Gerald Trutnau Department of Mathematical Sciences, SNU

Financial Mathematics 1 - Spring term 2015

Exercise sheet no.4 (2.4.2015)

Exercise 1:

(i) Show the statement "Then, there exists a r.v. $X \neq 0$ with $\mathbb{E}^*[XY] = 0$ for all $Y \in \tilde{V}$." on page 19 of the lecture.

(ii) Show the claimed identity $E^{**}[\sum_{n=1}^{N} \phi_n \cdot \Delta \widetilde{S}_n] = 0$ on page 19 of the lecture.

(iii) Show that $\mathcal{F}_n = \sigma(S_0, ..., S_n)$, $n = 0, ..., N$, as claimed on page 22 of the lecture.

(iv) Show that the conditional expectation is monotone, i.e. if $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathcal{P})$, $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{F}$ and $X \leq Y$ P-a.s, then $E[X|\mathcal{A}] \leq E[Y|\mathcal{A}]$ P-a.s.

Exercise 2: Consider the dicrete-time financial model from section 2.1 of the lecture. A *numéraire* is an adated sequence $W = (W_n)_{n=0,...,N}$ with $W_0 = 1$, $W_n > 0$ for $n = 1, ..., N$, and $W_n = V_n(\theta)$, $n = 0, ..., N$ for some admissible startegy θ. Define $S^W := \frac{S_n}{W_n}$, $n = 0, ..., N$ (the W-discounted price process).

(i) Show that a predictable sequence $\phi = (\phi_n)_{n=0,...,N}$ with values in $\mathbb{R}^{d+1}$ is self-financing, if and only if

$$V_n^W(\phi) := \frac{V_n(\phi)}{W_n} = V_0(\phi) + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^W, \quad n = 1, ..., N.$$

(ii) Prove that for $n = 1, ..., N$ we have $\sum_{j=1}^{n} \theta_j \cdot \Delta S_j^W = 0$.

(iii) Prove that for any predictable sequence $\phi = (\phi_n)_{n=0,...,N}$ with values in $\mathbb{R}^{d+1}$ and any $V_0 \in \mathbb{R}$ there exists a self-financing strategy $\hat{\phi}$ such that

$$\hat{\phi}_n \cdot S_n^W = V_0 + \sum_{j=1}^{n} \phi_j \cdot \Delta S_j^W, \quad n = 0, ..., N.$$

(iv) Prove that the market is viable, if and only if there exists a probability $P^W \approx P$, such that S^W is a P^W-martingale.

Remark: One can further show (here you are not asked to) that in a viable market there is at most one deterministic numéraire. And if the market is additionally complete and P^* denotes the unique risk-neutral measure for wich $(\widetilde{S}_n)_{n=0,...,N}$ is a martingale, then P^W is also unique and satisfies $\frac{dP^W}{dP^*} = \frac{W_N}{S_N^0}$. Moreover the fair price for a contingent claim h at time n is $W_n E^W[\frac{h}{W_N} \,|\, \mathcal{F}_n]$.

Please drop the solutions into the homework box of the lecture until 9.4.2015, 6 pm