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Financial Mathematics 2 - Fall term 2015

Exercise sheet no.9 (19.11.2015)

Exercise 1: Let $(V_n)_{n\geq 1}$ be a sequence independent, identically distributed, integrable random variables. Let N be a random variable taking values in $\mathbb{N}\cup\{0\}$, integrable and independent of the sequence $(V_n)_{n\geq 1}$. Let

$$S=\sum_{n=1}^{N} V_n$$

(i) Prove that S is integrable and that $\mathbb{E}[S]=\mathbb{E}[N]\mathbb{E}[V_1]$.

(ii) Assume N and V_1 are square-integrable. Then show that S is square-integrable and that its variance is $\mathrm{var}(S)=\mathbb{E}[N]\mathrm{var}(V_1)+\mathrm{var}(N)(\mathbb{E}[V_1])^2$.

(iii) Deduce that if N follows a Poisson law with parameter λ, $\mathbb{E}[S]=\lambda\mathbb{E}[V_1]$ and $\mathrm{var}(S)=\lambda\mathbb{E}[V_1^2]$.

Exercise 2: The hypotheses and notations are those of Exercise 4, Exercise sheet no.8. Suppose that the V_j's take values in $\{\alpha,\beta\}$, with $\alpha,\beta\in\mathbb{R}$, and let $p=P(V_1=\alpha)=1-P(V_1=\beta)$. Prove that S has the same law as $\alpha N_1+\beta N_2$, where N_1 and N_2 are two independent random variables with Poisson distributions with respective parameters λp and $\lambda(1-p)$.

Exercise 3: The hypotheses and notations are those of Chapter 7 of the lecture notes. For $\theta\in\mathbb{R}$ and $u>-1$ define

$$Z_t^{\theta,u}:=e^{\theta W_t-\frac{\theta^2}{2}t}(1+u)^{N_t}e^{-\lambda ut}.$$

(i) Prove that $(Z_t^{\theta,u})_{t\in[0,T]}$ is a martingale.

(ii) Define the probability $\hat{P}$ with density $d\hat{P}/dP=Z_T^{\theta,u}$. Prove that under $\hat{P}$, the σ-algebras generaterd by (W_t), (N_t), (U_j) remain independent, and that the process $(N_t)_{t\in[0,T]}$ is a Poisson process with intensity $\hat{\lambda}=\lambda(1+u)$.

(iii) In the jump-diffusion model, the discounted price at time t is given by $\tilde{X}_t=X_0\left(\prod_{j=1}^{N_t}(1+U_j)\right)e^{\sigma W_t+\left(\mu-r-\frac{\sigma^2}{2}\right)t}$. Prove that $(\tilde{X}_t)_{t\in[0,T]}$ is a $\hat{P}$-martingale, if and only if

$$\mu+\sigma\theta=r-\lambda(1+u)\mathbb{E}[U_1].$$

If $\sigma>0$ and $\mathbb{E}[U_1]\neq 0$, this equality is satisfied for infinitely many pairs (θ,u). Therefore, the set of risk neutral probability measures is infinite.

Please drop the solutions into the homework box of the lecture until 26.11.2015, 6 pm