence $\Delta_n(t)$ of simple processes such that

$$\lim_{n \to \infty} E \int_0^T |\Delta_n(t) - \Delta(t)|^2 dt = 0.$$ 

Step 2: For each adopted simple process $\Delta_n$, we define an Ito's integral $I_n$:

$$I_n := \int_0^T \Delta_n(t) dW(t).$$
Step 3: Define an Ito’s integral $I(T)$ as a limit of $I_n$, i.e.,

$$\int_0^T \Delta(t) dW(t) := \lim_{n \to \infty} \int_0^T \Delta_n(t) dW(t).$$
• **Theorem.** Ito’s integral \( I(t) = \int_0^t \Delta(s)dW(s) \) satisfies

1. **(Continuity):** The sample paths of \( I(t) \) are continuous.

2. **(Adaptivity):** For each \( t \), \( I(t) \) is \( \mathcal{F}(t) \)-measurable.

3. **(Linearity):** For every constants \( \lambda, \mu \),

\[
\int_0^t (\lambda \Delta_1(s)+\mu \Delta_2(s))dW(s) = \lambda \int_0^t \Delta_1(s)dW(s)+\mu \int_0^t \Delta_2(s)dW(s).
\]
1. (Martingale): \( I(t) \) is a martingale.

2. (Ito’s isometry):

\[
E(I^2(t)) = E \int_0^t \Delta^2(s)ds.
\]

3. (Quadratic variation):

\[
[I, I](t) = \int_0^t \Delta^2(s)ds.
\]
Question: We want to differentiate $f(W(t))$, $f$ is a differentiable function and $W(s)$ is a B.M.

- **Heuristic explanation.**

For one-dimensional case $n = 1$, consider a SDE:

$$dX(t) = A(t)dt + B(t)dW,$$

$$X(0) = X_0.$$

Let $f : \mathbb{R} \to \mathbb{R}$ and define

$$Y(t) = f(X(t)).$$
• (Wrong answer). If $f$ is differentiable,

$$\frac{d}{dt} Y(t) = f'(X(t))X'(t), \quad \text{or}$$

$$dY(t) = f'(X(t))X'(t)dt$$

$$= f'(X(t))A(t)dt + f'(X(t))B(t)dW(t).$$
• Right approach: We use a heuristic principle \( dW = (dt)^{\frac{1}{2}} \) and "Taylor expansion" to find

\[
dY(t) = df(X(t)) \\
= f'(X(t))dX(t) + \frac{1}{2}f''(X(t))dX(t)^2 + \frac{1}{6}f'''(X(t))dX(t)^3 + \cdots \\
= f'(X(t))\left(A(t)dt + B(t)dW(t)\right) \\
+ \frac{1}{2}f''(X(t))\left(A(t)dt + B(t)dW(t)\right)^2 + \cdots
\]
Note that

\[
\left( A(t)dt + B(t)dW(t) \right)^2
\]

\[
= A(t)^2 dt^2 + 2A(t)B(t)dt dW(t) + B(t)^2 dW(t)^2
\]
This is the Ito-Doeblin’s formula in differential form. Integrating this, we also obtain a mathematically meaningful form:

\[
Y(t) - Y(0) = \int_0^t \underbrace{f'(X(s))B(s)dW(s)}_{\text{Ito’s integral}} + \int_0^t \left( f'(X(s))A(s) + \frac{1}{2}B(s)^2f''(X(s)) \right) ds .
\]
(Higher dimensions)

\[ dX(t) = A(t)dt + \sigma dW(t), \quad t \geq 0, \]
\[ X(0) = x^0, \]

where \( X(t) = (x_1(t), \ldots, x_n(t))^T. \)

For \( f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) and \( Y(t) = f(X(t), t) \), we have

\[ dY(t) = df(X(t), t) \]
\[ = f_t(X(t), t)dt + \sum_{i=1}^{n} f_{x_i}(X(t), t)dx_i(t) \]
\[ + \frac{1}{2} \sum_{i,j} f_{x_i x_j}(X(t), t) dx_i(t) dx_j(t), \]

\[ dW_i = (dt)^{\frac{1}{2}}, \quad dW_i dW_j = \begin{cases} dt, & i = j, \\ 0, & i \neq j. \end{cases} \]
Hence we have

\[
dY(t) = f_t(X(t), t)dt + \sum_{i=1}^{n} f_{x_i}(X(t), t)dx_i(t) \\
+ \frac{1}{2} \sum_{i,j} f_{x_ix_j}(X(t), t)dx_i(t)dx_j(t) \\
= f_t(X(t), t)dt + \sum_{i=1}^{n} f_{x_i}(X(t), t)(A_i(t)dt + \sigma dW_i(t)) \\
+ \frac{\sigma^2}{2} \sum_{i} f_{x_ix_i}(X(t), t)dt \\
= f_t(X(t), t)dt + \nabla_x f(X(t), t) \cdot (A(t)dt + \sigma dW(t)) \\
+ \frac{\sigma^2}{2} \Delta f(X(t), t)dt.
\]
Theorem.

