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Financial Mathematics 1 - Spring term 2015

Exercise sheet no. 10 (21.5.2015)

Exercise 1: Write the following processes as Itô processes and precise their drift and diffusion coefficients ((B_t) is a standard 1-dimensional BM):

(i) $X_t = t + e^{B_t}$

(ii) $X_t = B_t^3 - 3tB_t$

(iii) $X_t = (B_t + t)\exp(-B_t - \frac{1}{2}t)$

Exercise 2: Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0}, P)$ be a standard BM.

(i) Let $(L_t)_{t\in[0,T]}$ be a $(\mathcal{F}_t)_{t\in[0,T]}$-martingale. Show that the probability measures $dP^{L_T} := L_T dP$ and $dP^{L_t} := L_t dP$ coincide on $\mathcal{F}_t$, $t \in [0,T]$.

(ii) Let ϑ_s be bounded and deterministic, i.e. $\vartheta_s = f(s)$ for some bounded measurable function f. Calculate

$$E_P\left[B_t \exp\big(\int_0^T \vartheta_s dB_s - \frac{1}{2}\int_0^T \vartheta_s^2 ds\big)\right], \quad t \geq 0.$$

Exercise 3: Let $(H_t)_{t\in[0,T]}$ be adapted, measurable, and such that $\int_0^T H_s^2 ds < \infty$ P-a.s. Consider the stochastic integral $M_t := \int_0^t H_s dW_s$. Show that if $E\big[\sup_{t\in[0,T]} |M_t|^2\big] < \infty$, then $E\big[\int_0^T H_s^2 ds\big] < \infty$ (i.e. $H \in \mathcal{H}$ and therefore (M_t) is a martingale).
Hint: Consider the stopping times

$$T_n := \inf\Big\{s \in [0,T] \mid \int_0^s H_u^2\, du \geq n\Big\}, \quad n \geq 1,$$

and show that $E\big[M_{t\wedge T_n}^2\big] = E\big[\int_0^{t\wedge T_n} H_s^2 ds\big]$. For this, you may have a look at the proof of Proposition 3.21 of the lecture.

Please drop the solutions into the homework box for the lecture until 28.5.2015, 6 pm