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Topics in Mathematics 1 - Spring term 2017

Exercise sheet no.1 (9.3.2017)

Exercise 1: (Sylvester's formula) Let $(\Omega, \mathcal{A}, P)$ be a probability space, I a finite index set, and $A_i \in \mathcal{A}$, $i \in I$. Show that

$$P\left(\bigcup_{i\in I} A_i\right) = \sum_{\substack{J\subset I\\ J\neq\emptyset}} (-1)^{|J|-1} \cdot P\left(\bigcap_{j\in J} A_j\right),$$

i.e. show that

$$P\left(\bigcup_{i\in I} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{1\le i_1<\ldots<i_k\le n} P\left(A_{i_1}\cap \ldots \cap A_{i_k}\right),$$

if $I = \{1, ..., n\}$ for some $n \in \mathbb{N}$.

Exercise 2: Let $(\Omega, \mathcal{A})$ be a measurable space, and $P : \mathcal{A} \to [0, \infty)$ a map with $P(\Omega) = 1$. Show that the following are equivalent:

(i) P is a probability measure.

(ii) P is additive, i.e. $A, B \in \mathcal{A}, A\cap B = \emptyset \Rightarrow P(A\cup B) = P(A)+P(B)$, and monotone continuous from above, i.e. for any sequence $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, with $A_{n+1} \subset A_n$ for all n, we have $P(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} P(A_n)$.

(iii) P is additive and monotone continuous from below, i.e. for any sequence $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, with $A_n \subset A_{n+1}$ for all n, we have $P(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} P(A_n)$.

Exercise 3: Let $(\Omega, \mathcal{A})$ be a measurable space and $A_n \in \mathcal{A}$, $n \in \mathbb{N}$. Show that

$$\limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} A_m = \{\omega \in \Omega | \omega \in A_n \text{ for infinitely many } n\},$$

$$\liminf_{n\to\infty} A_n := \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} A_m = \{\omega \in \Omega | \omega \in A_n \text{ for all but finitely many } n\},$$

and $\liminf_{n\to\infty} A_n$, $\limsup_{n\to\infty} A_n \in \mathcal{A}$ with $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$.

Please drop the solutions into the homework box for the lecture until 16.3.2017, 6 pm