Let $f = f(t, x)$ be a $C^{1,2}_{t,x}$-function, and let $W(t)$ be a B.M. Then for every $T > 0$, we have

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t))dt + \int_0^T f_x(t, W(t))dW(t) + \frac{1}{2} \int_0^T f_{xx}(t, W(t))dt.$$
Remark.

\[ \int_0^t W(s)dW(s) =? \]

If \( W \) is differentiable, then we might expect

\[ \int_0^t W(s)dW(s) = \int_0^t W(s)W'(s)ds = \int_0^t \left( \frac{1}{2}W(s)^2 \right)'ds = \frac{1}{2}W(t)^2. \]

Of course, this is not true. We now apply Ito and Doeblin’s formula for \( f(x) = \frac{1}{2}x^2 \) to find

\[
\frac{1}{2}W^2(T) = f(W(T)) - f(W(0)) \\
= \int_0^T f'(W(t))dW(t) + \frac{1}{2} \int_0^T f''(W(t))dt \\
= \int_0^T W(t)dW(t) + \frac{1}{2}T.
\]
Hence

\[ \int_0^T W(t) dW(t) = \frac{1}{2} W^2(T) - \frac{T}{2}. \]
Ito’s process

- **Definition.** Let $W(t), \ t > 0$ be a Brownian motion, and let $\mathcal{F}(t)$ be an associated filtration. An Ito’s process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(s)dW(s) + \int_0^t \Theta(s)ds,$$

where $X(0)$ is nonrandom, and $\Delta(s)$ and $\Theta(s)$ are adapted processes.

• Theorem (Quadratic variation).

\[ [X, X](t) = \int_0^t \Delta^2(s) ds. \]

Formal heuristic proof. Rewrite Ito’s process in differential form

\[ dX(t) = \Delta(t) dW(t) + \Theta(t) dt. \]

Then we use \( dW(t) dW(t) = dt, dW(t) dt = dt dt = 0 \) to get

\[

dX(t) dX(t) = \Delta^2(t) dW(t) dW(t) + 2 \Delta(t) \Theta(t) dW(t) dt + \Theta^2(t) dt dt \\
= \Delta^2(t) dt.
\]
Integral with respect to Ito process

- **Definition.** Let $X(t), t \geq 0$ be an Ito process, and let $\Gamma(t), t \geq 0$ be an adopted process. Define the integral with respect to Ito’s process

\[
\int_0^t \Gamma(s)dX(s) = \int_0^t \Gamma(s)\Delta(s)dW(s) + \int_0^t \Gamma(s)\Theta(s)ds.
\]
• **Theorem.** (Ito-Doeblin formula for an Ito's process)

Let $X(t), t \geq 0$ be an Ito process, and let $f$ be a $C^{1,2}_{t,x}$-function. Then for any $T \geq 0$, we have

$$f(T, X(T)) = f(0, X(0)) + \int_0^T f_t(t, X(t))dt$$
$$+ \int_0^T f_x(t, X(t))dX(t) + \frac{1}{2} \int_0^T f_{xx}(t, X(t))d[X, X](t)$$
$$= f(0, X(0)) + \int_0^T f_t(t, X(t))dt + \int_0^T f_x(t, X(t))\Delta(t)dW(t)$$
$$+ \int_0^T f_x(t, X(t))\Theta(t)dt + \frac{1}{2} \int_0^T f_{xx}(t, X(t))\Delta^2(t)dt.$$
Examples for Ito’s processes

1. Geometric Brownian Motion.

\[ dS(t) = \alpha S(t)dt + \sigma S(t) dW(t), \quad \alpha, \sigma: \text{constants} \]

Apply Ito’s formula to \( \ln S(t) \), i.e.,

\[
\begin{align*}
    d\ln S(t) &= \frac{dS(t)}{S(t)} - \frac{1}{2S(t)^2} \sigma^2 S^2(t) dt \\
    &= \left( \alpha - \frac{1}{2} \sigma^2 \right) dt + \sigma dW(t).
\end{align*}
\]

We integrate the above equality from 0 to \( t \) to get

\[ S(t) = S(0)e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W(t)}. \]
2. Generalized geometric Brownian Motion.

\[ dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t). \]

As before, we apply Ito’s formula to \( \ln S(t) \) to find

\[ d\ln S(t) = \left( \alpha(t) - \frac{1}{2}\sigma(t)^2 \right)dt + \sigma(t)dW(t). \]

Direct integration yields

\[ S(t) = S(0)e^{\int_0^t(\alpha(s) - \frac{\sigma(s)^2}{2})ds + \int_0^t\sigma(s)dW(s)}. \]
3. Vasicek interest rate model.

\[ dR(t) = (\alpha - \beta R(t))dt + \sigma dW(t). \]

Here \( \alpha, \beta \) and \( \sigma \) are positive constants.

