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Financial Mathematics 2 - Fall term 2015

Exercise sheet no.7 (26.10.2015)

Exercise 1: Let $r(\theta)$ be as at the end of Section 6.1.2 of the lecture notes. Calculate the law of $r(\theta)$ under P^θ, and the law of $r(\theta)$ under $P^{\theta,T}$ using Proposition 6.15.

Exercise 2: ("Stochastic Fubini") Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Consider a process with two indices $(H(t,s))_{t,s\in[0,T]}$ satisfying the following properties: for any ω, the map $(t,s)\mapsto H(t,s)(\omega)$ is continuous and for any $s\in[0,T]$, the process $(H(t,s))_{t\in[0,T]}$ is adapted. We would like to justify the equality

$$\int_0^T \left(\int_0^T H(t,s)dW_t\right) ds = \int_0^T \left(\int_0^T H(t,s)ds\right) dW_t.$$

For simplicity, we assume that $\int_0^T \mathbb{E}\left[\int_0^T H^2(t,s)dt\right] ds < \infty$ (which is sufficient to justify (108) of the lecture notes).

(i) Prove that

$$\int_0^T \mathbb{E}\left[\left|\int_0^T H(t,s)dW_t\right|\right] ds \leq \int_0^T \mathbb{E}\left[\int_0^T H^2(t,s)dt\right]^{1/2} ds$$

Deduce that the integral $\int_0^T \left(\int_0^T H(t,s)dW_t\right) ds$ exists.

(ii) Let $0 = t_0 < t_1 < ... < t_N = T$ be a partition of $[0,T]$. Explain why

$$\int_0^T \left(\sum_{i=0}^{N-1} H(t_i,s)(W_{t_{i+1}} - W_{t_i})\right) ds = \sum_{i=0}^{N-1}\left(\int_0^T H(t_i,s)ds\right)(W_{t_{i+1}} - W_{t_i})$$

and justify why one can take the linit to obtain the desired equality.

Exercise 3: In the HJM-model assume the the function σ is a positive constant.

(i) Show that the solution of (111) of the lecture notes is given by

$$f(t,T) = f(0,T) + \sigma^2 t(T - t/2) + \sigma\tilde{W}_t.$$

(ii) Compute the volatilities of the zero-coupon bonds $(\sigma_t^T)_{t\in[0,T]}$.

(iii) Find the price at time 0 of a call with maturity θ and strike price K, on a zero-coupon bond with maturity $T > \theta$.

Please drop the solutions into the homework box of the lecture until 12.11.2015, 6 pm