Gerald Trutnau Department of Mathematical Sciences, SNU

Topics in Mathematics 1 - Spring term 2017

Exercise sheet no.11 (25.5.2017)

Exercise 1: For $a > 0$ and

$$T_n = \sum_{k:k>(1+a)n} \frac{n^k}{k!}$$

determine the asymptotic behavior of $T_n^{\frac{1}{n}}$, i.e. its limit as $n \to \infty$.

Exercise 2: Let S be a finite set, μ the uniform distribution on S, $U : S \to \mathbb{R}$ an arbitrary map and $(\mu_\lambda)_{\lambda \in \Lambda}$ the corresponding exponential family. Show with the help of Cramér's Theorem that

$$\lim_{n\to\infty} \frac{1}{n} \log |\{\bar{x} = (x_1, \ldots, x_n) | \frac{1}{n} \sum_{i=1}^{n} U(x_i) \in [m, m+\varepsilon)\}| = H(\mu_{\lambda(m)})$$

for $m > \int U \, d\mu$ and $\varepsilon > 0$.

Exercise 3: Let S be a finite set, $(\Omega, \mathcal{A})$ a measurable space and $X_k : \Omega \to S$ $\mathcal{A}/\mathcal{P}(\mathcal{S})$–measurable, $1 \leq k \leq n$. Show with the help of Exercise 1, no. 4: for any $A_n \in \sigma(X_1, \ldots, X_n)$ there exists $C_n \in S^n$ with

$$A_n = \{\omega \in \Omega | (X_1(\omega), \ldots, X_n(\omega)) \in C_n\} \ .$$

For Exercises 4-6 let S be a separable metric space with coressponding Borel σ–algebra $\mathcal{B}(S)$.

Exercise 4: Let $\mathcal{M}_1(S)$ be the set of all probability measures on S equipped with the weak topology, $\mathcal{B}(\mathcal{M}_1(S))$ the coressponding Borel σ–algebra. Let further $(X_k)_{1 \leq k \leq n}$ be a family of random variables on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ with values in S. Show with the help of Exercise 6 (next page), that the empirical distribution

$$\varrho_n : \Omega \to \mathcal{M}_1(S), \, \omega \mapsto \varrho_n(\omega) := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(\omega)},$$

is $\mathcal{A}/\mathcal{B}(\mathcal{M}_1(S))$–measurable.

Exercise 5: Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. on $(\Omega, \mathcal{A}, \mathcal{P})$ with distribution μ on S and $\varrho_n(\omega) := \frac{1}{n}\sum_{i=1}^n \delta_{X_i(\omega)}$, $n \in \mathbb{N}$, the sequence of empirical distributions. Show with the help of Exercise 6, that

$$\varrho_n \longrightarrow \mu \text{ weakly } P\text{–a.s.}$$

Exercise 6: Let μ_n, μ be probability measures on S. Show:

(i) There are open subsets $U_1, U_2, \dots$ of S with

$$\lim_{n\to\infty} \mu_n = \mu \text{ weakly } \Leftrightarrow \liminf_{n\to\infty} \mu_n(U_k) \geq \mu(U_k) \text{ für alle } k \in \mathbb{N}\ .$$

(Hint: Let $S_0 \subset S$ be a countable dense subset, $\mathcal{U}_0 = \{B_q(s_0) | q \in \mathbb{Q}^+\, , s_0 \in S_0\}$ and $\mathcal{U} = \{\bigcup_{n=1}^m V_n | m \in \mathbb{N}, V_n \in \mathcal{U}_0,\ 1 \leq n \leq m\}$. Show that there exists for any open subset $U \subset S$ a sequence $(V_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ with $V_n \subset V_{n+1}$ and $U = \bigcup_{n\in\mathbb{N}} V_n$. Show then, that $\liminf_{n\to\infty} \mu_n(V_k) \geq \mu(V_k)$ for all $k \in \mathbb{N}$ implies $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ and apply Theorem I.10.3.)

(ii) There are functions $g_1, g_2, \dots \in C_b(S)$ with

$$\lim_{n\to\infty} \mu_n = \mu \text{ weakly } \Leftrightarrow \lim_{n\to\infty} \int g_k\, d\mu_n = \int g_k\, d\mu \text{ for all } k \in \mathbb{N}.$$

(Hint: Let $U_1, U_2, \dots$ be as in (i). For each U_k construct a sequence of continuous functions $(f_n^k)_{n\in\mathbb{N}}$ with $0 \leq f_n^k \leq 1_{U_k}$ and $\lim_{n\to\infty} f_n \nearrow 1_{U_k}$ pointwise.)

Please drop the solutions into the homework box for the lecture until 9.6.2017, 6 pm