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Financial Mathematics 1 - Spring term 2015

Exercise sheet no.9 (14.5.2015)

Exercise 1: Complete the proof of Proposition 3.24 of the lecture (webpage), i.e. show that

$$\tilde{J}(H) = J(H) \quad \forall H \in \mathcal{H},$$

and show the uniqueness of $\tilde{J}$.

Exercise 2: Let $(M_t)_{t\in[0,T]}$ be a continuous $(\mathcal{F}_t)_{t\in[0,T]}$-martingale with $M_t = \int_0^t K_s ds$, $t \in [0,T]$, where $(K_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process.

(i) Suppose that $\int_0^T |K_s|ds < const.$ P-a.s. Prove that if we write $t_i^n = \frac{Ti}{n}$ for $i = 0, ..., n$, then

$$\lim_{n\to\infty} E\left[\sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2\right] = 0.$$

(ii) Under the same assumptions as in (i), prove that

$$\lim_{n\to\infty} E\left[\sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2\right] = E\left[M_T^2 - M_0^2\right].$$

Conclude that $M_T = 0$ P-a.s., and thus $P(M_t = 0\ \forall t \in [0,T]) = 1$.

(iii) Now assume that only $\int_0^T |K_s|ds < \infty$ P-a.s. (You may use without proof that $\int_0^t |K_s|ds$ is $\mathcal{F}_t$-measurable). Show that

$$T_n := \inf\{t \in [0,T] \mid \int_0^t |K_s|ds \geq n\}, \qquad \inf\emptyset := T,$$

is a stopping time. Prove that P-a.s. $\lim_{n\to\infty} T_n = T$. Considering the sequence of martingales $(M_{t\wedge T_n})_{t\in[0,T]}$ (you may accept that $(M_{t\wedge T_n})_{t\in[0,T]}$ is again a continuous martingale for any n without proof) prove that $P(M_t = 0\ \forall t \in [0,T]) = 1$)

(iv) Let M_t be a martingale of the form $\int_0^t H_s dW_s + \int_0^t K_s ds$ with adapted processes satisfying $\int_0^t H_s^2 ds < \infty$ and $\int_0^t |K_s|ds < \infty$ P-a.s. Define the sequence of stopping times $T_n := \inf\{t \in [0,T] \mid \int_0^t H_s^2 ds \geq n\}$ in order to prove $K = 0$ $dt \otimes P$-a.s.

Exercise 3: Proceed as in Remark 3.28 of the lecture (webpage) in order to show that (27) of the lecture holds.

Please drop the solutions into the homework box of the lecture until 21.5.2015, 6 pm