We apply Ito’s formula to \( e^{\beta t} R(t) \) to get

\[
\begin{align*}
d(e^{\beta t} R(t)) &= \beta e^{\beta t} R(t) dt + e^{\beta t} dR(t) \\
&= \alpha e^{\beta t} dt + \sigma e^{\beta t} dW(t).
\end{align*}
\]

We now integrate the above relation from 0 to \( t \) and find

\[
R(t) = e^{-\beta t} R(0) + \frac{\alpha}{\beta} (1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta (t-s)} dW(s).
\]
4. Cox-Ingersoll-Ross (CIR) interest rate model.

\[ dR(t) = (\alpha - \beta R(t))dt + \sigma \sqrt{R(t)}dW(t). \]

We apply Ito’s formula to \( e^{\beta t}R(t) \) to find

\[ d(e^{\beta t}R(t)) = \alpha e^{\beta t}dt + \sigma e^{\beta t} \sqrt{R(t)}dW(t). \]

We integrate the above relation to get

\[ e^{\beta t}R(t) = R(0) + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta u} \sqrt{R(t)}dW(u). \]
Black-Scholes-Merton equation

- Derivation of B-S-M equation

Please see the separate note.
**Definition.** A stochastic differential equation (in short SDE) is an equation of the form

$$dX(s) = \beta(s, X(s))ds + \gamma(s, X(s))dW(s), \quad s \geq t,$$

$$X(t) = X_0.$$ 

where

$$\beta(s, x) : \text{drift}, \quad \gamma(s, x) : \text{diffusion}.$$ 

or equivalently,

$$X(T) = x + \int_t^T \beta(s, X(s))ds + \int_t^T \gamma(s, X(s))dW(s).$$
Consider one-dimensional linear SDE:

\[ dX(s) = (a(s) + b(s)X(s))ds + (\gamma(s) + \sigma(s)X(s))dW(s), \]

where \( a, b, \gamma, \sigma \) are nonrandom function of time \( s \).

- **Examples**
  1. Geometric Brownian motion.
     \[ dS(t) = \alpha S(t)dt + \sigma S(t)dW(t). \]
  2. Hull-White interest rate model.
     \[ dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)d\tilde{W}(t). \]
Consider SDE:

\[
dX(s) = \beta(s, X(s))ds + \gamma(s, X(s))dW(s), \quad s \geq t,
\]
\[
X(t) = X_0.
\]

Let \(0 \leq t \leq T\) be given, and let \(h(y)\) be a Borel-measurable function. We denote by

\[
g(t, x) := E^{t,x} h(X(T)),
\]

where \(X(T)\) is the solution of SDE with initial data \(X(t) = x\).
• **Theorem.** Let $X(s), s \geq 0$ be a solution to the stochastic differential equation with initial condition given at time 0. Then for $0 \leq t \leq T$,

$$E[h(X(T))|\mathcal{F}(t)] = g(t, X(t)).$$

• **Corollary.**

Solutions to SDE are Markov process.
Theorem. Consider the stochastic differential equation

\[ dX(s) = \beta(s, X(s))ds + \gamma(s, X(s))dW(s). \]

Let \( h(y) \) be a Borel-measurable function. Fix \( T > 0 \), and let \( t \in [0, T] \) be given. Define the function

\[ g(t, x) = E^{t,x} h(X(T)). \]

Then \( g(t, x) \) satisfies the following PDE of parabolic type:

\[ g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0, \]

with the terminal condition:

\[ g(T, x) = h(x), \quad \text{for all} \quad x. \]
• **Lemma.** Let \( X = X(s) \) be a solution to the SDE:

\[
dX(s) = \beta(s, X(s))ds + \gamma(s, X(s))dW(s),
\]

with initial condition given at time 0. Let \( h(y) \) be a Borel-measurable function. Fix \( T > 0 \), and let \( g = g(t, x) \) be given as before. Then stochastic process

\[
g(t, X(t)), \quad 0 \leq t \leq T, \quad \text{is a martingale.}
\]
Outline of proof of Feynman-Kac’s formula: Let \( X(t) \) be the solution to the SDE starting at time zero. Since \( g(t, X(t)) \) is a martingale, the net \( dt \) in the differential \( g(t, X(t)) \) must be zero.

\[
    dg(t, X(t)) = g_t dt + g_x dX + \frac{1}{2} g_{xx} dX dX
    = g_t dt + \beta g_x dt + \gamma g_x dW + \frac{1}{2} \gamma^2 g_{xx} dt
    = \left[ g_t + \beta g_x + \frac{1}{2} \gamma^2 g_{xx} \right] dt + \gamma g_x dW.
\]

Hence we have

\[
    g_t(t, X(t)) + \beta(t, X(t))g_x(t, X(t)) + \frac{1}{2} \gamma^2(t, X(t))g_{xx}(t, X(t)) = 0,
\]
along every path of \( X \). Therefore, we have

\[
    g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2} \gamma^2(t, x)g_{xx}(t, x) = 0.
\]